The generalized multiplier space and its Köthe-Toeplitz and null duals

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Abstract. The purpose of the present study is to generalize the multiplier space for introducing the concepts of $\alpha$-$B$, $\beta$-$B$, $\gamma$-$B$-duals and $NB$-duals, where $B = (b_{n,k})$ is an infinite matrix with real entries. Moreover, these duals are computed for the sequence spaces $X$ and $X(\Delta)$, where $X \in \{l_p, c, c_0\}$ and $1 \leq p \leq \infty$.

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1. Introduction

Let $\omega$ denote the space of all real-valued sequences. Any vector subspace of $\omega$ is called a sequence space. For $1 \leq p < \infty$, denote by $l_p$ the space of all real sequences $x = (x_n) \in \omega$ such that

$$
\|x\|_p = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} < \infty.
$$

For $p = \infty$, $(\sum_{n=1}^{\infty} |x_n|^p)^{1/p}$ is interpreted as $\sup_{n \geq 1} |x_n|$. We write $c$ and $c_0$ for the spaces of all convergent and null sequences, respectively. Also, $bs$ and $cs$ are used for the spaces of all bounded and convergent series, respectively. Kizmaz [8, 9] defined the forward and backward difference sequence spaces. In this paper, we focus on the backward difference space

$$
X(\Delta) = \{x = (x_k) : \Delta x \in X\},
$$

for $X \in \{l_\infty, c, c_0\}$, where $\Delta x = (x_k - x_{k-1})_{k=1}^{\infty}$, $x_0 = 0$. Observe that $X(\Delta)$ is a Banach space with the norm

$$
\|x\|_\Delta = \sup_{k \geq 1} |x_k - x_{k-1}|.
$$

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In summability theory, the $\beta$-dual of a sequence space is very important in connection with inclusion theorems. The idea of dual sequence space was introduced by Köthe and Toeplitz [10], and it is generalized to the vector-valued sequence spaces by Maddox [11]. For the sequence spaces $X$ and $Y$, the set $M(X, Y)$ defined by

$$M(X, Y) = \{ z = (z_k) \in \omega : (z_k x_k)_{k=1}^{\infty} \in Y \ \forall x = (x_k) \in X \}$$

is called the multiplier space of $X$ and $Y$. With the above notation, the $\alpha$, $\beta$, $\gamma$ and $N$-duals of a sequence space $X$, which are respectively denoted by $X^\alpha$, $X^\beta$, $X^\gamma$ and $X^N$, are defined by

$$X^\alpha = M(X, l_1), \ X^\beta = M(X, cs), \ X^\gamma = M(X, bs), \ X^N = M(X, c_0).$$

For a sequence space $X$, the matrix domain $X(A)$ of an infinite matrix $A$ is defined by

$$X(A) = \{ x = (x_n) \in \omega : Ax \in X \}, \quad (1)$$

which is a sequence space. The new sequence space $X(A)$ generated by the limitation matrix $A$ from a sequence space $X$ can be the expansion or the contraction and the overlap of the original space $X$.

In the past, several authors studied Köthe-Toeplitz duals of sequence spaces that are the matrix domains in classical spaces $l_p$, $l_\infty$, $c$ and $c_0$. For instance, some matrix domains of the difference operator were studied in [4]. The domain of the backward difference matrix in the space $l_p$ was investigated for $1 \leq p \leq \infty$ by Başar and Altay in [3] and was studied for $0 < p < 1$ by Altay and Başar in [1]. Recently the Köthe-Toeplitz duals were computed for some new sequence spaces by Erfanmanesh and Foroutannia [5], [6] and Foroutannia [7]. For more details on the domain of triangle matrices in some sequence spaces, the reader may refer to Chapter 4 of [2].

In the present study, the concept of the multiplier space is generalized and the $\alpha B$-, $\beta B$-, $\gamma B$- and $NB$-duals are determined for the classical sequence spaces $l_p$, $c$ and $c_0$, where $1 \leq p \leq \infty$. Moreover, the $\dagger B$-dual are investigated for the difference sequence spaces $X(\Delta)$, where $X \in \{l_\infty, c, c_0\}$ and $\dagger \in \{\alpha, \beta, N\}$.

2. The $\alpha B$-, $\beta B$-, $\gamma B$- and $NB$-duals of sequence spaces

In this section, we generalize the concept of multiplier space to introduce new generalizations of Köthe-Toeplitz duals and null duals of sequence spaces. Furthermore, we obtain these duals for the sequence spaces $l_p$, $c$ and $c_0$, where $1 \leq p \leq \infty$.

Let $A = (a_{n,k})$ and $B = (b_{n,k})$ be two infinite matrices of real numbers and $X$ and $Y$ two sequence spaces. We write $A_n = (a_{n,k})_{k=1}^{\infty}$ for the sequence in the $n$-th row of $A$. We say that $A$ defines a matrix mapping from $X$ into $Y$, and denote it by $A : X \rightarrow Y$, if and only if $A_n \in X^\beta$ for all $n$ and $Ax \in Y$ for all $x \in X$. If we consider the matrix $AB^t$, where $B^t$ is the transpose of matrix $B$, then the matrix $AB^t$ defines a matrix mapping from $X$ into $Y$, if and only if $(AB^t)_n \in X^\beta$ for all $n$ and $(AB)x \in Y$ for all $x \in X$. Note that the condition $(AB^t)_n \in X^\beta$ implies that

$$\sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} a_{n,i} b_{i,k} \right) < \infty.$$
Based on this fact, we generalize the multiplier space \( M(X, Y) \).

**Definition 1.** Suppose that \( B = (b_{n,k}) \) is an infinite matrix with real entries. For the sequence spaces \( X \) and \( Y \), the set \( M_B(X, Y) \) defined by

\[
M_B(X, Y) = \left\{ z \in \omega : \sum_{k=1}^{\infty} b_{n,k} z_k < \infty, \forall n \text{ and } \left( \sum_{k=1}^{\infty} b_{n,k} z_k \right)_{n=1}^{\infty} \in Y, \forall x \in X \right\}
\]

is called the generalized multiplier space of \( X \) and \( Y \).

The \( \alpha B-, \beta B-, \gamma B- \) and \( NB- \) duals of a sequence space \( X \), which are denoted by \( X^{\alpha B}, X^{\beta B}, X^{\gamma B} \) and \( X^{NB} \), respectively, are defined by

\[
X^{\alpha B} = M_B(X, l_1), \quad X^{\beta B} = M_B(X, cs), \quad X^{\gamma B} = M_B(X, bs), \quad X^{NB} = M_B(X, c_0).
\]

It should be noted that in the special case \( B = I \), we have \( M_B(X, Y) = M(X, Y) \). So

\[
X^{\alpha B} = X^\alpha, \quad X^{\beta B} = X^\beta, \quad X^{\gamma B} = X^\gamma, \quad X^{NB} = X^N.
\]

**Theorem 1.** If \( B = (b_{n,k}) \) is an invertible matrix, then \( M_B(X, Y) \simeq M(X, Y) \).

**Proof.** With the map \( T : M_B(X, Y) \to M(X, Y) \), which is defined by

\[
Tz = \left( \sum_{k=1}^{\infty} b_{n,k} z_k \right)_{n=1}^{\infty},
\]

the proof is obvious. \( \square \)

We determine the generalized multiplier space for some sequence spaces. In order to do this, we state the following lemma which is essential in the study.

**Lemma 1.** If \( X, Y, Z \subset \omega \), then

(i) \( X \subset Z \) implies \( M_B(Z, Y) \subset M_B(X, Y) \),

(ii) \( Y \subset Z \) implies \( M_B(X, Y) \subset M_B(X, Z) \).

**Proof.** The proof is elementary and so omitted. \( \square \)

**Remark 1.** If \( B = I \), we have Lemma 1.25 from [12].

**Corollary 1.** Suppose that \( X, Y \subset \omega \) and \( \dagger \) denotes either of the symbols \( \alpha, \beta, \gamma \) or \( N \). Then

(i) \( X^{\alpha B} \subset X^{\beta B} \subset X^{\gamma B} \subset \omega \); in particular, \( X^{\dagger B} \) is a sequence space.

(ii) \( X \subset Z \) implies \( Z^{\dagger B} \subset X^{\dagger B} \).

**Remark 2.** If \( B = I \), we have Corollary 1.26 from [12].
With the notation of (1), we can define the spaces $X(B)$ for $X \in \{l_p, c, c_0\}$ and $1 \leq p \leq \infty$, as follows:

$$X(B) = \left\{ x = (x_n) \in \omega : \left( \sum_{k=1}^{\infty} b_{n,k}x_k \right)_{n=1}^\infty \in X \right\}.$$  

**Theorem 2.** We have the following statements.

(i) $M_B(c_0, X) = l_\infty(B)$, where $X \in \{l_\infty, c, c_0\}$,

(ii) $M_B(l_\infty, X) = c_0(B)$, where $X \in \{c, c_0\}$,

(iii) $M_B(c, X) = c(B)$, where $X \in \{c, c_0\}$.

**Proof.** (i): Since $c_0 \subset c \subset l_\infty$, by applying Lemma 1(ii), we have

$$M_B(c_0, c) \subset M_B(c_0, c) \subset M_B(c_0, l_\infty).$$

So it is sufficient to verify $l_\infty(B) \subset M_B(c_0, c)$ and $M_B(c_0, l_\infty) \subset l_\infty(B)$. Suppose that $z \in l_\infty(B)$ and $x \in c_0$. We have

$$\lim_{n \to \infty} \left( x_n \sum_{k=1}^{\infty} b_{n,k}z_k \right) = 0.$$  

This means that $z \in M_B(c_0, c)$. Thus $l_\infty(B) \subset M_B(c_0, c)$.

Now we assume $z \notin l_\infty(B)$. Then there is a subsequence $\left( \sum_{k=1}^{\infty} b_{n,j,k}z_k \right)_{n=1}^{\infty}$ of the sequence $\left( \sum_{k=1}^{\infty} b_{n,j,k}z_k \right)_{n=1}^{\infty}$ such that

$$\left| \sum_{k=1}^{\infty} b_{n,j,k}z_k \right| > j^2,$$

for $j = 1, 2, \cdots$. If the sequence $x = (x_i)$ is defined by

$$x_i = \begin{cases} \frac{(-1)^{i-j}}{\sum_{k=1}^{\infty} b_{i,k}z_k}, & \text{if } i = n_j \\ 0, & \text{otherwise}. \end{cases}$$

for $i = 1, 2, \cdots$, we have $x \in c_0$ and $x_n \sum_{k=1}^{\infty} b_{n,j,k}z_k = (-1)^{i-j}$, for all $j$. Hence

$$\left( x_n \sum_{k=1}^{\infty} b_{n,k}z_k \right)_{n=1}^{\infty} \notin l_\infty.$$  

This shows $M_B(c_0, l_\infty) \subset l_\infty(B)$.

(ii): We have

$$M_B(l_\infty, c_0) \subset M_B(l_\infty, c),$$

by applying Lemma 1(ii). It is sufficient to prove $c_0(B) \subset M_B(l_\infty, c_0)$ and $M_B(l_\infty, c) \subset c_0(B)$. Suppose that $z \in c_0(B)$. We have

$$\lim_{n \to \infty} \left( x_n \sum_{k=1}^{\infty} b_{n,k}z_k \right) = 0.$$
for all \( x \in l_\infty \), that is, \( z \in M_B(l_\infty, c_0) \). Thus \( c_0(B) \subset M_B(l_\infty, c_0) \).

Now we assume \( z \notin c_0(B) \). Then there are a real number as \( b > 0 \) and a subsequence \( \left( \sum_{k=1}^{\infty} b_{n,j,k} z_k \right)_{j=1}^{\infty} \) of the sequence \( \left( \sum_{k=1}^{\infty} b_{n,k} z_k \right)_{n=1}^{\infty} \) such that

\[
\left| \sum_{k=1}^{\infty} b_{n,j,k} z_k \right| > b,
\]

for all \( j = 1, 2, \ldots \). If the sequence \( x = (x_i) \) is defined by

\[
x_i = \begin{cases} (-1)^j \sum_{k=1}^{\infty} b_{n,j,k} z_k, & \text{if } i = n_j, \\ 0, & \text{otherwise} \end{cases}
\]

for all \( i \in \mathbb{N} \), then we have \( x \in l_\infty \) and

\[
\left( x_n \sum_{k=1}^{\infty} b_{n,k} z_k \right)_{n=1}^{\infty} \notin c,
\]

which implies \( z \notin M_B(l_\infty, c) \). This shows that \( M_B(l_\infty, c) \subset c_0(B) \).

(iii): Suppose that \( z \in c(B) \). We deduce that \( \lim_{n \to \infty} \left( x_n \sum_{k=1}^{\infty} b_{n,k} z_k \right) \) exists for all \( x \in c_0 \). So \( z \in M_B(c, c_0) \) and \( (c(B) \subset M_B(c, c_0) \).

Conversely, we assume \( z \in M_B(c, c) \). Let \( x = (1, 1, \ldots) \). It is obvious that \( x \in c \) and

\[
\left( \sum_{k=1}^{\infty} b_{n,k} z_k \right)_{k=1}^{\infty} = \left( x_n \sum_{k=1}^{\infty} b_{n,k} z_k \right)_{k=1}^{\infty} \in c.
\]

So \( z \in c(B) \). This shows \( M_B(c, c) \subset c(B) \).

Remark 3. If \( B = I \), we have Example 1.28 from [12].

Corollary 2. We have \( c_0^N(B) = l_\infty(B) \), \( l_\infty^N = c_0(B) \) and \( c^N = c_0(B) \).

Below we recall the concept of normal and similarly to the Köthe-Toeplitz duals, we show that \( X^{\alpha B} = X^{\beta B} = X^{\gamma B} \) when \( X \) is a normal set.

Definition 2. A subset \( X \) of \( \omega \) is said to be normal if \( y \in X \) and \( |x_n| \leq |y_n| \), for \( n = 1, 2, \ldots \), together imply \( x \in X \).

Example 1. The sequence spaces \( c_0 \) and \( l_\infty \) are normal, but \( c \) is not normal.

Theorem 3. Let \( X \) be a normal subset of \( \omega \). We have

\[
X^{\alpha B} = X^{\beta B} = X^{\gamma B}.
\]

Proof. Obviously, \( X^{\alpha B} \subset X^{\beta B} \subset X^{\gamma B} \), by Corollary 1(i). To prove the statement, it is sufficient to verify \( X^{\gamma B} \subset X^{\alpha B} \). Let \( z \in X^{\gamma B} \) and \( x \in X \) be given. We define the sequence \( y \) such that

\[
y_n = \left( \text{sgn} \sum_{k=1}^{\infty} b_{n,k} z_k \right) |x_n|,
\]
for \( n = 1, 2, \ldots \). It is clear \(|y_n| \leq |x_n|\), for all \( n \). Consequently, \( y \in X \) since \( X \) is normal. So

\[
\sup_n \left| \sum_{k=1}^n \left( y_n \sum_{k=1}^\infty b_{n,k} z_k \right) \right| < \infty.
\]

Furthermore, by the definition of the sequence \( y \),

\[
\sum_{n=1}^\infty \left| x_n \sum_{k=1}^\infty b_{n,k} z_k \right| < \infty.
\]

Since \( x \in X \) was arbitrary, \( z \in X^{\alpha B} \). This finishes the proof of the theorem. \( \Box \)

**Remark 4.** If \( B = I \) and \( X \) is a normal subset of \( \omega \), we have

\[
X^\alpha = X^\beta = X^\gamma,
\]

hence Remark 1.27 from [12].

Now, we investigate the \( \alpha B \)-, \( \beta B \)- and \( \gamma B \)-duals for the sequence spaces \( l_\infty \), \( c \) and \( c_0 \).

**Theorem 4.** Suppose that \( \dagger \) denotes either of the symbols \( \alpha \), \( \beta \) or \( \gamma \). We have

\[
c_0^{\dagger B} = c^{\dagger B} = l_\infty^{\dagger B} = l_1(B).
\]

**Proof.** We only prove the statement for the case \( \dagger = \beta \); the other cases are proved by Theorem 3. Obviously, \( l_\infty^{\beta B} \subset c^{\beta B} \subset c_0^{\beta B} \) by Corollary 1(ii). So it is sufficient to show that \( l_1(B) \subset l_\infty^{\beta B} \) and \( c_0^{\beta B} \subset l_1(B) \).

Let \( z \in l_1(B) \) and \( x \in l_\infty \) be given. Hence

\[
\sum_{n=1}^\infty \left| x_n \sum_{k=1}^\infty b_{n,k} z_k \right| \leq \sup_n \left| x_n \right| \sum_{n=1}^\infty \left| \sum_{k=1}^\infty b_{n,k} z_k \right| < \infty,
\]

which shows \((x_n \sum_{k=1}^\infty b_{n,k} z_k)_{n=1}^\infty \in cs \). Thus \( z \in l_\infty^{\beta B} \) and \( l_1(B) \subset l_\infty^{\beta B} \). Now let \( z \notin l_1(B) \). We may choose an index subsequence \((n_j)\) in \( \mathbb{N} \) with \( n_0 = 0 \) and

\[
\sum_{n=n_j-1}^{n_{j-1}} \left| \sum_{k=1}^\infty b_{n,k} z_k \right| > j, \quad j = 1, 2, \ldots .
\]

We define the sequence \( x \in c_0 \) such that

\[
x_n = \begin{cases} 
\frac{1}{j} \text{sgn} \left( \sum_{k=1}^\infty b_{n,k} z_k \right), & \text{if } n_{j-1} \leq n < n_j \\
0, & \text{otherwise}
\end{cases}
\]

We get

\[
\sum_{n=n_j-1}^{n_{j-1}} \left( x_n \sum_{k=1}^\infty b_{n,k} z_k \right) = \frac{1}{j} \sum_{n=n_j-1}^{n_{j-1}} \left| \sum_{k=1}^\infty b_{n,k} z_k \right| > 1,
\]

for \( j = 1, 2, \ldots \). Therefore \((x_n \sum_{k=1}^\infty b_{n,k} z_k)_{k=1}^\infty \notin cs \), and \( z \notin c_0^{\beta B} \). This completes the proof of the theorem. \( \Box \)
Remark 5. If $B = I$ and $\dagger$ denotes either of the symbols $\alpha$, $\beta$ or $\gamma$. We have
\[ c_0^\dagger = c^\dagger = l_\infty^\dagger = l_1, \]
hence Theorem 1.29 from [12].

In the next theorem, we examine the $\alpha B$-, $\beta B$- and $\gamma B$-duals for the sequence space $l_p$.

Theorem 5. If $1 < p < \infty$ and $q = p/(p-1)$, then
\[ l_p^\alpha B = l_p^\beta B = l_p^\gamma B = l_q(B). \]
Moreover for $p = 1$, we have $l_1^\alpha B = l_1^\beta B = l_1^\gamma B = l_\infty(B)$.

Proof. We only prove the statement for the case $1 < p < \infty$; the case $p = 1$ is proved similarly. Let $z \in l_q(B)$ be given. By Hölder’s inequality, we have
\[ \left| \sum_{k=1}^{\infty} \left( x_k \sum_{j=1}^{\infty} b_{k,j} z_j \right) \right| \leq \left( \sum_{k=1}^{\infty} \left( \sum_{j=1}^{\infty} b_{k,j}^q z_j^q \right) \right)^{1/q} \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{1/p} < \infty, \tag{3} \]
for all $x \in l_p$. This shows $z \in l_p^\beta B$ and hence $l_q(B) \subset l_p^\beta B$.

Now, let $z \in l_p^\beta B$ be given. We consider the linear functional $f_n : l_p \to \mathbb{R}$ defined by
\[ f_n(x) = \sum_{k=1}^{n} \left( x_k \sum_{j=1}^{n} b_{k,j} z_j \right), \quad x \in l_p, \]
for $n = 1, 2, \cdots$. Similarly to (3), we obtain
\[ |f_n(x)| \leq \left( \sum_{k=1}^{n} \left( \sum_{j=1}^{n} b_{k,j} z_j \right)^q \right)^{1/q} \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}, \]
for every $x \in l_p$. So the linear functional $f_n$ is bounded and
\[ \|f_n\| \leq \left( \sum_{k=1}^{n} \left( \sum_{j=1}^{n} b_{k,j} z_j \right)^q \right)^{1/q}, \]
for all $n$. We now prove the reverse of the above inequality. We define the sequence $x = (x_k)$ such that
\[ x_k = \left( \text{sgn} \sum_{j=1}^{n} b_{k,j} z_j \right) \left( \sum_{j=1}^{n} b_{k,j} z_j \right)^{q-1}, \]
for \( 1 \leq k \leq n \), and put the remaining elements zero. Obviously, \( x \in l_p \), so
\[
\|f_n\| \geq \frac{|f_n(x)|}{\|x\|_p} = \frac{\sum_{k=1}^{n} \left| \sum_{j=1}^{n} b_{k,j} z_j \right|^{q}}{\left( \sum_{k=1}^{n} \left| \sum_{j=1}^{n} b_{k,j} z_j \right|^{q/p} \right)^{1/p}} = \left( \sum_{k=1}^{n} \left| \sum_{j=1}^{n} b_{k,j} z_j \right|^{q} \right)^{1/q},
\]
for \( n = 1, 2, \cdots \). Since \( z \in l_{\beta B}^p \), the map \( f_z : l_p \to \mathbb{R} \) defined by
\[
f_z(x) = \sum_{k=1}^{\infty} \left( \sum_{j=1}^{\infty} b_{k,j} z_j \right) x_k, \quad x \in l_p,
\]
is well-defined and linear, and also the sequence \((f_n)\) is pointwise convergent to \( f_z \). By using the Banach-Steinhaus theorem, it can be shown that
\[
\|f_z\| \leq \sup_n \|f_n\| < \infty,
\]
so \( z \in l_q(B) \). This establishes the proof of the theorem. \( \square \)

**Remark 6.** If \( B = I \) and \( 1 < p < \infty \) and \( q = p/(p - 1) \). Then we have
\[
l_{1}^{\alpha} = l_{1}^{\beta} = l_{1}^{\gamma} = l_{\infty}.
\]
Moreover, for \( p = 1 \), we have \( l_{1}^{\alpha} = l_{1}^{\beta} = l_{1}^{\gamma} = l_{\infty} \).

### 3. The \( \alpha B, \beta B \) and \( NB \)-duals of sequence spaces \( X(\Delta) \)

The purpose of this section is to compute the \( \dagger B \)-dual of the difference sequence spaces \( X(\Delta) \), where \( X \in \{l_\infty, c, c_0\} \) and \( \dagger \in \{\alpha, \beta, N\} \). In order to do this, we first give a preliminary lemma.

**Lemma 2.**

(i) If \( x \in l_\infty(\Delta) \), then \( \sup_k |\frac{x_k}{k}| < \infty \).

(ii) If \( x \in c(\Delta) \), then \( \frac{x_k}{k} \to \xi (k \to \infty) \), where \( \Delta x_k \to \xi (k \to \infty) \).

(iii) If \( x \in c_0(\Delta) \), then \( \frac{x_k}{k} \to 0 (k \to \infty) \).

**Proof.** The proof is trivial and so omitted. \( \square \)

For convenience of the notations, we use \( X^{\dagger B}(\Delta) \) instead of \( X(\Delta)^{\dagger B} \), where \( \dagger \) denotes either of the symbols \( \alpha, \beta \) or \( N \).

**Theorem 6.** Define the set as follows:
\[
d_1 = \left\{ z = (z_k) : \left( n \sum_{k=1}^{\infty} b_{n,k} z_k \right)_{n=1}^{\infty} \in c_0 \right\}.
\]
Then
\[
c^{NB}(\Delta) = l^{NB}_{\infty}(\Delta) = d_1.
\]
Proof. By using Corollary 1(ii), we have $l_1^{NB}(\Delta) \subset c^{NB}(\Delta)$. So it is sufficient to show that $d_1 \subset l_\infty^{NB}(\Delta)$ and $c^{NB}(\Delta) \subset d_1$.

Let $z \in d_1$ and $x \in l_\infty(\Delta)$. By Lemma 2 $\sup_n |\frac{z_n}{n}| < \infty$, so

$$\lim_{n \to \infty} x_n \sum_{k=1}^\infty b_{n,k} z_k = \lim_{n \to \infty} n \sum_{k=1}^\infty b_{n,k} x_n \frac{n}{n} = 0.$$ 

This implies that $z \in l_\infty^{NB}(\Delta)$. Now suppose that $z \in c^{NB}(\Delta)$, we have

$$\lim_{n \to \infty} x_n \sum_{k=1}^\infty b_{n,k} z_k = 0,$$

for all $x \in c(\Delta)$. If $x = (1, 2, 3, \cdots)$, we have $x \in c(\Delta)$ and

$$\lim_{n \to \infty} n \sum_{k=1}^\infty b_{n,k} z_k = 0.$$ 

So $z \in d_1$ and the proof of the theorem is finished. \qed

Remark 7. If $B = I$, we have $c^N(\Delta) = l_\infty^N(\Delta) = \{ z = (z_k) : (ka_k) \in c_0 \}$, [9].

Now, we recall the following theorem from [12] which is important to continue the discussion. Let $A = (a_{n,k})$ be an infinite matrix of real numbers $a_{n,k}$, where $n, k \in \mathbb{N} = \{1, 2, \cdots\}$. We consider the conditions

$$\sup_n \left( \sum_{k=1}^\infty |a_{n,k}| \right) < \infty,$$ \hspace{1cm} (4)

$$\lim_{n \to \infty} a_{n,k} = 0, \hspace{0.5cm} k = 1, 2, \ldots,$$ \hspace{1cm} (5)

$$\lim_{n \to \infty} a_{n,k} = l_k, \hspace{0.5cm} \text{for some } l_k \in \mathbb{R}, k = 1, 2, \ldots.$$ \hspace{1cm} (6)

By $(X, Y)$, we denote the class of all infinite matrices $A$ such that $A : X \to Y$.

Theorem 7 (see [13]). We have

(i) $A \in (c_0, c_0)$ if and only if conditions (4) and (5) hold;

(ii) $A \in (c_0, c)$ if and only if conditions (4) and (6) hold.

Theorem 8. Define the set as follows:

$$d_2 = \left\{ z = (z_k) : \left( n \sum_{k=1}^\infty b_{n,k} z_k \right)_{n=1}^\infty \in l_\infty \right\}.$$

Then $c_0^{NB}(\Delta) = d_2$. 

Proof. Suppose that \( z \in d_2 \). For \( x \in c_0(\Delta) \), by Lemma 2 we have \( \lim_{n \to \infty} \frac{x_n}{n} = 0 \). So
\[
\lim_{n \to \infty} x_n \sum_{k=1}^{\infty} b_{n,k} z_k = \lim_{n \to \infty} n \sum_{k=1}^{\infty} b_{n,k} z_k \frac{x_n}{n} = 0,
\]
this implies that \( z \in c_0^{NB}(\Delta) \).
Now let \( z \in c_0^{NB}(\Delta) \). We define the matrix \( D = (d_{n,j}) \) by
\[
d_{n,j} = \begin{cases} \sum_{k=1}^{\infty} b_{n,k} z_k, & \text{if } 1 \leq j \leq n \\ 0, & j > n, \end{cases}
\]
and prove that \( D \in (c_0, c_0) \). To do this, we show that \( D_n = (d_{n,j})_{j=1}^{\infty} \in c_0^\beta \) for all \( n \) and moreover \( Dy \in c_0 \) for all \( y \in c_0 \).
Since \( z \in c_0^{NB}(\Delta) \), we deduce that \( \sum_{k=1}^{\infty} b_{n,k} z_k < \infty \) for all \( n \); hence for \( y \in c_0 \)
\[
\sum_{j=1}^{\infty} d_{n,j} y_j = \sum_{j=1}^{n} b_{n,k} z_k y_j = \left( \sum_{j=1}^{n} y_j \right) \left( \sum_{k=1}^{\infty} b_{n,k} z_k \right) < \infty,
\]
for all \( n \), so \( D_n \in c_0^\beta \) for all \( n \). Moreover, \( z \in c_0^{NB}(\Delta) \) implies that
\[
\lim_{n \to \infty} x_n \sum_{k=1}^{\infty} b_{n,k} z_k = 0,
\]
for all \( x \in c_0(\Delta) \). There exists one and only one \( y = (y_k) \in c_0 \) such that \( x_n = \sum_{j=1}^{n} y_j \). So
\[
\lim_{n \to \infty} \sum_{j=1}^{\infty} d_{n,j} y_j = \lim_{n \to \infty} \sum_{j=1}^{n} \sum_{k=1}^{\infty} b_{n,k} z_k y_j = \lim_{n \to \infty} x_n \sum_{k=1}^{\infty} b_{n,k} z_k = 0,
\]
for all \( y \in c_0 \). Hence \( \lim_{n \to \infty} D_n y = 0 \) and \( Dy \in c_0 \) for all \( y \in c_0 \).
By applying Theorem 7(i) for \( D \in (c_0, c_0) \), we obtain
\[
\sup_{n} \left| \sum_{k=1}^{\infty} b_{n,k} z_k \right| = \sup_{n} \left| \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} b_{n,k} z_k \right| = \sup_{n} \left| \sum_{j=1}^{\infty} d_{n,j} \right| < \infty.
\]
This completes the proof of the theorem. \( \square \)

Remark 8. If \( B = I \), we have \( c_0^\beta(\Delta) = \{ z = (z_k) : (kz_k) \in l_\infty \} \), Lemma 2 from [9].

In what follows, we consider the \( \alpha B \)-dual for the sequence spaces \( c(\Delta) \) and \( l_\infty(\Delta) \).

Theorem 9. Define the set \( d_3 \) as follows:
\[
d_3 = \left\{ z = (z_k) : \left( n \sum_{k=1}^{\infty} b_{n,k} z_k \right) \in l_1 \right\}.
\]
Then \( c^{\alpha B}(\Delta) = l_\infty^{\alpha B}(\Delta) = d_3 \).

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Proof. By applying Corollary 1(ii), we have $l_1^B(\Delta) \subset c_0^B(\Delta)$. So it is sufficient to show that $d_3 \subset l_1^B(\Delta)$ and $c_0^B(\Delta) \subset d_3$.

Let $z \in d_3$ and $x \in l_\infty(\Delta)$ be given. By Lemma 2 sup$_n |x| / n < \infty$, so

$$
\sum_{n=1}^\infty \left| \sum_{k=1}^\infty b_{n,k} x_k \right| \leq \sup_{n} \left| \frac{x_n}{n} \right| \sum_{n=1}^\infty \left| \sum_{k=1}^\infty b_{n,k} x_k \right| < \infty,
$$

which shows $z \in c_0^B(\Delta)$ and $d_3 \subset l_1^B(\Delta)$. Now suppose that $z \in c_0^B(\Delta)$. Since $x = (1, 2, 3, \cdots) \in c(\Delta)$, we conclude that

$$
\sum_{n=1}^\infty \left| x_n \sum_{k=1}^\infty b_{n,k} z_k \right| = \sum_{n=1}^\infty \left| x_n \sum_{k=1}^\infty b_{n,k} z_k \right| < \infty,
$$

So $z \in d_3$, and this completes the proof of the theorem. \qed

Remark 9. If $B = I$, we have $c_0(\Delta) = l_1(\Delta) = \{ z = (z_k) : (kz_k) \in l_1 \}$. In order to investigate the $\beta B$-dual of the difference sequence space $c_0(\Delta)$, we need the following lemma.

Lemma 3 (see [9, Lemma 1]). If $z \in l_1, x \in c_0(\Delta)$ and $\lim_{k \to \infty} |z_k x_k| = L$, then $L = 0$.

For the next result we introduce the sequence $(R_k)$ given by

$$
R_k = \sum_{t=k}^\infty \sum_{j=1}^\infty b_{t,j} z_j.
$$

Theorem 10. If

$$
d_4 = \{ a = (a_k) \in l_1(B) : (R_k) \in l_1 \cap c_0^N(\Delta) \},
$$

then we have $c_0^\beta(\Delta) = d_4$.

Proof. Suppose that $z \in d_4$ and $x \in c_0(\Delta)$, by using Abel’s summation formula, we have

$$
\sum_{n=1}^m \left( x_n \sum_{k=1}^\infty b_{n,k} z_k \right) = \sum_{n=1}^m \left( \sum_{t=1}^n \sum_{j=1}^\infty b_{t,j} z_j \right) (x_n - x_{n+1}) + \left( \sum_{n=1}^m \sum_{k=1}^\infty b_{n,k} z_k \right) x_{m+1}
$$

$$
= \sum_{n=1}^m (R_{n+1} - R_n) (x_n - x_{n+1}) + (R_1 - R_{m+1}) x_{m+1}
$$

$$
= \sum_{n=1}^m R_n (x_n - x_{n-1}) - R_{m+1} x_{m+1}. \quad (7)
$$

This implies that

$$
\sum_{n=1}^\infty \left( x_n \sum_{k=1}^\infty b_{n,k} z_k \right)
$$

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is convergent, so \( z \in c_0^{\beta B}(\Delta) \).

Let \( z \in c_0(\Delta)^{\beta B} \), by applying Corollary 1(ii) and Theorem 4 we have \( c_0(\Delta)^{\beta B} \subset c_0^{\beta B} = l_1(B) \); hence \( z \in l_1(B) \). If \( x \in c_0(\Delta) \), then there exists \( y = (y_k) \in c_0 \) such that \( x_n = \sum_{j=1}^{k} y_j \). By applying Abel’s summation formula

\[
\sum_{n=1}^{m} R_n y_n = \sum_{n=1}^{m} (R_n - R_{n+1}) \left( \sum_{j=1}^{n} y_j \right) + \sum_{n=1}^{m} R_{m+1} y_n
\]

Thus

\[
\sum_{n=1}^{m} \left( \sum_{k=1}^{\infty} b_{n,k} z_k x_n \right) = \sum_{n=1}^{m} (R_n - R_{m+1}) y_n = \sum_{n=1}^{m} \left( \sum_{i=1}^{n} \sum_{j=1}^{\infty} b_{i,j} z_j \right) y_n.
\]  (8)

Now we define the matrix \( D = (d_{n,k}) \) by

\[
d_{n,k} = \begin{cases} \sum_{i=k}^{n} \sum_{j=1}^{\infty} b_{i,j} z_j, & \text{if } 1 \leq k \leq n \\ 0, & \text{if } k > n \end{cases}
\]

and we prove that \( D \in (c_0, c) \). To do this, we show that \( D_n = (d_{n,j})_{j=1}^{\infty} \in c_0^{\beta} \) for all \( n \), and moreover \( Dy \in c \) for all \( y \in c_0 \).

Since \( z \in c_0^{\beta B}(\Delta) \), we deduce that

\[
\sum_{k=1}^{\infty} b_{n,k} z_k < \infty,
\]

for all \( n \); hence for \( y \in c_0 \)

\[
\sum_{k=1}^{\infty} d_{n,k} y_k = \sum_{k=1}^{n} \left( \sum_{i=k}^{n} \sum_{j=1}^{\infty} b_{i,j} z_j \right) y_k < \infty,
\]

for all \( n \). So \( D_n \in c_0^{\beta} \) for all \( n \). Moreover, \( z \in c_0^{\beta B}(\Delta) \) implies that

\[
\sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} b_{n,k} z_k \right)
\]

is convergent for all \( x \in c_0(\Delta) \). Hence by (8), we deduce that

\[
\lim_{n \to \infty} D_n y = \lim_{n \to \infty} \sum_{k=1}^{n} d_{n,k} y_k = \lim_{n \to \infty} \sum_{i=1}^{n} x_i \left( \sum_{j=1}^{\infty} b_{i,j} z_j \right)
\]
exists. So $Dy \in c_0$ for all $y \in c_0$ and $D \in (c_0, c)$. This implies that
\[
\sup_n \sum_{k=1}^{\infty} |d_{n,k}| = \sup_n \sum_{k=1}^{n} \sum_{j=k}^{\infty} b_{i,j} z_j < \infty,
\]
by Theorem 7(ii). Thus we get
\[
\sum_{k=1}^{\infty} |R_k| < \infty.
\]
Furthermore, (7) implies that $\lim_{n \to \infty} R_{n+1} x_{n+1}$ exists for each $x \in c_0(\Delta)$; hence $(R_n) \in c_0^N(\Delta)$ by Lemma 3. This completes the proof of the theorem. \(\square\)

Remark 10. If $B = I$, then we have
\[
c_0^\beta(\Delta) = \{ z = (z_k) \in l_1 : (R_k) \in l_1 \cap c_0^N(\Delta) \},
\]
where the sequence $(R_k)$ given by $R_k = \sum_{i=k}^{\infty} z_i$, hence Lemma 3 from [9] is resulted.

References