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Numerical integration of singularly perturbed delay differential equations using exponential integrating factor

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Abstract. In this paper, we propose a numerical integration technique with an exponential integrating factor for the solution of singularly perturbed differential-difference equations with a negative shift, namely the delay differential equation, with layer behavior. First, the negative shift in the differentiated term is approximated by Taylor's series, provided the shift is of $o(\varepsilon)$. Subsequently, the delay differential equation is replaced by an asymptotically equivalent first order neutral type delay differential equation. An exponential integrating factor is introduced into the first order delay equation. Then the trapezoidal rule along with linear interpolation has been employed to get a three-term recurrence relation. The resulting tri-diagonal system is solved by the Thomas algorithm. The proposed technique is implemented on model examples, for different values of delay parameter δ and perturbation parameter ε. Maximum absolute errors are tabulated and compared to validate the technique. Convergence of the proposed method has also been discussed.

AMS subject classifications: 65L10, 65L11, 65L12

Key words: singularly perturbed differential-difference equation, negative shift, boundary layer, exponential integrating factor, numerical integration

1. Introduction

In mathematical modeling of a physical system, just like in control theory, the presence of small time parasitic parameters like moments of inertia, resistances, inductances and capacitances increases the order and stiffness of these systems. The suppression of these small constants results in the reduction of the order of the system. Such systems are termed as singular perturbation systems and when these systems take into account both the past history and the present state of the physical system, then they are called singularly perturbed delay differential equations. Delay differential equations arise in first-exit time problems in neurobiology and in mathematical formulation of various practical phenomena in biosciences. The study of differential-difference equations, with the presence of shift terms, which induce large amplitudes and exhibit oscillations, resonance, turning point behavior, boundary and interior layers was carried out by Lange and Miura [7, 8, 9]. Extensive numerical work has been initiated by Kadalbajoo and Sharma [4], Kadalbajoo and Ramesh

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[5], Kadalbajoo and et al. [6]. Mohapatra and Natesan [10] proposed a numerical method comprising an upwind finite difference operator on an adaptive grid with an arc-length monitor function, to approximate the solutions of singularly perturbed differential-difference equations with small delay and shift terms. Geng and et al. [3] presented an improved kernel method to obtain an accurate approximation of solutions for singularly perturbed differential-difference equations with small delay.

With this motivation, in this paper we employed a numerical integration technique with the exponential integrating factor for the solution of singularly perturbed delay differential equations, with layer behavior. A numerical scheme for the solution of a singularly perturbed delay differential equation with the left-end boundary layer and the right-end boundary layer is described in Section 2. In Section 3, convergence of the proposed method is analyzed. To demonstrate the efficiency of the method, numerical experiments are carried out for several test problems and the results are tabulated and compared in Section 4. Finally, the discussions and conclusion are given in the last section.

2. Numerical scheme

Consider the delay differential equation with layer behavior:

$$\varepsilon y''(x) + a(x)y'(x-\delta) + b(x)y(x) = f(x) \tag{1}$$

under the interval $(0, 1)$ and subject to the conditions

$$y(x) = \phi(x) \quad \text{for} \quad -\delta \leq x \leq 0 \tag{2}$$

$$y(1) = \beta, \tag{3}$$

where ε is a small positive parameter $0 < \varepsilon \ll 1$, $a(x)$, $b(x)$, $f(x)$ and $\phi(x)$ are sufficiently smooth functions and β is a positive constant. Furthermore, $\delta = o(\varepsilon)$, where δ is a delay parameter. When δ is zero, equation (1) reduces to a singular perturbation problem which, with small ε, exhibits layers and turning points depending upon the coefficient of convection term. The layer behavior of the problem under consideration is maintained only for $\delta \neq 0$ but sufficiently small, i.e., δ is of $o(\varepsilon)$. When the delay parameter exceeds the perturbation parameter, i.e., δ is of $O(\varepsilon)$, the layer behavior of the solution is no longer maintained, rather the solution exhibits an oscillatory behavior or diminished behavior (Lange and Miura [7]).

2.1. Left-end boundary layer problems

In general, the solution of problem (1)-(3) exhibits a boundary layer at one end of the interval depending on the sign of $a(x)$. We assume that $a(x) \geq M > 0$ throughout the interval $[0, 1]$, for some positive constant M. This assumption merely implies that the boundary layer will be in the neighbourhood of x=0.
By using a Taylor series expansion of the retarded term $y'(x-\varepsilon)$ in the neighbourhood of the point x, we have

$$y'(x-\varepsilon) \approx y'(x) - \varepsilon y''(x) \tag{4}$$

Consequently, equation (1) is replaced by an asymptotically equivalent first order delay neutral type differential equation

$$y'(x) + b(x)y(x) = f(x) + y'(x-\varepsilon) - a(x)y'(x-\delta) \tag{5}$$

with $y(0) = \phi(0)$; $y(1) = \beta$. Since $0 < \delta \ll 1$, the transition from (1) to (5) is admitted. This replacement is significant from the computational point of view. For more details on the validity of this transition, one can refer to El'sgolts and Norkin [2]. Thus, the solution of equation (5) provides a good approximation to the solution of equation (1). Here, for consolidation of our ideas, we assume $a(x)$ and $b(x)$ to be constants.

By applying an integrating factor e^{bx} to (5) (as in Brian J. McCartin [1]):

$$\frac{d}{dx}\left\{e^{bx}y(x)\right\} = e^{bx}\left\{f(x) + y'(x-\varepsilon) - ay'(x-\delta)\right\}. \tag{6}$$

Discretizing the interval $[0,1]$ into N equal subintervals of mesh size $h = 1/N$, let $0 = x_0, x_1, \ldots, x_N = 1$ be the mesh points. Then we have $x_i = ih$, for $i = 0,\ 1, \ldots, N$. Integrating (6) with respect to x from x_i to x_{i+1}, we get

$$\int_{x_i}^{x_{i+1}} \frac{d}{dx}\left(e^{bx}y(x)\right) = \int_{x_i}^{x_{i+1}} e^{bx}\left\{f(x) + y'(x-\varepsilon) - ay'(x-\delta)\right\}dx \tag{7}$$

$$\begin{aligned} e^{bx_{i+1}}y(x_{i+1}) - e^{bx_i}y(x_i) &= \int_{x_i}^{x_{i+1}} e^{bx}\left\{f(x) + y(x-\varepsilon) - ay(x-\delta)\right\}dx \\ &\quad + e^{bx_{i+1}}y(x_{i+1}-\varepsilon) - e^{bx_i}y(x_i-\varepsilon) \\ &\quad - ae^{bx_{i+1}}y(x_{i+1}-\delta) + ae^{bx_i}y(x_i-\delta). \end{aligned} \tag{8}$$

Using the trapezoidal rule to evaluate the integrals in (8), we get

$$\begin{aligned} &e^{bx_{i+1}}y(x_{i+1}) - e^{bx_i}y(x_i) \\ &\quad = \frac{h}{2}\left[\, e^{bx_i}f(x_i) + e^{bx_{i+1}}f(x_{i+1}) \,\right] + e^{bx_{i+1}}y(x_{i+1}-\varepsilon) \\ &\quad - e^{bx_i}y(x_i-\varepsilon) - \frac{bh}{2}\left(e^{bx_i}y(x_i-\varepsilon) + e^{bx_{i+1}}y(x_{i+1}-\varepsilon)\right) - ae^{bx_{i+1}}y(x_{i+1}-\delta) \\ &\quad + ae^{bx_i}y(x_i-\delta) - \frac{abh}{2}\left(e^{bx_i}y(x_i-\delta) + e^{bx_{i+1}}y(x_{i+1}-\delta)\right). \end{aligned} \tag{9}$$

Further, since $0 < \varepsilon \ll 1$ and $\delta = o(\varepsilon)$, to tackle the terms containing delay, the Taylor series approximations (given by Kadalbajoo and Sharma [4] and Lange and Miura [7]) have been used. Thus, by the Taylor series expansion and then approximating $y'(x)$ by linear interpolation, we get

$$y(x_i-\delta) =y(x_i)-\delta y'(x_i) = \left(1-\frac{\delta}{h}\right) y_i + \frac{\delta}{h} y_{i-1} \tag{10}$$

$$y(x_{i+1}-\delta) =y(x_{i+1})-\delta y'(x_{i+1}) = \left(1-\frac{\delta}{h}\right) y_{i+1} + \frac{\delta}{h} y_{i} \tag{11}$$

$$y(x_i-\varepsilon) =y(x_i)-\varepsilon y'(x_i) = \left(1-\frac{\varepsilon}{h}\right) y_i + \frac{\varepsilon}{h} y_{i-1} \tag{12}$$

$$y(x_{i+1}-\varepsilon) =y(x_{i+1})-\varepsilon y'(x_{i+1}) = \left(1-\frac{\varepsilon}{h}\right) y_{i+1} + \frac{\varepsilon}{h} y_{i}. \tag{13}$$

By using equations (10) - (13) in equation (9) and simplifying, we get

$$\begin{aligned}
&e^{bh}\left(\frac{\varepsilon}{h}+\frac{bh}{2}\left(1-\frac{\varepsilon}{h}\right)+a\left(1-\frac{\delta}{h}\right)+\frac{abh}{2}\left(1-\frac{\delta}{h}\right)\right)y_{i+1}\\
&\quad-\left(e^{bh}(1+\frac{\varepsilon}{h}-\frac{b\varepsilon}{2}-\frac{a\delta}{2})-\left(1-\frac{\varepsilon}{h}\right)\left(1+\frac{bh}{2}\right)+a\left(1-\frac{\delta}{h}\right)+\frac{abh}{2}\right)y_i\\
&\quad+\left(\frac{\varepsilon}{h}+\frac{b\varepsilon}{2}-\frac{a\delta}{h}+\frac{ab\delta}{2}\right)y_{i-1}=\frac{h}{2}\left(e^{bh}f_i+f_{i+1}\right).
\end{aligned} \tag{14}$$

The resulting three-term recurrence relation of (14) is of the form

$$E_i y_{i-1} - F_i y_i + G y_{i+1} = H_i \tag{15}$$

where

$$\begin{aligned}
E_i =&\frac{\varepsilon}{h}+\frac{b\varepsilon}{2}-\frac{a\delta}{h}+\frac{ab\delta}{2}\\
F_i =&\frac{\varepsilon}{h}+\frac{\varepsilon}{h}e^{bh}-\frac{b\varepsilon}{2}e^{bh}-\frac{bh}{2}\left(1-\frac{\varepsilon}{h}\right)-\frac{a\delta}{h}e^{bh}+a\left(1-\frac{\delta}{h}\right)\\
&+\frac{abh}{2}-\frac{ab\delta}{2}+\frac{ab\delta}{2}e^{bh}\\
G_i =&\frac{\varepsilon}{h}e^{bh}+\frac{bh}{2}\left(1-\frac{\varepsilon}{h}\right)e^{bh}+a\left(1-\frac{\delta}{h}\right)e^{bh}+\frac{abh}{2}\left(1-\frac{\delta}{h}\right)e^{bh}\\
H_i =&\frac{h}{2}\left(f_i+e^{bh}f_{i+1}\right).
\end{aligned}$$

This diagonal dominant tri-diagonal system is solved by the Thomas algorithm.

2.2. Right-end boundary layer problems

Now, assume $a(x) \leq -M < 0$ throughout the interval $[0,1]$, for some positive constant M. This assumption implies that the boundary layer will be at the right end, that is, in the neighbourhood of $x = 1$. The evaluation of the right-end boundary layer problems for equations (1)-(3) is similar to that of the left-end boundary layer, except for some differences worth noting. By using the Taylor series expansion of the term $y'(x+\varepsilon)$ around the point x, we obtain

$$y'(x+\varepsilon) \approx y'(x) + \varepsilon y''(x). \tag{16}$$

Consequently, (1) is replaced by an asymptotically equivalent first order delay neutral type differential equation:

$$y'(x) - b(x)y(x) = y'(x+\varepsilon) + a(x)y'(x-\delta) - f(x). \tag{17}$$

By applying an integrating factor e^{-bx} to (17) (as in Brian J. McCartin [1]):

$$\frac{d}{dx}\left\{e^{-bx}y(x)\right\} = e^{-bx}\left\{y'(x+\varepsilon) + a(x)y'(x-\delta) - f(x)\right\}. \tag{18}$$

Discretizing the interval $[0,1]$ into N equal subintervals of mesh size $h = 1/N$, let $0 = x_0, x_1, \ldots, x_N = 1$ be the mesh points. Then $x_i = ih, \text{for} i = 0, 1, \ldots, N$. Taking integrals from x_{i-1} to x_i in (18), we have

$$\int_{x_{i-1}}^{x_i} \frac{d}{dx}\left(e^{-bx}y(x)\right) = \int_{x_{i-1}}^{x_i} e^{-bx}(y'(x+\varepsilon) + ay'(x-\delta) - f(x))dx. \tag{19}$$

Again, by integrating (19) with respect to x, we obtain

$$\begin{aligned}
&e^{-bx_i}y(x_i) - e^{-bx_{i-1}}y(x_{i-1}) \\
&\quad = e^{-bx_i}y(x_i+\varepsilon) - e^{-bx_{i-1}}y(x_{i-1}+\varepsilon) + ae^{-bx_i}y(x_i-\delta) \\
&\qquad - ae^{-bx_{i-1}}y(x_{i-1}-\delta) + b\int_{x_{i-1}}^{x_i} e^{-bx}(y(x+\varepsilon) + ay(x-\delta) - f(x))dx
\end{aligned} \tag{20}$$

Now, employing the trapezoidal rule to evaluate the integrals in (20), we get

$$\begin{aligned}
&e^{-bx_i}y(x_i) - e^{-bx_{i-1}}y(x_{i-1}) \\
&\quad = e^{-bx_i}y(x_i+\varepsilon) - e^{-bx_{i-1}}y(x_{i-1}+\varepsilon) + ae^{-bx_i}y(x_i+\varepsilon) - ae^{-bx_{i-1}}y(x_{i-1}+\varepsilon) \\
&\qquad + \frac{bh}{2}\left(e^{-bx_{i-1}}y(x_{i-1}+\varepsilon) + e^{bx_i}y(x_i+\varepsilon)\right) \\
&\qquad + \frac{abh}{2}\left(e^{-bx_{i-1}}y(x_{i-1}-\delta) + e^{-bx_i}y(x_i-\delta)\right) \\
&\qquad - \frac{h}{2}\left[e^{-bx_{i-1}}f(x_{i-1}) + e^{bx_i}f(x_i)\right].
\end{aligned} \tag{21}$$

Again, by means of the Taylor series expansion and then approximating $y'(x)$ by linear interpolation, we get

$$y(x_i-\delta) = y(x_i) - \delta y'(x_i) = \left(1+\frac{\delta}{h}\right)y_i - \frac{\delta}{h}y_{i+1} \tag{22}$$

$$y(x_{i-1}-\delta) = y(x_{i-1}) - \delta y'(x_{i-1}) = \left(1+\frac{\delta}{h}\right)y_{i-1} - \frac{\delta}{h}y_i \tag{23}$$

$$y(x_i+\varepsilon) = y(x_i) + \varepsilon y'(x_i) = \left(1-\frac{\varepsilon}{h}\right)y_i + \frac{\varepsilon}{h}y_{i+1} \tag{24}$$

$$y(x_{i-1}+\varepsilon) = y(x_{i-1}) + \varepsilon y'(x_{i-1}) = \left(1-\frac{\varepsilon}{h}\right)y_{i-1} + \frac{\varepsilon}{h}y_i. \tag{25}$$

By making use of (22)- (25) in (21) and simplifying, we obtain:

$$\left(-\frac{\varepsilon}{h}-\frac{bh}{2}\left(1-\frac{\varepsilon}{h}\right)+a\left(1+\frac{\delta}{h}\right)-\frac{abh}{2}\left(1+\frac{\delta}{h}\right)\right)y_{i-1}$$
$$-\left(\left(-\frac{\varepsilon}{h}+\frac{bh}{2}\left(1-\frac{\varepsilon}{h}\right)+a\left(1+\frac{\delta}{h}\right)+\frac{abh}{2}\left(1+\frac{\delta}{h}\right)\right)e^{-bh}-\frac{ab\delta}{2}-\frac{\varepsilon}{h}-\frac{b\varepsilon}{2}+\frac{a\delta}{h}\right)y_i$$
$$+\left(-\frac{e^{-bh}\varepsilon}{h}-\frac{b\varepsilon}{2}e^{-bh}+\frac{a\delta}{h}e^{-bh}+\frac{ab\delta}{2}e^{-bh}\right)y_{i+1}=\frac{h}{2}\left[e^{-bx_{i-1}}f(x_{i-1})+e^{bx_i}f(x_i)\right] \quad (26)$$

By rearranging, (26) can be written as the three-term recurrence relation

$$E_iy_{i-1}-F_iy_i+Gy_{i+1}=H_i, \quad (27)$$

where

$$E_i=-\frac{\varepsilon}{h}-\frac{bh}{2}\left(1-\frac{\varepsilon}{h}\right)+a\left(1+\frac{\delta}{h}\right)-\frac{abh}{2}\left(1+\frac{\delta}{h}\right)$$
$$F_i=-\frac{\varepsilon}{h}+\frac{bh}{2}\left(1-\frac{\varepsilon}{h}\right)+a\left(1+\frac{\delta}{h}\right)+\frac{abh}{2}\left(1+\frac{\delta}{h}\right))e^{-bh}-\frac{ab\delta}{2}-\frac{\varepsilon}{h}-\frac{b\varepsilon}{2}+\frac{a\delta}{h}$$
$$G_i=-\frac{e^{-bh}\varepsilon}{h}-\frac{b\varepsilon}{2}e^{-bh}+\frac{a\delta}{h}e^{-bh}+\frac{ab\delta}{2}e^{-bh}$$
$$H_i=-\frac{h}{2}\left(e^{-bh}f_i+f_{i-1}\right).$$

The resulting diagonal dominant tri-diagonal system is solved using the Thomas algorithm.

3. Thomas algorithm

A brief description of the algorithm is given as follows: Consider the tri-diagonal system

$$E_iy_{i-1}-F_iy_i+G_iy_{i+1}=H_i \;\;;\; i=1,2,\ldots,N-1 \quad (28)$$

subject to the boundary conditions

$$y_0=y(0)=\varphi_0 \quad (29)$$
$$y_N=y(1)=\gamma_1. \quad (30)$$

We set

$$y_i=W_iy_{i+1}+T_i \;\;;\; i=N-1,N-2,\ldots,2,1, \quad (31)$$

where $W_i=W(x_i)$ and $T_i=T(x_i)$, which are to be determined. From (31), we have:

$$y_{i-1}=W_{i-1}y_i+T_{i-1}. \quad (32)$$

By substituting (33) in (28) and comparing with (31) we obtain the recurrence relations given as follows:

$$W_i=\left(\frac{G_i}{F_i-E_iW_{i-1}}\right) \quad (33)$$

$$T_i = \left(\frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}}\right). \tag{34}$$

To solve these recurrence relations for $i = 1, 2, \ldots, N-1$, initial conditions for W_0 and T_0 are required. For this, we take $y_0 = \varphi_0 = W_0 y_1 + T_0$ and choose $W_0 = 0$ so that the value of $T_0 = \varphi_0$. With these initial values, W_i and T_i for $i = 1, 2, \ldots N-1$ are computed from (33) and (34) in the forward process, and y_i is obtained in the backward process from (30) and (31). For further discussions, about the Thomas algorithm one can refer to Angel and Bellman. The stability of the algorithm is guaranteed under the conditions $E_i > 0$, $G_i > 0$, $F_i \geq E_i + G_i$ and $|E_i| \leq |G_i|$.

4. Convergence analysis

Writing the tri-diagonal system of equations (15) in matrix-vector form, we get

$$AY = C, \tag{35}$$

where $A = (m_{i,j})$, $1 \leq i, j \leq N$-1 is a tri-diagonal matrix of order $N-1$, with

$$\begin{aligned} m_{ii+1} =& \varepsilon e^{bh} + \frac{bh^2}{2} e^{bh} - \frac{bh\varepsilon}{2} e^{bh} + ahe^{bh} - \frac{ah\delta}{2} e^{bh} - \frac{abh^2}{2} e^{bh} + \frac{ab\delta h}{2} e^{bh} \\ m_{ii} =& \varepsilon e^{bh} + \varepsilon - \frac{b\varepsilon h}{2} e^{bh} - \frac{bh^2}{2} + \frac{bh\varepsilon}{2} - a\delta e^{bh} + ah - a\delta + \frac{abh^2}{2} - \frac{abh\delta}{2} + \frac{abh\delta}{2} e^{bh} \\ m_{ii-1} =& \varepsilon + \frac{bh\varepsilon}{2} - a\delta - \frac{abh\delta}{2} \end{aligned}$$

and $C = (d_i)$ is a column vector with $d_i = \frac{h^2}{2}\left(f_i + e^{bh} f_{i+1}\right)$, where i = 1,2,…,N-1 with the local truncation error $T_{\text{i}}(h) = \frac{h^2}{2} K + O(h^3)$, where $K = \left[\left(2ab\delta - \frac{a\delta}{2}\right) y'_i + \frac{h}{2}\left(-2ab\delta - ab^2 - b^2 - 2b\varepsilon - 4\varepsilon - 2a\right) y''_i\right]$.
We also have

$$A\overline{Y} - T(h) = C, \tag{36}$$

where $\overline{Y} = (\overline{y_0}, \overline{y_1}, \ldots, \overline{y_N})^T$ and $T\left(h\right) = \left(T_0(h), T_1(h), \ldots, T_N(h)\right)^T$ are the actual solution and the local truncation error respectively.
From (35) and (36), we have

$$A\ \left(\bar{Y} - Y\right) = T(h). \tag{37}$$

Thus, we obtain the error equation

$$AE = T(h), \tag{38}$$

where $E = \bar{Y} - Y = (e_0, e_1, e_2, \ldots, e_N)^T$. Let the i-th row element sum of matrix A be S_i then we have

$$\begin{aligned} S_i =& -\varepsilon + a\delta + hB'_i + O\left(h^2\right) \text{ for } i = 1, \text{ where } B'_i = -a - \frac{b\varepsilon}{2} - \frac{ab\delta}{2} + ae^{bh} \\ S_i =& B''_i + O\left(h^2\right) \text{ for } i = 2,\ 3, \ldots, N-2, \text{ where } B''_i = h(2ab + b) \\ S_i =& e^{bh}(-\varepsilon + a\delta) + hB'''_i + O\left(h^2\right) \text{ for } i = N-1, \text{ where } B'''_i = -a + \frac{b\varepsilon e^{bh}}{2} + \frac{ab\delta e^{bh}}{2}. \end{aligned}$$

For a given h, A is irreducible and monotone [10], since $0 < \varepsilon << 1$ and $\delta = o(\varepsilon)$. Hence from (38), we get

$$E = A^{-1}T(h) \tag{39}$$

and

$$\|E\| \leq \left\|A^{-1}\right\| . \|T(h)\| . \tag{40}$$

Also from the theory of matrices we have

$$\sum_{i=1}^{N-1} \bar{m}_{k,i}\ S_i = 1\ ,\ k = 1, \ldots, N-1, \tag{41}$$

where $\bar{m}_{ki} \geq 0$ is $(k,i)^{\text{th}}$ element of the matrix A^{-1}, therefore

$$\sum_{i=1}^{N\text{-}1} \bar{m}_{k,i} \leq \frac{1}{\min\limits_{1 \leq\ i\ \leq N-1} S_i} = \frac{1}{B_{i_o}} \leq \frac{1}{|B_{i_o}|} \tag{42}$$

for some i_o between 1 and $N-1$ and

$$B_{i_o} = \begin{cases} B'_i, & i = 1, \ldots, n-1 \\ B''_i, & i = n \\ B'''_i, & i = n+1, \ldots, N-1 \end{cases} . \tag{43}$$

Using (39), (40) and (42), we get

$$e_j = \sum_{i=1}^{N-1} \bar{m}_{k.i}\ \ T_i(h)\quad ;\quad j = 1, 2, \ldots, N-1 \tag{44}$$

which implies

$$e_j \leq \frac{Kh}{|B_{i_o}|}. \tag{45}$$

Therefore, using (45) we have

$$\|E\| = O(h),$$

i.e., our method is uniform convergent on uniform mesh.

5. Numerical examples

The proposed technique is implemented on model examples of the type by equations (1)-(3), for different values of delay parameter δ and perturbation parameter ε. Maximum absolute errors are computed, tabulated and compared with the results of Kadalbajoo and Sharma [4] for the left-end boundary layer problems and Y. N. Reddy et al. [11] for the right-end boundary layer problems.

Example 1. *Consider the singularly perturbed differential difference equation exhibiting left-end boundary layer [4]:*

$$\varepsilon y''(x) + y'(x-\delta) - y(x) = 0 \quad ; \quad 0 < x < 1$$

$\delta \backslash h$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
Present method				
0.1ϵ	0.011721	0.001225	0.000123	$1.230e-005$
0.3ϵ	0.015056	0.001589	0.000159	$1.599e-005$
0.6ϵ	0.025750	0.002812	0.000283	$2.840e-005$
0.8ϵ	0.047807	0.005629	0.000573	$5.747e-005$
$CPUtime(sec)$	0.031973	0.070347	1.069150	124.111747
Results by the method in [4]				
0.1ϵ	0.011824	0.001229	0.000123	$1.236e-005$
0.3ϵ	0.015155	0.001593	0.000160	$1.603e-005$
0.6ϵ	0.025847	0.002816	0.000284	$2.845e-005$
0.8ϵ	0.083131	0.011103	0.001151	$5.748e-005$
$CPUtime(sec)$	0.090661	0.094502	1.391478	277.361489

Table 1: *The maximum absolute error of Example 1 for ε =0.1 and different values of δ*

$\delta \backslash h$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
Present method				
0.1ϵ	0.090733	0.012286	0.001279	$1.284e-004$
0.3ϵ	0.108033	0.015622	0.001644	$1.653e-004$
0.6ϵ	0.127778	0.026309	0.002870	$2.897e-004$
0.8ϵ	0.100404	0.048338	0.005688	$5.794e-004$
$CPUtime(sec)$	0.031375	0.070726	1.093657	125.8793
Results by the method in [4]				
0.1ϵ	0.090928	0.012290	0.001279	$1.284e-004$
0.3ϵ	0.108362	0.015626	0.001644	$1.653e-004$
0.6ϵ	0.128454	0.026314	0.002870	$2.897e-004$
0.8ϵ	0.101499	0.048347	0.005688	$5.794e-004$
$CPUtime(sec)$	0.062357	0.082070	1.281464	290.5346

Table 2: *The maximum absolute error of Example 1 for ε =0.01 and different values of δ*

with boundary conditions $y(0) = 1$, $-\delta \leq x \leq 0$ and $y(1) = 1$.

The exact solution to this problem is given by

$$y(x) = \frac{(1 - e^{m_2})\, e^{m_1 x} + (e^{m_1} - 1)\, e^{m_2 x}}{e^{m_1} - e^{m_2}},$$

where

$$m_1 = \frac{-1 - \sqrt{1 + 4(\varepsilon - \delta)}}{2(\varepsilon - \delta)}$$

and

$$m_2 = \frac{-1 + \sqrt{1 + 4(\varepsilon - \delta)}}{2(\varepsilon - \delta)}.$$

Maximum absolute errors are presented in Tables 1 and 2 for $\varepsilon = 0.1, 0.01$ *and for different values of* δ*, respectively. The effect of* δ *on the boundary layer solutions is presented in Figure 1.*

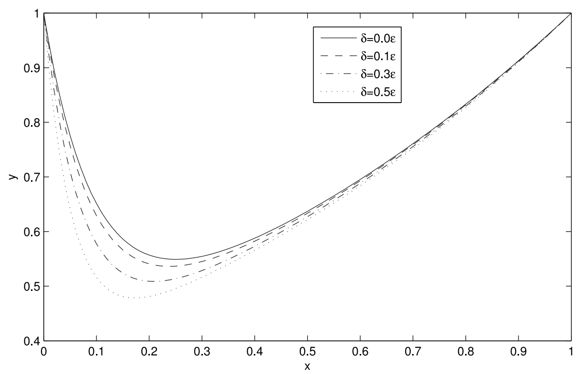

Figure 1: *Left layer solution of Example 1 for* $\epsilon = 0.1$, $N = 10^2$ *and different values of* δ

Example 2. *Consider an example with a variable coefficient singularly perturbed differential-difference equation exhibiting left-end boundary layer [4]:*

$$\varepsilon y''(x) + e^{-0.5x} y'(x - \delta) - y(x) = 0 \quad ; \quad 0 < x < 1$$

with boundary conditions $y(0) = 1$, $-\delta \leq x \leq 0$ *and* $y(1) = 1$.
The exact solution is not known for this problem. Maximum absolute errors are presented in Table 3 for $\varepsilon = 0.1$ *and different values of* δ*, respectively. The effect of* δ *on the boundary layer solutions is presented in Figure 2.*

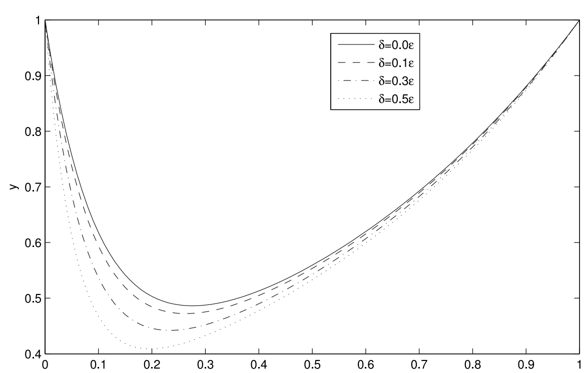

Figure 2: *Left layer solution of Example 2 for* $\epsilon = 0.1$, $N = 10^2$ *and different values of* δ

$\delta\backslash h$	10^{-2}	10^{-3}	10^{-4}
Present method			
0.1ϵ	$6.2687e-003$	$6.6646e-004$	$6.7072e-005$
0.3ϵ	$8.0060e-003$	$8.6458e-004$	$8.7156e-005$
0.6ϵ	$1.342e-002$	$1.5282e-003$	$1.5493e-004$
0.8ϵ	$2.3860e-002$	$3.0459e-003$	$3.1280e-004$
$CPUtime(sec)$	0.044040	0.156487	4.371006
Results by the method in [4]			
0.1ϵ	$6.3299e-003$	$6.7426e-003$	$6.7871e-005$
0.3ϵ	$8.1591e-003$	$8.8825e-004$	$8.8986e-005$
0.6ϵ	$1.3847e-002$	$1.5797e-003$	$1.1602e-004$
0.8ϵ	$2.4771e-002$	$3.1732e-003$	$3.2602e-004$
$CPUtime(sec)$	0.157124	0.402152	6.964002

Table 3: *The maximum absolute error of Example 2 for ε =0.1 and different values of δ*

Example 3. *Consider the singularly perturbed differential-difference equation with right-end boundary layer [4]:*

$$\varepsilon y''(x) - y'(x-\delta) - y(x) = 0 \quad ; \quad 0 < x < 1$$

with boundary conditions $y(0) = 1$, $-\delta \leq x \leq 0$ and $y(1) = -1$.

$\delta\backslash h$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
Present method				
0.1ϵ	0.165949	0.022109	0.002285	0.000229
0.15ϵ	0.158945	0.021173	0.002186	0.000219
0.25ϵ	0.146034	0.019539	0.002013	0.000201
$CPUtime(sec)$	0.076865	0.153582	1.789081	283.2369
Results by the method in [4]				
0.1ϵ	0.1634318	0.0217617	0.002249	0.000225
0.15ϵ	0.1564526	0.0208227	0.002151	0.000215
0.25ϵ	0.1435932	0.0191881	0.001977	0.000198
$CPUtime(sec)$	0.079840	0.102861	1.641013	312.8349

Table 4: *The maximum absolute error of Example 3 for ε =0.01 and different values of δ*

The exact solution is given by

$$y(x) = \frac{(1+e^{m_2})\, e^{m_1 x} - (e^{m_1}+1)\, e^{m_2 x}}{e^{m_2} - e^{m_1}},$$

$\delta \backslash h$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
Present method				
0.1ϵ	$1.94e-006$	$1.95e-007$	$1.95e-008$	$1.71e-009$
0.15ϵ	$2.05e-006$	$2.06e-007$	$2.06e-008$	$2.88e-009$
0.25ϵ	$2.26e-006$	$2.27e-007$	$2.27e-008$	$1.95e-009$
$CPUtime(sec)$	0.076334	0.138354	1.765720	275.5426
Results by the method in [4]				
0.1ϵ	$1.76e-003$	$1.78e-004$	$1.78e-005$	$1.78e-006$
0.15ϵ	$1.78e-003$	$1.79e-004$	$1.79e-005$	$1.79e-006$
0.25ϵ	$1.80e-003$	$1.81e-005$	$1.81e-005$	$1.81e-006$
$CPUtime(sec)$	0.070809	0.164937	1.577380	309.4581

Table 5: *The maximum absolute error of Example 3 for ε =0.001 and different values of δ*

where

$$m_1 = \frac{1-\sqrt{1+4(\varepsilon+\delta)}}{2(\varepsilon+\delta)}$$

and

$$m_2 = \frac{1+\sqrt{1+4(\varepsilon+\delta)}}{2(\varepsilon+\delta)}.$$

Maximum absolute errors are presented in Tables 4 and 5 for $\varepsilon = 0.01, 0.001$ and for different values of δ, respectively. The effect of δ on the boundary layer solutions is presented in Figure 3.

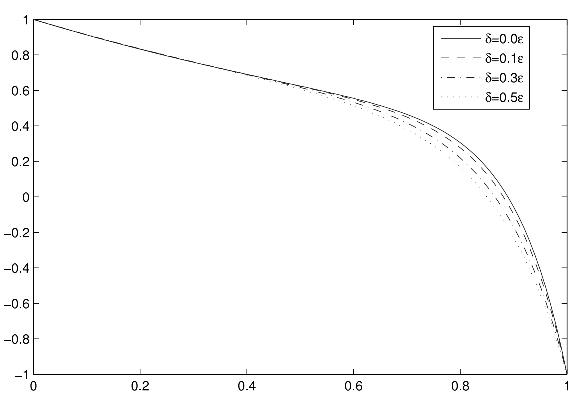

Figure 3: *Right layer solution of Example 3 for $\epsilon = 0.1$, $N = 10^2$ and different values of δ*

Example 4. *Consider an example with a variable coefficient singularly perturbed differential-difference equation exhibiting left-end boundary layer [4]:*

$$\varepsilon y''(x) - e^x y'(x-\delta) - y(x) = 0$$

$\delta \backslash h$	10^{-2}	10^{-3}	10^{-4}
Present method			
0.1ϵ	$7.7065e-003$	$8.5743e-004$	$8.6724e-005$
0.3ϵ	$5.5572e-003$	$6.0006e-004$	$6.0487e-005$
0.6ϵ	$3.8911e-003$	$4.1085e-004$	$4.1314e-005$
0.8ϵ	$3.2241e-003$	$3.3750e-004$	$3.3908e-005$
$CPUtime(sec)$	0.185853	0.376056	7.192782
Results by the method in [4]			
0.1ϵ	$1.3426e-002$	$1.5027e-003$	$1.5211e-004$
0.3ϵ	$9.6462e-003$	$1.0470e-003$	$1.0560e-004$
0.6ϵ	$6.7256e-003$	$7.1364e-004$	$7.1798e-005$
0.8ϵ	$5.5701e-003$	$5.8554e-004$	$5.8854e-005$
$CPUtime(sec)$	0.188576	0.379035	7.271094

Table 6: *The maximum absolute error of Example 4 for* ε *=0.1 and different values of* δ

with $y(0) = 1, \;\; y(1) = 1.$
The exact solution is not known for this problem. Maximum absolute errors are presented in Tables 6 for $\varepsilon = 0.1$ *and for different values of* δ*, respectively. The effect of* δ *on the boundary layer solutions is presented in Figure 4.*

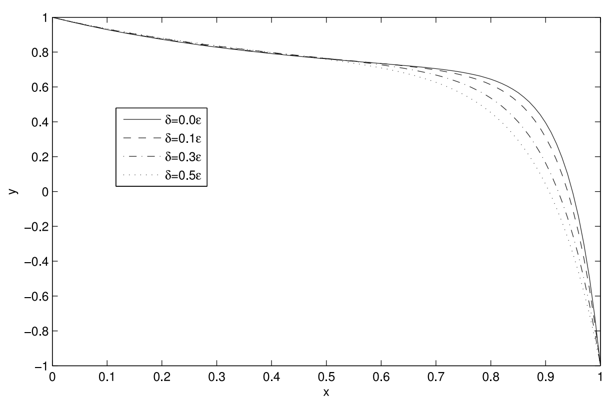

Figure 4: *Right layer solution of Example 4 for* $\epsilon = 0.1$, $N = 10^2$ *and different values of* δ

6. Discussions and conclusion

A numerical integration technique with an exponential integrating factor has been presented for solving singularly perturbed delay differential equations, whose solutions exhibit layer behavior on one (left/right) end of the interval. To validate the proposed method, for the examples with the exact solution, the maximum absolute errors are compared with the results of Kadalbajoo and Sharma [4] (in Tables 1, 2, 4 and 5). For the problems without exact solutions, the double mesh principle has been used to calculate the maximum errors to compare them with the results of Reddy and et al. [11] (in Tables 3 and 6). The effect of a negative shift on the boundary layer solutions has been investigated and presented in graphs (Figures 1-4). It is observed that, as the value of the negative shift δ increases, the thickness of the layer decreases in the left-end boundary layer problems and increases in right-end boundary layer problems. The disadvantage of the proposed method is that it is applicable only for the constant coefficient boundary value problem. Moreover, our method does not depend on a asymptotic expansion as well as on the matching of coefficients. And, hence the technique is simple, easy and it is an alternative to solving singularly perturbed delay differential equations with modest amount of computational effort.

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