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Positive periodic and subharmonic solutions of second order singular differential equations with impulsive effects

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Abstract. In this paper, we study the existence and multiplicity of positive periodic and subharmonic solutions of second order singular differential equations with impulsive effects. The proof is based on a generalized version of the Poincaré-Birkhoff twist theorem due to Ding and the phase plane analysis methods.

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1. Introduction

In this paper, we are concerned with the existence and multiplicity of positive periodic and subharmonic solutions of the second order singular differential equation with impulsive effects

$$\begin{cases} \ddot{u}+f(u)=p(t), & t\neq t_j; \\ \Delta u(t_j)=\tilde{I}_j\big(u(t_j^-),\dot{u}(t_j^-)\big), & \\ \Delta \dot{u}(t_j)=\tilde{L}_j\big(u(t_j^-),\dot{u}(t_j^-)\big), & j=1,2,\ldots, \end{cases} \tag{1}$$

where $f:\mathbb{R}^+\to\mathbb{R}$ is locally Lipschitz continuous and has a singularity at the origin, $p:\mathbb{R}\to\mathbb{R}$ is a continuous T-periodic function, $0<t_1<t_2<\cdots<t_k<T$, $t_{j+k}=t_j+T$, $\Delta u(t_j)=u(t_j^+)-u(t_j^-)$, $\Delta\dot{u}(t_j)=\dot{u}(t_j^+)-\dot{u}(t_j^-)$, $\tilde{I}_j:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}^+$ and $\tilde{L}_j:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ are continuous maps with $\tilde{I}_{j+k}=\tilde{I}_j$ and $\tilde{L}_{j+k}=\tilde{L}_j$ for $j=1,2,\ldots$. For simplicity, we shall only consider right-continuous solutions, i.e., $u(t_j^+)=u(t_j)$ and $\dot{u}(t_j^+)=\dot{u}(t_j)$, $j=1,2,\ldots$.

In this paper, we consider system (1) under the following conditions:

(A_1) $\lim\limits_{u\to+\infty}\frac{f(u)}{u}=+\infty$;

(A_2) $\lim\limits_{u\to 0^+}f(u)=-\infty$;

(A_3) $\lim\limits_{u\to 0^+}F(u)=+\infty$, where $F(u)=\int_1^u f(s)\mathrm{d}s$;

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(A$_4$) $\lim\limits_{u\to 0^+} \tilde{I}_j(u,v) = 0$ uniformly for $v \in \mathbb{R}$, $j = 1, 2, \ldots$.

Impulsive differential equations, which were initiated by the work of Mil'man and Myshkis in [11], are basic models in the study of evolution processes of real life phenomena that are subjected to abrupt changes in their state, and have an extensive physical, chemical, biological, engineering background and realistic mathematical model. Therefore, impulsive differential equations are very important in the theory of differential equations. During the past few years, the periodic problem of impulsive differential equations has been widely studied the in literature. We just refer the reader to classical papers [8, 14, 15, 16] for the existence of periodic solutions of impulsive differential equations via fixed point theory, [7, 12, 19] via topological degree theory and [17, 23, 24, 27] via variational methods.

Recently, in [18] Qian, Chen and Sun studied the existence and multiplicity of periodic solutions for the superlinear impulsive second order differential equation of the form

$$\begin{cases} \ddot{x} + g(x) = p(t, x, \dot{x}), & t \neq t_j; \\ \Delta x(t_j) = \bar{I}_j\big(x(t_j^-), \dot{x}(t_j^-)\big), & \\ \Delta \dot{x}(t_j) = \bar{J}_j\big(x(t_j^-), \dot{x}(t_j^-)\big), & j = \pm 1, \pm 2, \ldots, \end{cases}$$

where $0 \le t_1 < t_2 < \cdots < t_k < 2\pi$, $t_{j+k} = t_j + 2\pi$, $\Delta x(t_j) = x(t_j^+) - x(t_j^-)$, $\Delta \dot{x}(t_j) = \dot{x}(t_j^+) - \dot{x}(t_j^-)$, $\bar{I}_j, \bar{J}_j : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous maps with $\bar{I}_{j+k} = \bar{I}_j$ and $\bar{J}_{j+k} = \bar{J}_j$ for $j = \pm 1, \pm 2, \ldots$. In addition, $g : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the superlinear growth condition

$$\lim_{|x|\to\infty} \frac{g(x)}{x} = +\infty,$$

and $p : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is bounded, continuous and 2π-periodic in the first variable. The proof is based on a generalized version of the Poincaré-Birkhoff twist theorem by Rebelo [21, Corollary 2 and Remark 2].

However, up to now, there are only a few works focused on the existence of positive periodic solutions for singular differential equations with impulsive effects [3, 25]. For instance, in [3] Chu and Nieto studied the existence of positive periodic solutions of the first order singular differential equation with impulsive effects

$$\begin{cases} \dot{x} + a(t)x = f(t, x) + e(t), & t \neq t_j; \\ \Delta x(t_j) = J_j\big(u(t_j^-)\big), & j = 1, 2, \ldots, \end{cases} \tag{2}$$

where a, e are continuous and 1-periodic functions, $0 = t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} = 1$. The nonlinearity $f(t, x)$ is continuous and presents a singularity at $x = 0$, $f(t_j^+, x)$ and $f(t_j^-, x)$ exist, $f(t_j^-, x) = f(t_j, x)$ and 1-periodic in t. The impulses $J_j : \mathbb{R} \to \mathbb{R}$, $k = 1, \ldots$ are continuous functions. Based on a nonlinear alternative principle of Leray-Schauder, together with a truncation technique, they obtained some existence results about positive periodic solutions of system (2). In [25], Sun and his co-authors analyzed the second order singular differential equation with impulsive effects

$$\begin{cases} \ddot{x} - \frac{1}{x^{\alpha}(t)} = e(t), & t \neq t_j; \\ \Delta \dot{x}(t_j) = J_j\big(u(t_j)\big), & j = 1, 2, \ldots, p-1, \end{cases} \tag{3}$$

where $\alpha \geq 1$, $e \in L^1([0,T])$ is T-periodic, $0 = t_0 < t_1 < t_2 < \cdots < t_{p-1} < t_p = T$, $t_{j+p} = t_j + T$, $J_j : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous maps with $J_{j+p} = J_j$ for $j = 1, 2, \ldots, p-1$. Based on the mountain-pass theorem, it was proved that system (3) has at least one positive periodic solution.

The aim of this paper is to study the existence and multiplicity of positive periodic and subharmonic solutions for the second order singular differential equation with impulsive effects (1). As far as we know, such problems have been rarely studied in the literature until now. Our proof is based on a generalized version of the Poincaré-Birkhoff twist theorem due to Ding [4], and it has been widely applied to study periodic problems of second order differential equations without impulsive effects and many nice results have been obtained. See [1, 2, 5, 6, 9, 10, 13, 20, 22, 26] and the references therein. Let us recall here the generalized Poincaré-Birkhoff twist theorem due to Ding [4].

Theorem 1 ([4, Generalized Poincaré-Birkhoff twist theorem])**.** *Let D denote an annular region in the (x, y)-plane. The bounds of D consist of two simple closed curves, the inner boundary curve C_1 and the outer boundary curve C_2. Let D_1 denote the simple connected open set bounded by C_1. Consider an area-preserving mapping $P : \mathbb{R}^2 \to \mathbb{R}^2$. Suppose that $P(D) \subset \mathbb{R}^2/\{O\}$. Let (r, θ) be the polar coordinate of (x, y). Assume the restriction $P|D$ is given by*

$$\theta^* = \theta + h(\theta, r), \quad r^* = \varphi(\theta, r),$$

where h and φ are continuous in (θ, r) and T-periodic in θ, and the following conditions hold:

(i) C_1 *and* C_2 *are star-shaped about the origin* O*;*

(ii) $O \in P(D_1)$*;*

(iii) $h(\theta, r) > 0 (< 0), (r\cos\theta, r\sin\theta) \in C_1$; $h(\theta, r) < 0 (> 0), (r\cos\theta, r\sin\theta) \in C_2$.

Then P has at least two fixed points in D.

The rest part of this paper is organized as follows. In Section 2, some preliminary results will be given in order to use the generalized Poincaré-Birkhoff twist theorem. The existence and multiplicity results of system (1) are stated and proved in Section 3.

2. Some lemmas

In order to use Theorem 1 and the phase plane analysis methods conveniently, we will not consider system (1) directly, in which f has a singularity at the origin.

Instead, we consider the equation with impulsive effects

$$\begin{cases} \ddot{x} + g(x) = p(t), & t \neq t_j; \\ \Delta x(t_j) = I_j\big(x(t_j^-), \dot{x}(t_j^-)\big), \\ \Delta \dot{x}(t_j) = L_j\big(x(t_j^-), \dot{x}(t_j^-)\big), & j = 1, 2, \dots, \end{cases} \tag{4}$$

where $g : (-1, +\infty) \to \mathbb{R}$ is locally Lipschitz continuous and has a singularity at $x = -1$ with $g(x) = f(x+1)$, $\dot{x}(t_j^+) = \dot{x}(t_j)$, $x(t_j^+) = x(t_j)$, $j = 1, 2, \dots$. $I_j : (-1, +\infty) \times \mathbb{R} \to (-1, +\infty)$ and $L_j : (-1, +\infty) \times \mathbb{R} \to \mathbb{R}$ are continuous maps with $I_j(x, \dot{x}) = \tilde{I}_j(x+1, \dot{x})$ and $L_j(x, \dot{x}) = \tilde{L}_j(x+1, \dot{x})$, $j = 1, 2, \dots$. In fact, we can take a parallel translation $u = 1 + x$ to achieve this aim. Under this transformation, assumptions (A_1)-(A_4) become

(H_1) $\lim\limits_{x \to +\infty} \frac{g(x)}{x} = +\infty$;

(H_2) $\lim\limits_{x \to -1^+} g(x) = -\infty$;

(H_3) $\lim\limits_{x \to -1^+} G(x) = +\infty$, where $G(x) = \int_0^x g(s)\mathrm{d}s$;

(H_4) $\lim\limits_{x \to -1^+} I_j(x, y) = 0$ uniformly for $y \in \mathbb{R}$, $j = 1, 2, \dots$.

Let us rewrite system (4) as its equivalent form

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -g(x) + p(t), & t \neq t_j; \\ \Delta x(t_j) = I_j\big(x(t_j^-), y(t_j^-)\big), \\ \Delta y(t_j) = L_j\big(x(t_j^-), y(t_j^-)\big), & j = 1, 2, \dots. \end{cases} \tag{5}$$

We denote by $(x, y) = \big(x(t, x_0, y_0), y(t, x_0, y_0)\big)$ the solution of (5) satisfying the initial condition

$$\begin{cases} x(0) = x_0, \\ y(0) = y_0. \end{cases} \tag{6}$$

Lemma 1. *Assume that conditions* (H_1) *and* (H_3) *hold. Then for any* $(x_0, y_0) \in (-1, +\infty) \times \mathbb{R}$, *system* (5) *has a unique solution* (x, y) *which satisfies the initial condition* (6). *Moreover, we have that*

$$P_t : (x_0, y_0) \mapsto \big(x(t, x_0, y_0), y(t, x_0, y_0)\big)$$

is continuous in (x_0, y_0) *for* $t \neq t_j$, $j = 1, 2, \dots$.

Proof. Define a potential function

$$W(t) = \frac{1}{2}y^2(t) + G(x(t)).$$

Then we have

$$\begin{aligned} W'(t) &= y(t)y'(t) + g(x(t))x'(t) \\ &= y(t)\big(y'(t) + g(x(t))\big) \\ &= y(t)p(t). \end{aligned}$$

Obviously, we have

$$\big|W'(t)\big| \le \frac{1}{2}y^2(t) + \frac{1}{2}p^2(t) \le \frac{1}{2}y^2(t) + A,$$

where $A = \max\limits_{\forall t\in\mathbb{R}} \frac{p^2(t)}{2}$. By ($\mathrm{H}_1$) and ($\mathrm{H}_3$), we know that there exists a constant $M \ge 0$ such that $G(x) + M \ge 0$, which yields

$$\big|W'(t)\big| \le \frac{1}{2}y^2(t) + A + G(x) + M = W(t) + A_1,$$

where $A_1 = A + M$. Hence, for all $t \in [t_{j-1}, t_j)$, $j = 1, 2, 3, \ldots$, we have

$$W(t_{j-1})e^{-\Delta_j} + A_1(e^{-\Delta_j} - 1) \le W(t) \le W(t_{j-1})e^{\Delta_j} + A_1(e^{\Delta_j} - 1), \tag{7}$$

where $t_0 = 0$, $\Delta_j = t_j - t_{j-1}$, $j = 1, 2, \ldots$. By (7), it is not difficult to verify that there is no blow-up of the solution $(x(t), y(t))$ for all $t \in [0, t_1)$ if $W(0)$ is bounded. Since I_1, L_1 are continuous maps and $W(t_1^-)$ is bounded, then $W(t_1)$ is bounded, which implies that there is no blow-up of the solution $(x(t), y(t))$ for $t \in [t_1, t_2)$ if $W(0)$ is bounded. Therefore, it is by analogy that there is no blow-up for the solution $\big(x(t), y(t)\big)$ for all $t \in [t_{j-1}, t_j)$, $j = 1, 2, \ldots$ if $W(0)$ is bounded.

Consequently, by the fact that g is locally Lipschitz continuous and there is no blow-up for the solution $(x(t), y(t))$ for all $t \in [t_{j-1}, t_j)$, $j = 1, 2, \ldots$, we know that system (5) has a unique solution (x, y) which satisfies the initial condition (6).

From the theorem on continuity with respect to the initial condition and when g is continuous in $(-1, +\infty) \times \mathbb{R}$, we know that

$$P_t : (x_0, y_0) \mapsto \big(x(t, x_0, y_0), y(t, x_0, y_0)\big)$$

is continuous in (x_0, y_0) for $t \ne t_j, \quad j = 1, 2, \ldots$. □

Now, we can define the Poincaré map $P : (-1, +\infty) \times \mathbb{R} \to (-1, +\infty) \times \mathbb{R}$ as

$$P : (x_0, y_0) \mapsto \big(x(T, x_0, y_0), y(T, x_0, y_0)\big).$$

By Lemma 1, we know that P is continuous in (x_0, y_0).

Let us denote by $\big(x(t, x_0, y_0), y(t, x_0, y_0)\big)$ the solution of (5) satisfying the initial condition $\big(x(0,x_0,y_0), y(0,x_0,y_0)\big) = (x_0,y_0)$ and define

$$\begin{aligned}
P_0 &: (x_0,y_0) \mapsto \big(x(t_1^-,x_0,y_0), y(t_1^-,x_0,y_0)\big),\\
P_1 &: (x_1,y_1) \mapsto \big(x(t_2^-,x_1,y_1), y(t_2^-,x_1,y_1)\big),\\
&\vdots\\
P_j &: (x_j,y_j) \mapsto \big(x(t_{j+1}^-,x_j,y_j), y(t_{j+1}^-,x_j,y_j)\big),\\
&\vdots\\
P_{k-1} &: (x_{k-1},y_{k-1}) \mapsto \big(x(t_k^-,x_{k-1},y_{k-1}), y(t_k^-,x_{k-1},y_{k-1})\big),\\
P_k &: (x_k,y_k) \mapsto \big(x(T,x_k,y_k), y(T,x_k,y_k)\big),
\end{aligned}$$

where

$$(x_j,y_j) = \big(x(t_j,x_{j-1},y_{j-1}), y(t_j,x_{j-1},y_{j-1})\big), \quad j = 1,2,\dots.$$

Then the Poincaré map

$$P : (x_0,y_0) \mapsto \big(x(T,x_0,y_0), y(T,x_0,y_0)\big)$$

can be expressed by

$$P = P_k \circ \Phi_k \circ \cdots \circ P_1 \circ \Phi_1 \circ P_0,$$

where

$$\Phi_j : (x,y) \mapsto \big(x + I_j(x,y), y + L_j(x,y)\big), \quad j = 1,2,\dots.$$

A map $P : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be an area-preserving map or symplectic if

$$\det P' = 1.$$

Lemma 2. *Assume that $P_j,\ j = 1,2,\dots$ are area-preserving maps. Then the finite compositions of maps $P_j,\ j = 1,2,\dots$ are also area-preserving maps.*

Proof. Set $\P_n = P_n \circ P_{n-1} \circ \cdots P_2 \circ P_1,\ n = 1,2,\dots$. It comes directly from the fact that

$$\det \P_2' = \det(P_2' \circ P_1)\det P_1' = 1,$$

which implies that $\P_2$ is an area-preserving map. Analogously, we can conclude that

$$\begin{aligned}
\det \P_3' &= \det(P_3' \circ \P_2)\det \P_2' = 1,\\
\det \P_4' &= \det(P_4' \circ \P_3)\det \P_3' = 1,\\
&\vdots\\
\det \P_n' &= \det(P_n' \circ \P_{n-1})\det \P_{n-1}' = 1.
\end{aligned}$$

Therefore $\P_n,\ n = 1,2,\dots$ are area-preserving maps. $\square$

Since the equation

$$\begin{cases} \dot{x} = y,\\ \dot{y} = -g(x) + p(t), \end{cases}$$

is conservative, we know that $P_j,\ j = 1, 2, \dots$ are area-preserving maps. Therefore, by Lemma 2, we know that the Poincaré map P is an area-preserving homeomorphism if $\Phi_j,\ j = 1, 2, \dots$ are global area-preserving homeomorphisms.

We define a function $\Gamma : (-1, +\infty) \times \mathbb{R} \to \mathbb{R}^+$ as follows:

$$\Gamma(x, y) = x^2 + \frac{1}{(1 + x)^2} + y^2.$$

Denote

$$\Gamma_a := \big\{(x, y)\big|x^2 + \frac{1}{(1 + x)^2} + y^2 < a^2\big\},$$

where $a > 1$. Obviously, $O(0, 0) \in \Gamma_a$. Using the same ideas as in [18, Lemma 2.1], we can easily obtain the following results.

Lemma 3. *Let P be a global orientation-preserving planar homeomorphism and let a be a constant large enough such that $O \in P(\Gamma_a)$; then there exists a well defined continuous polar lifting of P outside Γ_a.*

Now we introduce a new assumption:

(H_5): $\Phi_j,\ j = 1, 2, \dots$ are global orientation-preserving and area-preserving homeomorphisms with the finite twist property.

Here we say that $\Phi_j,\ j = 1, 2, \dots$ have the finite twist property if there exist uniform constants $M_{\Phi_j},\ j = 1, 2, \dots$ such that

$$|\Delta\theta_{\Phi_j(z)}| = |arg(\Psi_j(z)) - arg(z)| \le M_{\Phi_j},\ j = 1, 2, \dots, \text{ for } \Gamma(z) \ge a,$$

where $z = (x, y)$, a is large enough such that $\Phi_j^{-1}(O) \in \Gamma_a,\ j = 1, 2, \dots$. If ($\mathrm{H}_5$) holds, then the Poincaré map P is an area-preserving homeomorphism in the phase plane. Clearly, the fixed points of the map P correspond to the periodic solutions of system (5).

Lemma 4. *Assume that conditions* (H_1), (H_3), (H_4) *and* (H_5) *hold. Then*

$$\Gamma(x(t), y(t)) \to +\infty \quad \text{uniformly} \quad \text{for} \quad t \in [t_{j-1}, t_j) \quad \text{as} \quad \Gamma(x_0, y_0) \to +\infty,$$

$j = 1, 2, \dots$.

Proof. By (7), we know that

$$\Gamma\big(x(t), y(t)\big) \to +\infty \quad \text{uniformly} \quad \text{for } t \in [0, t_1) \quad \text{as } \Gamma(x_0, y_0) \to +\infty. \tag{8}$$

If $\Gamma(x(t_1^-), y(t_1^-)) \to +\infty$, we know that at least one of the following cases holds

- $x(t_1^-) \to -1^+$;
- $x(t_1^-) \to +\infty$.

Then by (H_4) and (H_5), we know that at least one of the following cases holds

- $x(t_1) \to -1^+$;

- $x(t_1) \to +\infty$,

which implies that

$$\Gamma\big(x(t_1), y(t_1)\big) \to +\infty.$$

Similarly to (8), we have

$$\Gamma\big(x(t), y(t)\big) \to +\infty \quad \text{uniformly} \quad \text{for } t \in [t_1, t_2) \quad \text{as } \Gamma(x_0, y_0) \to +\infty.$$

Analogously, by (7), we can conclude that $\Gamma\big(x(t), y(t)\big) \to +\infty$ uniformly for $t \in [t_{j-1}, t_j)$ if $\Gamma(x_0, y_0) \to +\infty$, $j = 1, 2, 3, \ldots$. □

By Lemma 4, we know that if $\Gamma(x_0, y_0)$ is sufficiently large, then

$$x^2(t) + y^2(t) > 0, \quad \forall t \in [t_{j-1}, t_j), \quad j = 1, 2, 3, \ldots.$$

Then, it follows from Lemma 3 that Φ_j, $j = 1, 2, 3, \ldots$ have a continuous polar lifting with polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. Denote $\big(\theta(t, \theta_0, r_0), r(t, \theta_0, r_0)\big)$ and let (θ_0, r_0) be the polar coordinates of $\big(x(t, x_0, y_0), y(t, x_0, y_0)\big)$ and (x_0, y_0), respectively. By calculating, we get that $\big(\theta(t, \theta_0, r_0), r(t, \theta_0, r_0)\big)$ satisfies

$$\begin{cases} \dfrac{\mathrm{d}\theta}{\mathrm{d}t} = -\sin^2\theta - \dfrac{g(r\cos\theta)\cos\theta}{r} + \dfrac{p(t)\cos\theta}{r}, \\ \dfrac{\mathrm{d}r}{\mathrm{d}t} = r\sin\theta\cos\theta - g(r\cos\theta)\sin\theta + p(t)\sin\theta, \end{cases} \tag{9}$$

for $t \in [0, T]$, $t \neq t_j$, $j = 1, 2, \ldots$. If $\Gamma(x_0, y_0)$ is sufficiently large, we can easily see that the Poincaré map P has a continuous polar lifting

$$\tilde{P} : (\theta_0, r_0) \mapsto \big(\theta(T, \theta_0, r_0), r(T, \theta_0, r_0)\big)$$

which has the form

$$\tilde{P} = \tilde{P}_k \circ \tilde{\Phi}_k \circ \cdots \circ \tilde{P}_1 \circ \tilde{\Phi}_1 \circ \tilde{P}_0,$$

where

$$\begin{aligned} \tilde{\Phi}_j &: (\theta_j^-, r_j^-) \mapsto (\theta_j, r_j), \quad j = 1, 2, \ldots; \\ \tilde{P}_{j-1} &: (\theta_{j-1}, r_{j-1}) \mapsto (\theta_j^-, r_j^-), \quad j = 1, 2, \ldots; \\ \tilde{P}_k &: (\theta_k, r_k) \mapsto \big(\theta(T, \theta_0, r_0), r(T, \theta_0, r_0)\big), \end{aligned}$$

and

$$\begin{aligned} (\theta_j^-, r_j^-) &= \big(\theta(t_j^-, \theta_0, r_0), r(t_j^-, \theta_0, r_0)\big), \\ (\theta_j, r_j) &= \big(\theta_j^- + \Delta\theta_{\Phi_j}\big(z(t_j^-, z_0)\big), \big|\Phi_j(z(t_j^-, z_0))\big|\big), \quad j = 1, 2, \ldots, \end{aligned}$$

where $z(t_j^-, z_0) = \big(x(t_j^-, x_0, y_0), y(t_j^-, x_0, y_0)\big)$.

Lemma 5. *Assume that conditions* (H_1), (H_2), (H_3) *and* (H_4) *hold. Then for any* $t \in [t_{j-1}, t_j)$, $j = 1, 2, 3, \ldots$, *there exists a positive constant* ρ_0 *such that if* $\Gamma(x_0, y_0) \geq \rho_0^2$, *then* $\theta'(t) < 0$.

Proof. It follows from (H_1) that there exist positive constants a_0 and b_0 such that

$$\frac{g(x)-p(t)}{x} \geq a_0,\ x \in (b_0,+\infty),\ t \in [t_{j-1},t_j),\ j=1,2,3,\ldots,$$

which implies that if $x > b_0$, we have

$$\begin{aligned} \theta'(t) &= -\sin^2\theta - \frac{g(x)-p(t)}{x}\cos^2\theta \\ &\leq -\sin^2\theta - a_0\cos^2\theta \\ &< 0, \end{aligned} \tag{10}$$

for all $t \in [t_{j-1},t_j), j = 1,2,3,\ldots$. Moreover, by ($\mathrm{H}_2$), we know that there exist positive constants a_1 and $b_1 < 1$ such that

$$\frac{g(x)-p(t)}{x} \geq a_1,\ x \in (-1,-b_1),\ t \in [t_{j-1},t_j),\ j=1,2,3,\ldots,$$

which implies that if $-1 < x < -b_1$, we have

$$\begin{aligned} \theta'(t) &= -\sin^2\theta - \frac{g(x)-p(t)}{x}\cos^2\theta \\ &\leq -\sin^2\theta - a_1\cos^2\theta \\ &< 0, \end{aligned} \tag{11}$$

for all $t \in [t_{j-1},t_j), j = 1,2,3,\ldots$. For $x \in [-b_1,b_0]$, we know from Lemma 4 that there exist positive constants $\tilde{\rho}_0$ and a_3 such that

$$\left|\frac{g(x)-p(t)}{x}\right| \leq a_3, \quad |\sin\theta| > \sqrt{\frac{a_3}{1+a_3}}, \quad \text{for} \quad \Gamma(x_0,y_0) \geq \rho_0^2,$$

for all $t \in [t_{j-1},t_j),\ j = 1,2,3,\ldots$. Consequently, if $x \in [-b_1,b_0]$, we have

$$\begin{aligned} \theta'(t) &= -\sin^2\theta - \frac{g(x)-p(t)}{x}\cos^2\theta \\ &\leq -\sin^2\theta + \left|\frac{g(x)-p(t)}{x}\right|\cos^2\theta \\ &\leq -\sin^2\theta + a_3\cos^2\theta \\ &= -(1+a_3)\sin^2\theta + a_3 \\ &< 0 \end{aligned} \tag{12}$$

for all $t \in [t_{j-1},t_j)$, $j = 1,2,3,\ldots$. By Lemma 4, (10), (11) and (12), we know that for any $t \in [t_{j-1},t_j)$, $j = 1,2,3,\ldots$, there exists a positive constant ρ_0 such that if $\Gamma(x_0,y_0) \geq \rho_0^2$, then $\theta'(t) < 0$. □

Denote by $\Delta(\theta_0,r_0)$ the time for the solution $(\theta(t),r(t))$ to make one turn around the origin. Proceeding as in the proof of [10, Lemma 2.6], we can easily obtain the following lemma which gives the estimate on $\Delta(\theta_0,r_0)$.

Lemma 6. *Assume that conditions* (H_1), (H_2), (H_3), (H_4) *and* (H_5) *hold. Then* $\Delta(\theta_0, r_0) = o(1)$ *as* $\Gamma(x_0, y_0) \to +\infty$.

By Lemma 6, we notice that

$$\lim_{\Gamma(x_0,y_0)\to+\infty} \frac{t_j^- - t_{j-1}}{\Delta(\theta_0, r_0)} = +\infty, \quad j = 1, 2, 3, \ldots,$$

where $t_0 = 0$. Then by Lemma 4 and Lemma 5, one can easily obtain the following lemma.

Lemma 7. *Assume that conditions* (H_1), (H_2), (H_3), (H_4) *and* (H_5) *hold. Then for any* $n_j \in \mathbb{N}^+$, $j = 1, 2, \ldots$, *there exist sufficiently large constants* $\rho_{n_j} \geq \rho_0$, $j = 1, 2, \ldots$, *such that if* $\Gamma(x_0, y_0) \geq \rho_{n_j}$, *then*

$$\theta(t_j^-; \theta_0, r_0) - \theta(t_{j-1}; \theta_0, r_0) \leq -2n_j\pi, \quad j = 1, 2, \ldots.$$

Finally, the following results allow us to apply Theorem 1 to prove our main results in Section 3.

Lemma 8. [10, Lemma 3.1] *There exists a constant* $c_0 > 0$ *such that if* $c > c_0$, *then* $\Gamma(x, y) = c$ *is star-shaped with respect to the origin.*

3. Main results

In this section, we state and prove our main results.

Theorem 2. *Assume that conditions* (A_1), (A_2), (A_3), (A_4) *and* (H_5) *hold. Then system* (1) *has infinitely many* T*-periodic solutions* $u = u_i$, $i = 1, 2, \ldots$, *which satisfy*

$$\lim_{i\to+\infty} \Big(\min_{t\in[0,T]} (u_i(t) + |u_i'(t)|) \Big) = 0, \tag{13}$$

and

$$\lim_{i\to+\infty} \Big(\max_{t\in[0,T]} (u_i(t) + |u_i'(t)|) \Big) = +\infty. \tag{14}$$

Proof. For every solution u of system (1), there exists a solution x of system (4) satisfying $u = x + 1$, so we just need to prove that system (4) has infinitely many T-periodic solutions x_i, $i = 1, 2, \ldots$, which satisfy

$$\lim_{i\to+\infty} (\min_{t\in[0,T]} (1 + x_i(t) + |x_i'(t)|)) = 0,$$
$$\lim_{i\to+\infty} (\max_{t\in[0,T]} (1 + x_i(t) + |x_i'(t)|)) = +\infty.$$

Let

$$\begin{aligned}\big(x, y\big) &= \big(x(t, x_0, y_0), y(t, x_0, y_0)\big)\\ &= \big(r(t, \theta_0, r_0)\cos\theta(t, \theta_0, r_0), r(t, \theta_0, r_0)\sin\theta(t, \theta_0, r_0)\big)\end{aligned}$$

be the solution of system (5) satisfying the initial condition $(x(0), y(0)) = (x_0, y_0) = (r_0 \cos\theta_0, r_0 \sin\theta_0)$. We consider

$$\Delta_1(T, \theta_0, r_0) = \theta(T, \theta_0, r_0) - \theta_0.$$

Since $\Delta_1(T, \theta_0, r_0)$ is continuous with respect to (θ_0, r_0), so if we choose a suitably large constant α_1 and define

$$n = \left[\sup_{\Gamma(r_0 \cos\theta_0, r_0 \sin\theta_0) = \alpha_1} \frac{-\Delta_1(T, \theta_0, r_0)}{2\pi} \right] + 1,$$

where $[q]$ stands for the integer part of the real number of q, then we have

$$\theta(T, \theta_0, r_0) - \theta(0, \theta_0, r_0) > -2n\pi, \quad \text{for} \quad \Gamma(r_0 \cos\theta_0, r_0 \sin\theta_0) = \alpha_1. \tag{15}$$

Moreover, we have

$$\begin{aligned} \theta(T, \theta_0, r_0) - \theta(0, \theta_0, r_0) =& \, \theta(T, \theta_0, r_0) - \theta(t_k, \theta_0, r_0) \\ &+ \sum_{j=1}^{k} \Big(\theta(t_j, \theta_0, r_0) - \theta(t_{j-1}, \theta_0, r_0) \Big). \end{aligned}$$

It is easy to see that when $\Gamma(x_0, y_0)$ is large enough, θ is decreases strictly, then we have

$$\begin{aligned} \theta(T, \theta_0, r_0) - \theta(0, \theta_0, r_0) \leq& \sum_{j=1}^{k} \Big(\theta(t_j, \theta_0, r_0) - \theta(t_{j-1}, \theta_0, r_0) \Big) \\ =& \sum_{j=1}^{k} \Big(\theta(t_j^-, \theta_0, r_0) - \theta(t_{j-1}, \theta_0, r_0) \Big) \\ &+ \sum_{j=1}^{k} \Delta\theta_{\Phi_j} \big(x(t_j^-, x_0, y_0) \big). \end{aligned}$$

By Lemma 7 and (H_5), we can choose constant β_1 with $\beta_1 > \alpha_1$, integers n_j, $j = 1, 2, \ldots$ and n such that

$$\theta(t_j^-, \theta_0, r_0) - \theta(t_{j-1}, \theta_0, r_0) < -2n_j\pi, \quad \text{for} \quad \Gamma(r_0 \cos\theta_0, r_0 \sin\theta_0) = \beta_1,$$

$j = 1, 2, \ldots, k$, and

$$\sum_{j=1}^{k} -2n_j\pi + \sum_{j=1}^{k} M_{\Phi_j} < -2n\pi,$$

which imply that

$$\theta(T, \theta_0, r_0) - \theta(0, \theta_0, r_0) < -2n\pi, \quad \text{for} \quad \Gamma(r_0 \cos\theta_0, r_0 \sin\theta_0) = \beta_1. \tag{16}$$

Now we can consider the Poincaré map

$$\tilde{P}(\theta_0, r_0) = \big(\theta(T, \theta_0, r_0), r(T, \theta_0, r_0) \big)$$

on the annulus

$$S_1 : \alpha_1 \leq \Gamma(x_0, y_0) \leq \beta_1.$$

It is obvious that the boundaries of the annulus S_1 are star-shaped with respect to the origin by Lemma 8.

Since $P_j,\ j = 1, 2, \dots$ are area-preserving maps, $\Phi_j,\ j = 1, 2, \dots$ are global orientation-preserving and area-preserving homeomorphisms with the finite twist property, thus the Poincaré map P is an area-preserving homeomorphism. Meanwhile, the Poincaré map

$$\tilde{P}(\theta_0, r_0) = (\theta(T, \theta_0, r_0), r(T, \theta_0, r_0))$$

is a continuous polar lifting of P and is also an area-preserving homeomorphism. Moreover, from (15) and (16), we know that the Poincaré map $\tilde{P}$ is a twist on the annulus S_1. Finally, it is obvious that $r(T, \theta_0, r_0) > 0$ if $\Gamma(x_0, y_0) \geq \alpha_1$. Hence, $O \in \tilde{P}(D_1)$, where D_1 is an open region with boundary $\Gamma(x_0, y_0) = \alpha_1$.

As a consequence, all conditions of Theorem 1 are satisfied. By Theorem 1, we know that the Poincaré map P has at least two fixed points in the annulus S_1, i.e., system (5) has at least two T-periodic solutions on the annulus S_1.

Using the same method as above, we can construct infinitely many annuli

$$S_i : \alpha_i \leq \Gamma(x_0, y_0) \leq \beta_i,\ i = 1, 2, \dots,$$

such that the Poincaré map P has at least two fixed points in the annuli S_i, where $\alpha_{i+1} > \beta_i$, $i = 1, 2, \dots$. Now, if we choose an annulus S with $\bigcup_{i=1}^{+\infty} S_i \subseteq S$, then the Poincaré map P has infinitely many fixed points in the annulus S, which implies that system (5) has infinitely many T-periodic solutions $\big(x_i, y_i\big)$ on the annulus S, $i = 1, 2, \dots$.

Moreover, by $\alpha_{i+1} > \beta_i$ and $\lim_{i \to +\infty} \alpha_i = +\infty$, we have either

$$\lim_{i \to +\infty} \big(\min_{t \in [0,T]} x_i(t) \big) = -1$$

or

$$\lim_{i \to +\infty} \big(\max_{t \in [0,T]} x_i(t) \big) = +\infty,$$

which implies that (13) and (14) hold. □

In order to state and prove our existence result of infinitely many m-order subharmonic solutions for system (1), we define the Poincaré map

$$P^m : (-1, +\infty) \times \mathbb{R} \to (-1, +\infty) \times \mathbb{R}$$

as

$$P^m : (x_0, y_0) \mapsto \big(x(mT, x_0, y_0), y(mT, x_0, y_0)\big).$$

By Lemma 1, the Poincaré map P^m is well defined, and it is a homeomorphism and continuous in (x_0, y_0). Let us denote by $\big(x(t, x_0, y_0), y(t, x_0, y_0)\big)$ the solution of (5) satisfying the initial condition $(x_0, y_0) = \big(x(0), y(0)\big)$ and define

$$\begin{aligned}
P_0 &: (x_0, y_0) \mapsto \big(x(t_1^-, x_0, y_0), y(t_1^-, x_0, y_0)\big),\\
P_1 &: (x_1, y_1) \mapsto \big(x(t_2^-, x_1, y_1), y(t_2^-, x_1, y_1)\big),\\
&\vdots\\
P_j &: (x_j, y_j) \mapsto \big(x(t_{j+1}^-, x_j, y_j), y(t_{j+1}^-, x_j, y_j)\big),\\
&\vdots\\
P_{mk-1} &: (x_{mk-1}, y_{mk-1}) \mapsto \big(x(t_{mk}^-, x_{mk-1}, y_{mk-1}), y(t_{mk}^-, x_{mk-1}, y_{mk-1})\big),\\
P_{mk} &: (x_{mk}, y_{mk}) \mapsto \big(x(mT, x_{mk}, y_{mk}), y(mT, x_{mk}, y_{mk})\big),
\end{aligned}$$

where $(x_j, y_j) = \big(x(t_j, x_{j-1}, y_{j-1}), y(t_j, x_{j-1}, y_{j-1})\big)$, $j = 1, 2, \ldots$. Then the Poincaré map

$$P^m : (x_0, y_0) \mapsto \big(x(mT, x_0, y_0), y(mT, x_0, y_0)\big)$$

can be expressed by

$$P^m = P_{mk} \circ \Phi_{mk} \circ \cdots \circ P_1 \circ \Phi_1 \circ P_0,$$

where

$$\Phi_j : (x, y) \mapsto \big(x + I_j(x, y), y + L_j(x, y)\big), \quad j = 1, 2, \ldots.$$

If

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -g(x) + p(t), \end{cases}$$

is conservative and $\Phi_j, j = 1, 2, \ldots$ are global area-preserving homeomorphisms, then P^m is an area-preserving homeomorphism. Moreover, if $\Gamma(x_0, y_0)$ is sufficiently large, P^m has a continuous polar lifting

$$\tilde{P^m} : (\theta_0, r_0) \mapsto (\theta(mT, \theta_0, r_0), r(mT, \theta_0, r_0)),$$

which has the form

$$\tilde{P}^m = \tilde{P}_{mk} \circ \tilde{\Phi}_{mk} \circ \cdots \circ \tilde{P}_1 \circ \tilde{\Phi}_1 \circ \tilde{P}_0,$$

where

$$\begin{aligned}
\tilde{\Phi}_j &: (\theta_j^-, r_j^-) \mapsto (\theta_j, r_j), && j = 1, 2, \ldots\\
\tilde{P}_{j-1} &: (\theta_{j-1}, r_{j-1}) \mapsto (\theta_j^-, r_j^-), && j = 1, 2, \ldots,\\
\tilde{P}_{mk} &: (\theta_{mk}, r_{mk}) \mapsto \big(\theta(mT, \theta_0, r_0), r(mT, \theta_0, r_0)\big)
\end{aligned}$$

and

$$\begin{aligned}
(\theta_j^-, r_j^-) &= \big(\theta(t_j^-, \theta_0, r_0), r(t_j^-, \theta_0, r_0)\big),\\
(\theta_j, r_j) &= \big(\theta_j^- + \Delta\theta_{\Phi_j}(z(t_j^-, z_0)), |\Phi_j(z(t_j^-, z_0))|\big), \quad j = 1, 2, \ldots,
\end{aligned}$$

where $z(t_j^-, z_0) = \big(x(t_j^-, x_0, y_0), y(t_j^-, x_0, y_0)\big)$.

Theorem 3. *Assume that conditions* (A_1), (A_2), (A_3), (A_4) *and* (H_5) *hold. Then for any given integer $m > 1$, system* (1) *has infinitely many m-order subharmonic solutions $u = u_i$, $i = 1, 2, \ldots$, which satisfy*

$$\lim_{i\to+\infty}\Big(\min_{t\in[0,mT]}(u_i(t) + |u_i'(t)|)\Big) = 0,$$
$$\lim_{i\to+\infty}\Big(\max_{t\in[0,mT]}(u_i(t) + |u_i'(t)|)\Big) = +\infty.$$

Proof. The proof is a straightforward modification of the proof of Theorem 2, with T replaced by mT. The only difference is that we need to prove that mT is the minimal period of subharmonic solutions.

We consider the Poincaré map

$$P^m : (x_0, y_0) \mapsto \big(x(mT, x_0, y_0), y(mT, x_0, y_0)\big)$$

and its continuous polar lifting

$$\tilde{P}^m(\theta_0, r_0) = \big(\theta(mT, \theta_0, r_0), r(mT, \theta_0, r_0)\big).$$

Proceeding as in the proof of Theorem 2, we can choose two suitably large constants ξ_1 and η_1 with $\xi_1 < \eta_1$; then there exists a prime positive integer n such that

$$\theta(mT, \theta_0, r_0) - \theta(0, \theta_0, r_0) > -2n\pi, \quad \text{for} \;\; \Gamma(r_0\cos\theta_0, r_0\sin\theta_0) = \xi_1, \tag{17}$$

$$\theta(mT, \theta_0, r_0) - \theta(0, \theta_0, r_0) < -2n\pi, \quad \text{for} \;\; \Gamma(r_0\cos\theta_0, r_0\sin\theta_0) = \eta_1. \tag{18}$$

Then we can consider the Poincaré map

$$\tilde{P}^m(\theta_0, r_0) = \big(\theta(mT, \theta_0, r_0), r(mT, \theta_0, r_0)\big)$$

on the annulus

$$\Psi_1 : \xi_1 \le \Gamma(x_0, y_0) \le \eta_1.$$

It is obvious that the boundaries of the annulus Ψ_1 are star-shaped with respect to the origin by Lemma 8.

Since P_j, $j = 1, 2, \ldots$ are symplectic, Φ_j, $j = 1, 2, \ldots$ are global orientation-preserving and area-preserving homeomorphisms with the finite twist property, thus the Poincaré map P^m is an area-preserving homeomorphism. Meanwhile, the Poincaré map

$$\tilde{P}^m(\theta_0, r_0) = (\theta(mT, \theta_0, r_0), r(mT, \theta_0, r_0))$$

is a continuous polar lifting of P^m and also an area-preserving homeomorphism. Moreover, from (17) and (18), we know that the Poincaré map $\tilde{P}^m$ is a twist on the annulus Ψ_1. Finally, it is obvious that $r(mT, \theta_0, r_0) > 0$ if $\Gamma(x_0, y_0) \ge \xi_1$. Hence, $O \in \tilde{P}^m(D_1)$, where D_1 is an open region with boundary $\Gamma(x_0, y_0) = \xi_1$.

As a consequence, all conditions of Theorem 1 are satisfied. By Theorem 1, we know that the Poincaré map P^m has at least two fixed points in the annulus Ψ_1, i.e., system (5) has at least two mT-periodic solutions on the annulus Ψ_1.

Next, we prove that mT is the minimal period of subharmonic solutions of system (5). We suppose that (x_0^*, y_0^*) is a fixed point of P^m in the annulus Ψ_1 and denote by

$$\big(x^*, y^*\big) = \big(x(t; x_0^*, y_0^*), y(t; x_0^*, y_0^*)\big)$$

the $2mT$-periodic solution of system (5) satisfying the initial value condition

$$\big(x(0; x_0^*, y_0^*), y(0; x_0^*, y_0^*)\big) = (x_0^*, y_0^*).$$

On the contrary, we suppose that the minimal period of (x^*, y^*) is lT, where $l < m$. Obviously, there exist two integers $k^* \geq 1$ and $0 \leq q < l$ such that $m = k^* l + q$. By

$$P^m(x_0^*, y_0^*) = (x_0^*, y_0^*), \quad P^l(x_0^*, y_0^*) = (x_0^*, y_0^*),$$

we have

$$P^q(x_0^*, y_0^*) = (x_0^*, y_0^*).$$

Then by periodic points (x_0^*, y_0^*) of P^m has a minimal period lT, we know that $m = k^* l$ with $k^* > 1$.

Moreover, by Lemma 6 and Lemma 7, we know that as long as ξ_1 is sufficiently large, there exists $N_0 > 1$ such that (x^*, y^*) turns clockwise N_0 times around the origin during $[0, lT]$ and turns clockwise $k^* N_0$ times around the origin during $[0, mT]$. Then $n = k^* N_0$, where $k^* > 1$ and $N_0 > 1$, which contradicts the fact that n is a prime positive integer.

The rest is similar to the proof of Theorem 2. $\square$

As an example, let us consider the system

$$\begin{cases} \ddot{u} + u^p - u^{-q} = e(t), & t \neq t_j; \\ \Delta u(t_j) = \alpha_j\big(u(t_j^-), \dot{u}(t_j^-)\big), & \\ \Delta \dot{u}(t_j) = \beta_j\big(u(t_j^-), \dot{u}(t_j^-)\big), & j = 1, 2, \ldots, \end{cases} \tag{19}$$

where $p > 1$, $q > 1$ and e is a continuous 2π-periodic function. Assume that α_j and β_j satisfy the condition (A_4) and (H_4). It is not difficult to verify that $f(u) = u^p - u^{-q}$ satisfies the conditions (A_1)-(A_3). Then by Theorems 2 and 3, we can obtain the existence of infinitely many periodic and subharmonic solutions for system (19).

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