

Pascal pyramid in the space $\mathbf{H}^2 \times \mathbf{R}$

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Abstract. In this article, we introduce a new type of Pascal pyramids. A regular squared mosaic in the hyperbolic plane yields an (h^2r) -cube mosaic in space $\mathbf{H}^2 \times \mathbf{R}$ and the definition of the pyramid is based on this regular mosaic. The levels of the pyramid inherit some properties from the Euclidean and hyperbolic Pascal triangles. We give the growing method from level to level and show some illustrating figures.

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Key words: Pascal's triangle, hyperbolic Pascal triangle, Pascal pyramid, regular mosaics, cubic honeycomb, Thurston geometries, prism tiling in space $\mathbf{H}^2 \times \mathbf{R}$, recurrence sequences

1. Introduction

There are several approaches to generalize the Pascal's arithmetic triangle (see, for instance [1, 3, 5, 14]). A new type of its variations is based on the hyperbolic regular mosaics denoted by Schläfli's symbol $\{p, q\}$, where $(p - 2)(q - 2) > 4$ ([7]). Each regular mosaic induces a so-called hyperbolic Pascal triangle (see [4]), following and generalizing the connection between the classical Pascal's triangle and the Euclidean regular square mosaic $\{4, 4\}$. For more details, see [4, 10, 11, 12], but here we also collect some necessary information. We use the attribute Pascal's (with the apostrophe) only to refer to the original, Euclidean arithmetic triangle and pyramid.

The hyperbolic Pascal triangle based on the mosaic $\{p, q\}$ can be envisaged as a digraph, where the vertices and the edges are the vertices and the edges of a well defined part of lattice $\{p, q\}$, respectively, and the vertices possess a value that give the number of different shortest paths from the base vertex to the given vertex. In this article, we build on the hyperbolic squared mosaics, thus the other properties hold just for mosaic $\{4, q\}$. Figure 1 illustrates the hyperbolic Pascal triangle when $\{p, q\} = \{4, 5\}$. Here the base vertex has two edges, the leftmost and the rightmost vertices have three, the others have q edges. The quadrilateral shape cells surrounded by the appropriate edges correspond to the squares in the mosaic. Apart from the winger elements, certain vertices (called "Type A") have 2 ascendants and $q - 2$ descendants, while the others ("Type B") have 1 ascendant and $q - 1$ descendants. In the figures, we denote vertices of type A by circles and vertices of type B by diamonds, while the wingers are denoted by white diamonds (according to the denotations in [4]). The vertices which are n -edge-long far from the base vertex are in row n . The general method of preparing the graph is the

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following: we go along the vertices of the j^{th} row, according to the type of the elements (winger, A , B), and we draw the appropriate number of edges downwards ($2, q - 2, q - 1$, respectively). Neighbour edges of two neighbour vertices of the j^{th} row meet in the $(j + 1)^{\text{th}}$ row, constructing a new vertex of type A . The other descendants of row j have type B in row $j + 1$. In the sequel, $|_k^n$ denotes the k^{th} element in row n , which is either the sum of the values of its two ascendants or the value of its unique ascendant. We note that the hyperbolic Pascal triangle has the property of vertical symmetry.

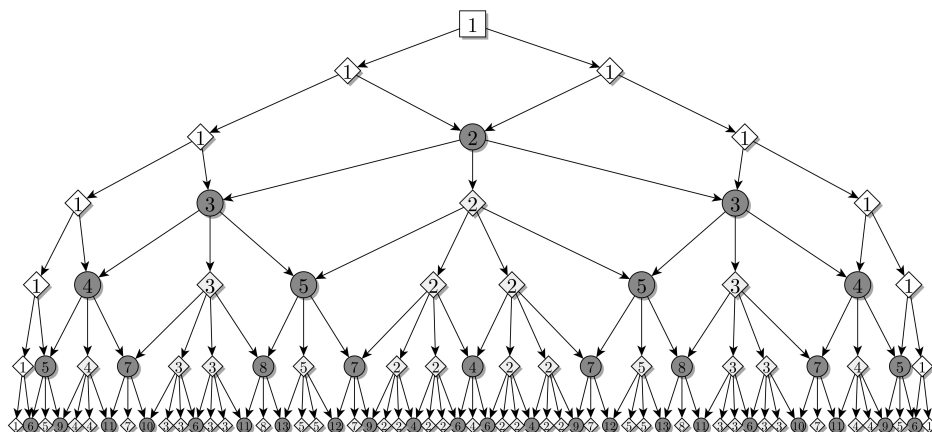


Figure 1: Hyperbolic Pascal triangle linked to $\{4, 5\}$ up to row 6

The 3-dimensional analogue of the original Pascal’s triangle is the well-known Pascal’s pyramid (or more precisely Pascal’s tetrahedron). Its levels are triangles and the numbers along the three edges of the n^{th} level are the numbers of the n^{th} line of Pascal’s triangle. Each number inside in any level is the sum of the three adjacent numbers on the level above (see [2, 6, 8, 9]). In the hyperbolic space based on the hyperbolic regular cube mosaic (cubic honeycomb) with Schläfli’s symbol $\{4, 3, 5\}$ was defined a hyperbolic Pascal pyramid (\mathcal{HPP}) as a generalisation of the hyperbolic Pascal triangle (\mathcal{HPT}) linked to mosaic $\{4, 5\}$ and the classical Pascal’s pyramid ([10]).

The space $\mathbf{H}^2 \times \mathbf{R}$ is one of the eight simply connected 3-dimensional maximal homogeneous Riemannian geometries (Thurston geometries [16], [17]). This Seifert fibre space is derived by the direct product of the hyperbolic plane \mathbf{H}^2 and the real line \mathbf{R} . For more details, see [15]. In what follows we define the Pascal pyramids in this space based on the so-called (h^2r)-cube mosaics similarly to the definition of \mathcal{HPP} (in [10]). The definition could also be extended to the other regular tiling, but we deal with the prism tilings with a square, because they are the most natural generalizations of the original Pascal’s triangle and pyramid. This work was suggested by Professor Emil Molnár.

2. Construction of the Pascal pyramid $\mathcal{PP}_{4,q}$

In the space $\mathbf{H}^2 \times \mathbf{R}$, we define an infinite number of so-called (h^2r) -cube mosaics. We take a hyperbolic plane Π as a reference plane and a regular squared mosaic with $\{4, q\}$ ($q \geq 5$) on it. Denote by d the common length of the sides of the squares in this mosaic. We consider the hyperbolic planes parallel to Π , where the distance between two consecutive ones is d . Let the same mosaic $\{4, q\}$ be defined on all planes and let the corresponding vertices of the mosaics be on the same Euclidean lines which are perpendicular to the hyperbolic planes. An (h^2r) -cube is the convex hull of two corresponding congruent squares on two consecutive hyperbolic mosaics. All (h^2r) -cubes yield an (h^2r) -cube mosaic in the space $\mathbf{H}^2 \times \mathbf{R}$ based on the regular hyperbolic planar mosaic $\{4, q\}$. (In [15], the (h^2r) -cube mosaics were called prism tilings with squares.) Figure 2 shows three consecutive hyperbolic planes with mosaic $\{4, 5\}$ and some lines perpendicular to these planes. Let V be a mosaic vertex on a hyperbolic plane. The vertex figure of V is a double q -gon based pyramid, where the vertices are the nearest mosaic vertices to V , all vertices are one-edge-long far from V . Their base vertices are on the same hyperbolic plane as V . The vertex figure of V is illustrated in Figure 2 (or Figure 10). We mention that the edges of vertex figures are not the edges of the mosaic, they are the diameters of its faces.

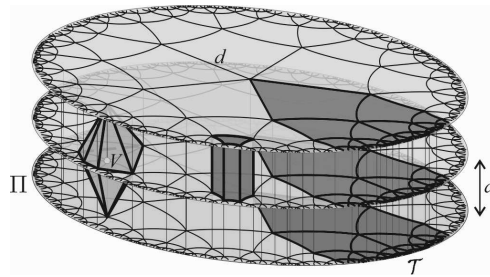


Figure 2: Construction of the (h^2r) -cube mosaic based on $\{4, 5\}$

Take the part \mathcal{T} of the mosaic $\{4, q\}$ on Π on which the hyperbolic Pascal triangle $\mathcal{HP}\mathcal{T}$ can be defined (see [4]) and let \mathcal{P} be the part of the (h^2r) -cube mosaic which contains \mathcal{T} and all its corresponding vertices on other hyperbolic planes that are "above" plane Π (the hyperbolic planes which are in the same half space bordered by Π). (Obviously, \mathcal{P} contains also the corresponding edges between the vertices.) The shape of this convex part of the mosaic resembles an infinite tetrahedron. This part \mathcal{P} is darkened in Figure 2.

Let V_0 be the base vertex of $\mathcal{HP}\mathcal{T}$ on plane Π . Let $\mathcal{G}_{\mathcal{P}}$ be the digraph directed according to the growing edge-distance from V_0 , in which the vertices and edges are the vertices and edges of \mathcal{P} . We label an arbitrary vertex V of $\mathcal{G}_{\mathcal{P}}$ by the number of different shortest paths along the edges of \mathcal{P} from V_0 to V . (We mention that all edges of the mosaic are equivalent.) Let the labelled digraph $\mathcal{G}_{\mathcal{P}}$ be the Pascal pyramid (more precisely, the Pascal tetrahedron) in space $\mathbf{H}^2 \times \mathbf{R}$ denoted by $\mathcal{PP}_{4,q}$. Some labelled vertices can be seen in Figure 3 in case $\mathcal{PP}_{4,5}$.

Let level 0 be the vertex V_0 . Level n consists of the vertices of $\mathcal{PP}_{4,q}$ whose

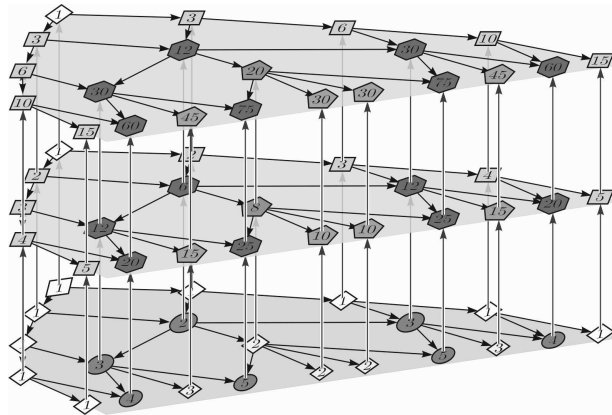


Figure 3: Pascal pyramid $\mathcal{PP}_{4,5}$ in space $\mathbf{H}^2 \times \mathbf{R}$

edge-distances from V_0 are an n -edge (the distance of the shortest path along the edges of \mathcal{P} is n). It is clear that one (infinite) face of $\mathcal{PP}_{4,q}$ is a \mathcal{HPT} in plane Π and the other two faces are Euclidean Pascal's triangles. Figure 4 shows the Pascal pyramid $\mathcal{PP}_{4,5}$ in $\mathbf{H}^2 \times \mathbf{R}$ up to level 4. Moreover, Figures 5–8 show the growing from a level to the next one in case of some lower levels. The colours and shapes of different types of vertices are different. (See the definitions later.) The numbers without colouring and shapes refer to vertices in the lower level in each figure. The graphs growing from a level to the new one contain graph-cycles with six nodes, which refer to convex hulls of parallel projections of the cubes from the mosaic, where the direction of the projection is not parallel to any edge of the cubes.

In what follows, we describe the method of the growing of $\mathcal{PP}_{4,q}$ and we give the sum of the paths connecting vertex V_0 and level n .

3. Growing of $\mathcal{PP}_{4,q}$

In the classical Pascal's pyramid the number of elements on level n is $(n+1)(n+2)/2$ and its growing from level n to $n+1$ is $n+2$, but in the hyperbolic Pascal pyramid it is more complex (see [10]).

As one face of $\mathcal{PP}_{4,q}$ is the hyperbolic Pascal triangle, then there are three types of vertices A , B and 1 corresponding to the Introduction. Their denotations are also the same. From all A and B only one edge starts to the inside of the pyramid; these are the Euclidean edges of the mosaic (see Figure 3). The types of the inside vertices of these edges differ from the types A and B ; let us denote them by type D and type E , respectively. The other two sides of the Pascal pyramid are Euclidean Pascal's triangles, which have two types of vertices, let us denote them by C and 1 , respectively. A vertex C connects three new vertices in the next level, two vertices C on the side of $\mathcal{PP}_{4,q}$ and one vertex of type D inside the pyramid. Sometimes, if it is important, we distinguish the types 1 . If a vertex of type 1 belongs to \mathcal{HPT} , we denote it by 1_h .

Their growing methods are illustrated in Figure 9 (cf. with the growing method

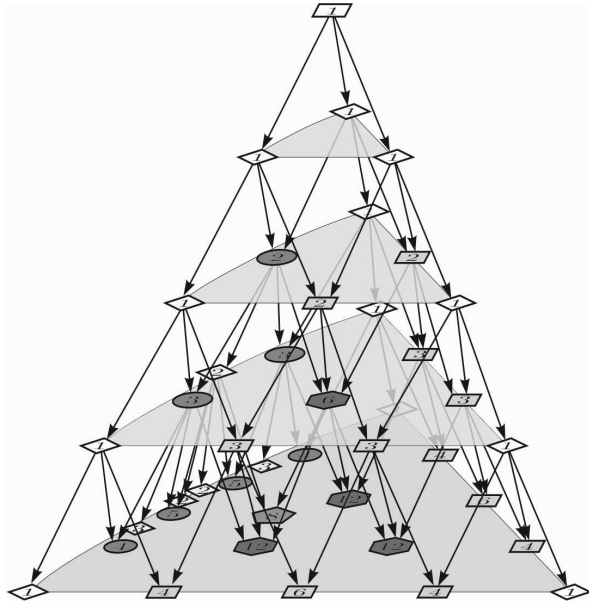


Figure 4: Pascal pyramid $\mathcal{PP}_{4,5}$

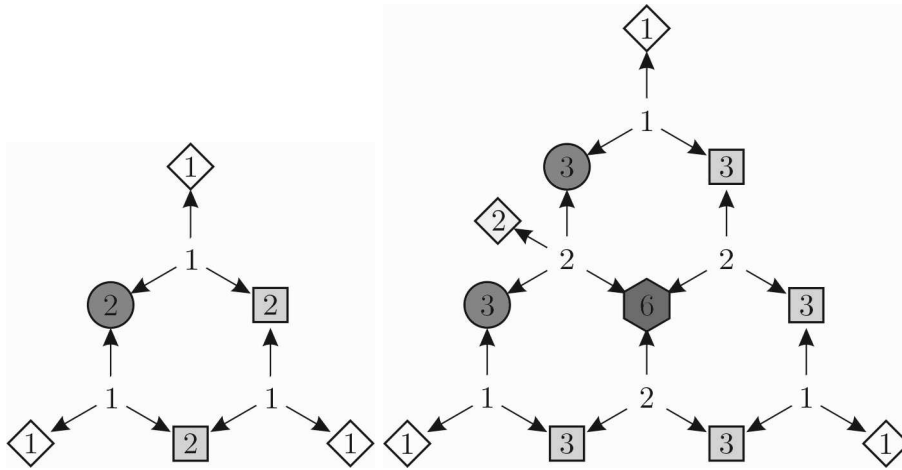


Figure 5: Connection between levels one, two and three in $\mathcal{PP}_{4,5}$

in [4] and [10]). In the figures, we denote vertices of type C by light grey squares (not diamonds) squares.

For the classification and the exact definitions of inner vertices we examine the vertex figures of the inner vertices. As the structure descends from a hyperbolic plane to the consecutive one, there are two types of vertex figures. During the growing (step from level $i - 1$ to level i) an arbitrary inner vertex V on level i can be reached from level $i - 1$ with three or two edges. This fact allows us to classify

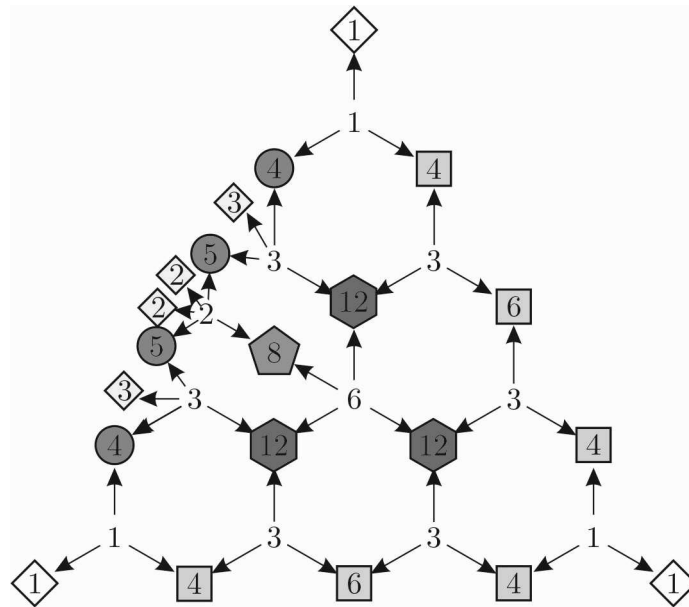


Figure 6: Connection between levels three and four in $PP_{4,5}$

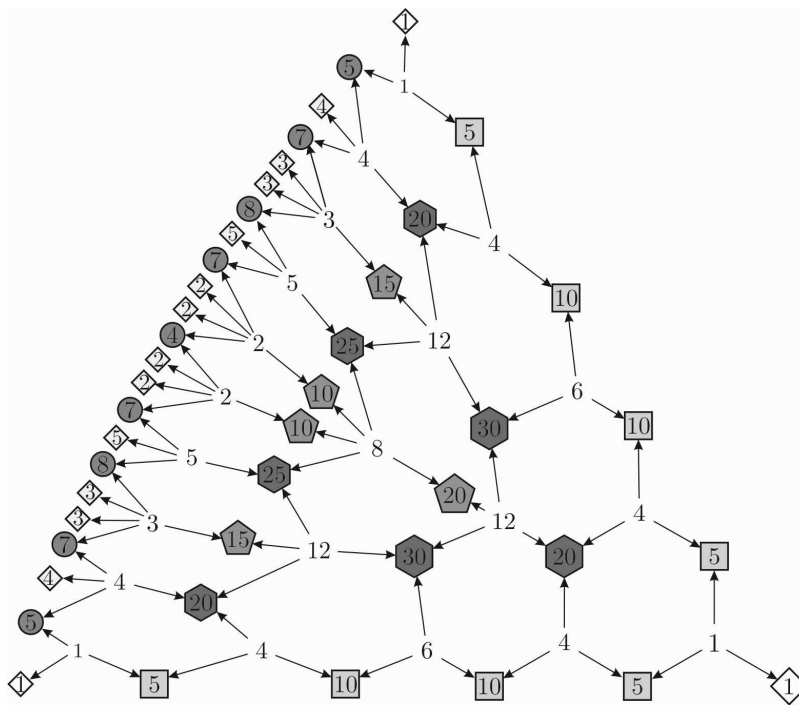


Figure 7: Connection between levels four and five in $PP_{4,5}$

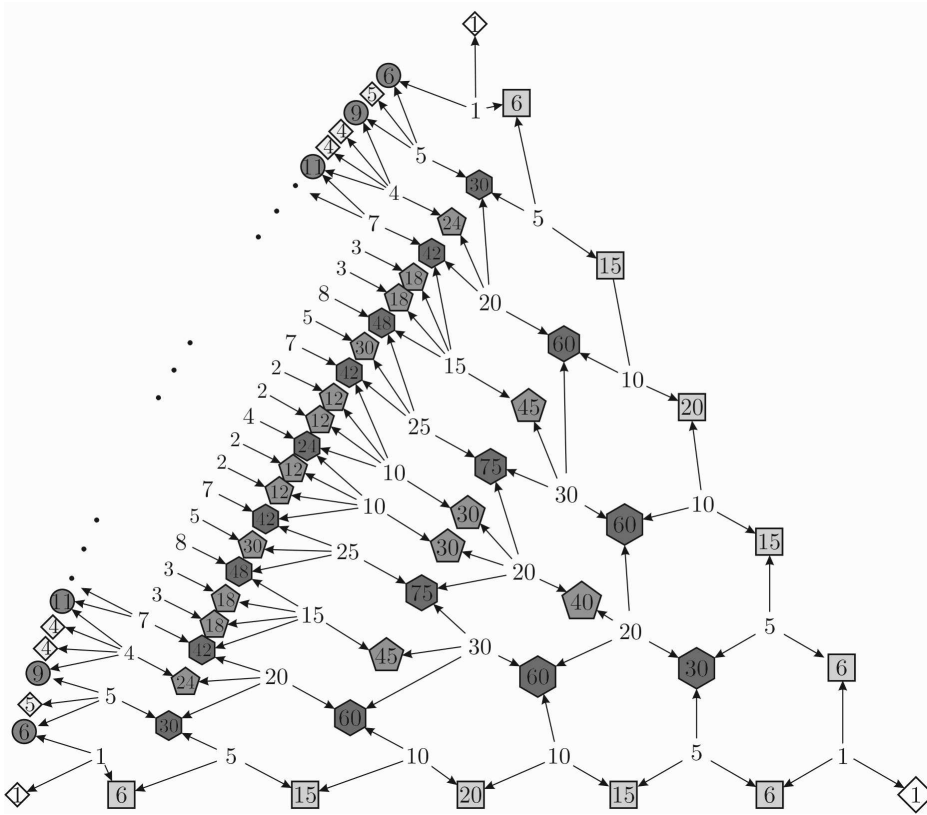


Figure 8: Connection between levels five and six in $\mathcal{PP}_{4,5}$

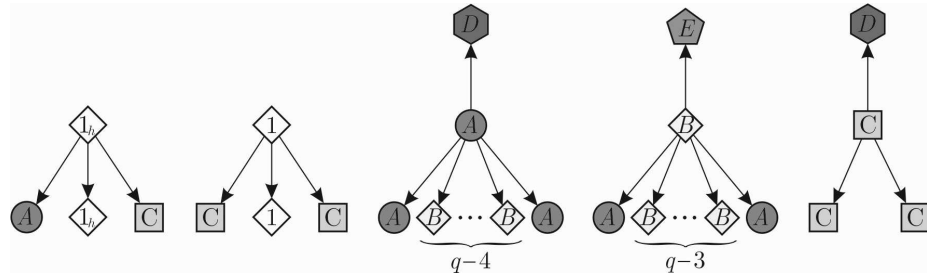


Figure 9: Growing method in case of the faces

the inner vertices. Let the type of a vertex on level i be D or E , respectively, if it has three or two joining edges to level $i - 1$ (as before). Figure 10 shows the vertex figures of the inner vertices of $\mathcal{PP}_{4,q}$. Vertices W_{i-1} (small grey circles) are on level $i - 1$, W_i and the centres are on level i . We do not know the types of W (or it is not important to know them). The other vertices of the double pyramid are on level $i + 1$ and their classification gives their types. An edge of the double pyramid and its centre V determine a square (a side-face of an (h^2r) -cube) from the mosaic.

(Recall that an edge of the vertex figure is a diagonal of a side-face of an (h^2r) -cube.) Since from a vertex of a square we can go to the opposite vertex in two ways, then a vertex D of the double pyramid, where D and a W_{i-1} are connected by an edge, can be reached with two paths from level $i - 1$. (For example, on the left-hand side in Figure 10, between a vertex W_{i-1} and D , there are the paths $W_{i-1} - D_i - D$ and $W_{i-1} - W_i - D$.)

So, the type of the third vertex of the faces on the double pyramid whose other two vertices are W_{i-1} is D . The others connect to only one W_{i-1} , they can be reached in two ways from level i , their types are E . See Figure 10. In the figures, we denote vertices of type D by hexagons and vertices of type E by pentagons.

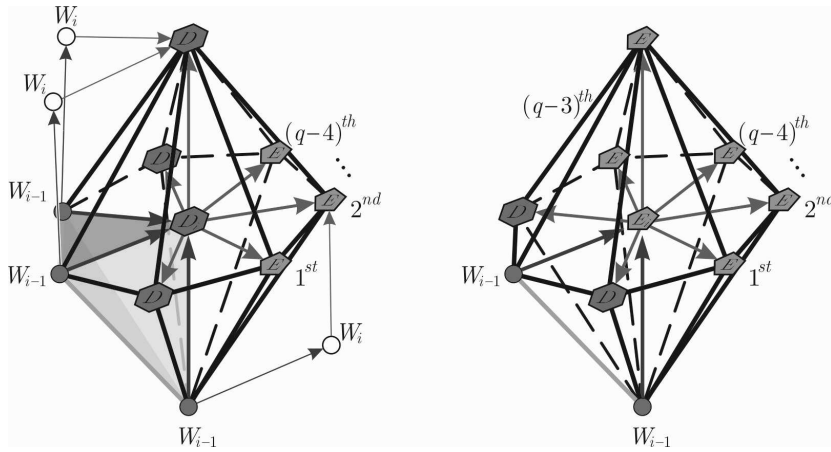


Figure 10: Growing method in case of the inner vertices with vertex figures

In Figure 11, the growing method is presented in case of the inner vertices. These vertices are the centres and some vertices of the double pyramids are presented in Figure 10.

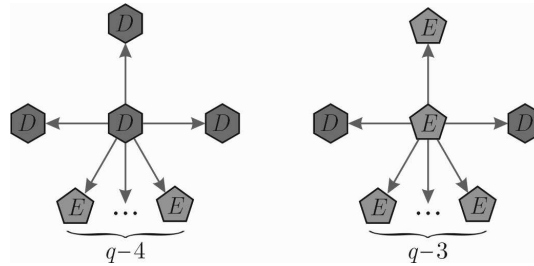


Figure 11: Growing method in case of the inner vertices

Finally, we denote the sums of vertices of types A , B , C , D and E on level n by a_n , b_n , c_n , d_n and e_n , respectively.

Summarising the details we prove Theorem 1.

Theorem 1. *The growing of the numbers of different types of vertices are described by the system of linear inhomogeneous recurrence sequences ($n \geq 1$)*

$$\begin{aligned} a_{n+1} &= a_n + b_n + 1, \\ b_{n+1} &= (q - 4)a_n + (q - 3)b_n, \\ c_{n+1} &= c_n + 2, \\ d_{n+1} &= a_n + d_n, \\ e_{n+1} &= b_n + e_n, \end{aligned} \tag{1}$$

with zero initial values.

We mention that c_n ($n \geq 1$) is an arithmetical sequence and $c_n = 2(n - 1)$.

Lemma 1. *For the sequences d_n and e_n ($n \geq 1$), there hold*

$$\begin{aligned} d_n &= -a_n + \frac{1}{q-4}b_n + (n - 1), \\ e_n &= a_n - (n - 1). \end{aligned} \tag{2}$$

Proof. Obviously, for $n = 1$, the equations hold. In case $n > 1$, we suppose that $d_{n-1} = -a_{n-1} + \frac{1}{q-4}b_{n-1} + (n - 2)$ and $e_{n-1} = a_{n-1} - (n - 2)$. Firstly, from the first, second and fourth row of (1) we have

$$\begin{aligned} a_n + d_n &= a_{n-1} + b_{n-1} + 1 + a_{n-1} + d_{n-1} = a_{n-1} + b_{n-1} + 1 + \frac{1}{q-4}b_{n-1} + n - 2 \\ &= \frac{q-4}{q-4}a_{n-1} + \frac{q-3}{q-4}b_{n-1} + n - 1 = \frac{1}{q-4}b_n + n - 1. \end{aligned}$$

Secondly, from the first and fifth row we obtain $a_n - e_n = a_{n-1} - e_{n-1} + 1 = (n - 2) + 1 = n - 1$. □

Moreover, let s_n ($n \geq 1$) be the number of all vertices on level n , so that $s_0 = 1$ and

$$\begin{aligned} s_n &= a_n + b_n + c_n + d_n + e_n + 3 \\ &= a_n + \frac{q-3}{q-4}b_n + 2n + 1. \end{aligned} \tag{3}$$

Table 1 shows the numbers of vertices on levels up to 10 in case $\mathcal{PP}_{4,5}$.

Theorem 2. *The sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{e_n\}$ and $\{s_n\}$ can be described by the same fourth order linear homogeneous recurrence sequence*

$$x_n = qx_{n-1} + 2(1 - q)x_{n-2} + qx_{n-3} - x_{n-4} \quad (n \geq 5), \tag{4}$$

the initial values be can obtained from Theorem 1 (in case $\mathcal{PP}_{4,5}$ from Table 1). The sequences $\{a_n\}$, $\{b_n\}$ can also be described by

$$x_n = (q - 1)x_{n-1} + (1 - q)x_{n-2} + x_{n-3} \quad (n \geq 4). \tag{5}$$

Moreover, the explicit formulae

$$\begin{aligned}
 a_n &= \left(\frac{2-q}{2} + \frac{D+2}{2D} \sqrt{D} \right) \alpha_1^n + \left(\frac{2-q}{2} - \frac{D+2}{2D} \sqrt{D} \right) \alpha_2^n + 1, \\
 b_n &= \left(\frac{q-3}{2} + \frac{1-q}{2q} \sqrt{D} \right) \alpha_1^n + \left(\frac{q-3}{2} - \frac{1-q}{2q} \sqrt{D} \right) \alpha_2^n - 1, \\
 d_n &= \left(\frac{q^2-5q+5}{2(q-4)} - \frac{q^2-3q-1}{2D} \sqrt{D} \right) \alpha_1^n + \left(\frac{q^2-5q+5}{2(q-4)} + \frac{q^2-3q-1}{2D} \sqrt{D} \right) \alpha_2^n \\
 &\quad + n - \frac{1}{q-4} + 1, \\
 e_n &= \left(\frac{2-q}{2} + \frac{D+2}{2D} \sqrt{D} \right) \alpha_1^n + \left(\frac{2-q}{2} - \frac{D+2}{2D} \sqrt{D} \right) \alpha_2^n - n + 2, \\
 s_n &= \left(\frac{q}{2} - \frac{\sqrt{D}}{2D} \right) \alpha_1^n + \left(\frac{q}{2} + \frac{\sqrt{D}}{2D} \right) \alpha_2^n + 2n - \frac{1}{q-4} + 1,
 \end{aligned}$$

hold, where $D = q(q-4)$, $\alpha_1 = (q-2 + \sqrt{D})/2$ and $\alpha_2 = (q-2 - \sqrt{D})/2$.

Proof. The sequences $\{a_n\}$ and $\{b_n\}$ are the same as $\{a_n\}$, $\{b_n\}$ in [4]. So they can be described by (5) and their explicit formulae hold. According to Lemma 1 and (3), more explicit formulae are derived by the combination of explicit formulae of $\{a_n\}$ and $\{b_n\}$.

Let us extend (5) to (4) considering the sequence $\{a_n\}$. Substitute a_n into (5) and sum a_n and a_{n-1} . Then we obtain implicit form (4) for $\{a_n\}$. Similarly, (4) is also the implicit form of $\{b_n\}$.

Now we prove that (4) holds for s_n also. Multiply equation (3) by q , $2(1-q)$, q and -1 , where $n = n, n-1, n-2$ and $n-3$, respectively. If we sum them, then we

n	0	1	2	3	4	5	6	7	8	9	10
a_n	0	0	1	2	4	9	22	56	145	378	988
b_n	0	0	0	1	4	12	33	88	232	609	1596
c_n	0	0	2	4	6	8	10	12	14	16	18
d_n	0	0	0	1	3	7	16	38	94	239	617
e_n	0	0	0	0	1	5	17	50	138	370	979
s_n	1	3	6	11	21	44	101	247	626	1615	4201

Table 1: Number of types of vertices ($n \leq 10$) in case $\mathcal{PP}_{4,5}$

obtain

$$\begin{aligned} & qs_n + 2(1 - q)s_{n-1} + qs_{n-2} - s_{n-3} \\ &= qa_n + 2(1 - q)a_{n-1} + qa_{n-2} - a_{n-3} + \frac{q-3}{q-4}(qb_n + 2(1 - q)b_{n-1} + qb_{n-2} - b_{n-3}) \\ &\quad + q(2n + 1) + 2(1 - q)(2(n - 1) + 1) + q(2(n - 2) + 1) - (2(n - 3) + 1) \\ &= a_{n+1} + \frac{q-3}{q-4}b_{n+1} + 2n + 3 = s_{n+1}. \end{aligned}$$

This proves (4) for $\{s_n\}$.

In case of $\{d_n\}$ and $\{e_n\}$, we can prove relation (4) similarly to $\{s_n\}$. \square

Remark 1. If $q = 4$, then the sequences give results of the Euclidean Pascal's pyramid. Its all faces are Pascal's triangle, thus $b_n = 0$, $e_n = 0$, a_n coincide c_n , in that way the growing equation system (1) is just $c_{n+1} = c_n + 3$, $d_{n+1} = c_n + d_n$.

Remark 2. The generating function of the sequence s_n is given by

$$\frac{1 - (q - 3)x - (q - 4)x^2}{1 - qx + (2q - 2)x^2 - qx^3 + x^4}.$$

In the case $\mathcal{PP}_{4,5}$ it is

$$\frac{1 - 2x - x^2}{1 - 5x + 8x^2 - 5x^3 + x^4},$$

which is not in the OEIS at present.

4. Sum of the values on levels in $\mathcal{PP}_{4,q}$

In this section, we determine the sum of the values of elements on level n .

Let $\hat{a}_n, \hat{b}_n, \hat{c}_n, \hat{d}_n$ and \hat{e}_n denote the sums of the values of vertices of type A, B, C, D and E on level n , respectively.

Theorem 3. If $n \geq 1$, then

$$\begin{aligned} \hat{a}_{n+1} &= 2\hat{a}_n + 2\hat{b}_n + 2, \\ \hat{b}_{n+1} &= (q - 4)\hat{a}_n + (q - 3)\hat{b}_n, \\ \hat{c}_{n+1} &= 2\hat{c}_n + 4, \\ \hat{d}_{n+1} &= \hat{a}_n + c_n + 3\hat{d}_n + 2\hat{e}_n, \\ \hat{e}_{n+1} &= \hat{b}_n + (q - 4)\hat{d}_n + (q - 2)\hat{e}_n \end{aligned} \tag{6}$$

with zero initial values.

Proof. From Figures 9 and 11 the results can be read directly. For example, all vertices of type A, B and 1 on level n generate two vertices of type A on level $n + 1$ and it follows from the first equation of (6). \square

Table 2 shows the sum of the values of the vertices on levels up to 10. Let \hat{s}_n be the sum of the values of all vertices on level n . Then $\hat{s}_0 = 1$ and

$$\hat{s}_n = \hat{a}_n + \hat{b}_n + \hat{c}_n + \hat{d}_n + \hat{e}_n + 3 \quad (n \geq 1).$$

The value \hat{s}_n also shows the number of paths from V_0 to level n .

n	0	1	2	3	4	5	6	7	8	9	10
\hat{a}_n	0	0	2	6	18	58	194	658	2242	7642	26114
b_n	0	0	0	2	10	38	134	462	1582	5406	18462
\hat{c}_n	0	0	4	12	28	60	124	252	508	1020	2044
d_n	0	0	0	6	36	170	768	3458	15596	70314	316296
\hat{e}_n	0	0	0	0	8	70	418	2156	10388	48342	220746
\hat{s}_n	1	3	9	29	103	399	1641	6989	30319	132735	583665

Table 2: Sum of values of types of vertices ($n \leq 10$) in case $\mathcal{PP}_{4,5}$

Theorem 4. The sequences $\{\hat{a}_n\}$, $\{\hat{b}_n\}$, $\{\hat{c}_n\}$, $\{\hat{d}_n\}$, $\{\hat{e}_n\}$ and $\{\hat{s}_n\}$ can be described by the same sixth order linear homogeneous recurrence sequence

$$x_n = (2q + 3)x_{n-1} + (-q^2 - 7q - 5)x_{n-2} + (4q^2 + 10q + 9)x_{n-3} + (-5q^2 - 13q - 10)x_{n-4} + (2q^2 + 12q + 12)x_{n-5} + (-4q - 8)x_{n-6}. \quad (7)$$

The initial values are from equation system (3). The sequences $\{\hat{a}_n\}$, $\{\hat{b}_n\}$ can also be described by

$$x_n = qx_{n-1} - (q + 1)x_{n-2} + 2x_{n-3} \quad (n \geq 3). \quad (8)$$

Moreover, for sequences $\{\hat{c}_n\}$, $\{\hat{s}_n\}$

$$\hat{c}_n = 3\hat{c}_{n-1} - 2\hat{c}_{n-2} \quad (n \geq 2), \quad (9)$$

$$\hat{s}_n = (q + 3)\hat{s}_{n-1} - (3q + 4)\hat{s}_{n-2} + (2q + 4)\hat{s}_{n-3} \quad (n \geq 3) \quad (10)$$

and the explicit formula of \hat{s}_n

$$\hat{s}_n = \left(-\frac{1}{2} + (q - 1)\frac{\sqrt{D}}{2D}\right)\alpha_1^n + \left(-\frac{1}{2} - (q - 1)\frac{\sqrt{D}}{2D}\right)\alpha_2^n + 2 \cdot 2^n \quad (11)$$

holds, where $D = q^2 - 2q - 7$, $\alpha_1 = \frac{1}{2}(1 + q + \sqrt{D})$ and $\alpha_2 = \frac{1}{2}(1 + q - \sqrt{D})$.

We do not give the implicit formulae for all sequences, because generally, they are complicated, but they can be calculated easily with computer.

Proof. Let $\hat{v}_n = 3$ ($n \geq 1$) be constant sequences and $\hat{v}_0 = 1$. The value \hat{v}_n gives the sum of the number of vertices of type “1” on level n . Substitute $2 = 2\hat{v}_n/3$ and

$4 = 4\hat{v}_n/3$ into the first and third equations of (3) and complete it with equation $\hat{v}_{n+1} = \hat{v}_n$. Then we have a system of linear homogeneous recurrence sequences ($n \geq 1$)

$$\begin{aligned}\hat{a}_{n+1} &= 2\hat{a}_n + 2\hat{b}_n + \frac{2}{3}v_n, \\ \hat{b}_{n+1} &= (q-4)\hat{a}_n + (q-3)\hat{b}_n, \\ \hat{c}_{n+1} &= 2\hat{c}_n + \frac{4}{3}v_n, \\ \hat{d}_{n+1} &= \hat{a}_n + \hat{c}_n + 3\hat{d}_n + 2\hat{e}_n, \\ \hat{e}_{n+1} &= \hat{b}_n + (q-4)\hat{d}_n + (q-2)\hat{e}_n, \\ \hat{v}_{n+1} &= \hat{v}_n\end{aligned}\tag{12}$$

and

$$\hat{s}_n = \hat{a}_n + \hat{b}_n + \hat{c}_n + \hat{d}_n + \hat{e}_n + \hat{v}_n \quad (n \geq 0).$$

A shorter form of linear homogeneous recurrence sequences (12) is

$$\vec{u}_{i+1} = \vec{M}\vec{u}_i,\tag{13}$$

where $\vec{u}_j = [\hat{a}_j \ \hat{b}_j \ \hat{c}_j \ \hat{d}_j \ \hat{e}_j \ \hat{v}_j]^T$ and

$$\vec{M} = \begin{pmatrix} 2 & 2 & 0 & 0 & 0 & \frac{2}{3} \\ q-4 & q-3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & \frac{4}{3} \\ 1 & 0 & 1 & 3 & 2 & 0 \\ 0 & 1 & 0 & q-4 & q-2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The characteristic polynomial of \vec{M} is

$$p_6(x) = (x-1)(x-2)(x^2 - (q+1)x + q+2)(x^2 + (1-q)x + 2)\tag{14}$$

and according to Theorem 3 from [10], equation (14) is the characteristic equation of all sequences $\hat{r}_{i+1} = \vec{y}^T \vec{u}_i$, where $\vec{y} = [y_1 \ y_2 \ \dots \ y_6]^T$. Thus (14) is the characteristic equation of $\hat{s}_n, \hat{a}_n, \dots, \hat{e}_n$ with $\vec{y} = (1, 1, 1, 1, 1, 1)$, $\vec{y} = (1, 0, 0, 0, 0, 0)$, \dots , $\vec{y} = (0, 0, 0, 0, 1, 0)$, respectively.

Equation (14) implies the six order linear recurrence sequence (7) for the considered sequences, but for the lower order implicit formulae we have to examine the first elements of the sequences. One can easily check that the polynomial

$$\begin{aligned}p_s(x) &= (x-2)(x^2 - (q+1)x + q+2) \\ &= x^3 - (q+3)x^2 + (3q+4)x - (2q+4);\end{aligned}\tag{15}$$

moreover, in case of \hat{a}_i and \hat{b}_i (see [4])

$$p_3(x) = (x-1)(x^2 + (1-q)x + 2) = x^3 - qx^2 + (1+q)x - 2,$$

in case of \hat{c}_i

$$p_2(x) = (x-1)(x-2) = x^2 - 3x + 2.$$

These polynomials imply recurrence relations (7)–(10), respectively.

The roots of polynomial (15) are $\alpha_1, \alpha_2, \alpha_3 = 2$ and

$$\hat{s}_n = \beta_1 \cdot \alpha_1^n + \beta_2 \cdot \alpha_2^n + \beta_3 \cdot 2^n \quad (n \geq 1)$$

provide the explicit formulae (see [13]). Solutions of the linear equation system from $n = 1, 2, 3$ determine the exact values of β 's. \square

Remark 3. In the case of $\mathcal{PP}_{4,5}$, the growing ratio of values is $\lim_{n \rightarrow \infty} \hat{s}_{n+1}/\hat{s}_n = \alpha_1 \approx 4.414$, where α_1 is the largest eigenvalue of matrix \tilde{M} . Recall that these growing ratios are 3 and ≈ 10.351 in the Euclidean and hyperbolic cases, respectively ([10]).

Remark 4. The generating function of the sequence \hat{s}_n is given by

$$\frac{1 - qx + 4x^2}{1 - (q+3)x + (3q+4)x^2 - (2q+4)x^3}.$$

In the case of $\mathcal{PP}_{4,5}$, it is

$$\frac{1 - 5x + 4x^2}{1 - 8x + 19x^2 - 14x^3},$$

which is not in the OEIS at present.

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