RAD HAZU. MATEMATIČKE ZNANOSTI
Vol. 21 = 532 (2017): 9-20
DOI: http://doi.org/10.21857/ygjwrc6j1y

COMPUTABILITY OF A WEDGE OF CIRCLES

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Dedicated to the memory of Professor Sibe Mardešić

Abstract. We examine conditions under which a semicomputable set in a computable metric space is computable. In particular, we focus on wedge of circles and prove that each semicomputable wedge of circles is computable.

1. Introduction

A compact subset S of Euclidean space is computable if we can effectively approximate S by a finite set of points with rational coordinates with arbitrary precision. On the other hand, S is semicomputable if we can effectively enumerate all rational open sets (i.e. finite unions of rational balls) which cover S. Each computable set is semicomputable, but a semicomputable set in general need not be computable. In fact, while each nonempty computable set contains computable points and they are dense in it (a point is computable if it can be effectively approximated by a rational point with arbitrary precision), there exists a nonempty semicomputable set in $\mathbb{R}$ which contains no computable point [14].

However, it turns out that under certain conditions we can conclude that a semicomputable set is computable. Topology plays here an important role. An important result in this direction was made by Miller [11] who showed that the implication

$$S \text{ semicomputable} \implies S \text{ computable} \tag{1.1}$$

holds in Euclidean space if S is a topological sphere or a cell whose boundary sphere is computable. A more general result was later proved in [6]: (1.1) holds in any computable metric space if S is a compact manifold whose boundary is computable.

Implication (1.1) holds not just when S is locally Euclidean. In [7, 8] it was proved that (1.1) holds if S is a circularly chainable continuum which is

2010 *Mathematics Subject Classification.* 03D78.
Key words and phrases. Computable set, semicomputable set, wedge of circles.

not chainable or if S is a continuum chainable from a to b, where a and b are computable points. Results related to (1.1) can also be found in [1,3,5,9,10].

In particular, by [6], the implication (1.1) holds if S is a topological circle. In this paper we examine spaces called wedge of circles [12]. We prove that (1.1) holds if S is a wedge of circles. The main idea in the proof of this result is to use the notion of the computability of a set at a point. We are going to prove that a semicomputable set S is computable at a point x if x has a "starlike neighbourhood" in S (Theorem 3.2). Using this, our main result (Theorem 4.2) will follow easily.

2. Preliminaries

In this section we give some basic notions of computable analysis (see [4], [13], [15], [16], [2]). Let $k \in \mathbb{N} \setminus \{0\}$. A function $f : \mathbb{N}^k \to \mathbb{Q}$ is said to be *computable* if there exist computable functions $a, b, c : \mathbb{N}^k \to \mathbb{N}$ such that $c(x) \neq 0$ and

$$f(x) = (-1)^{a(x)} \frac{b(x)}{c(x)}$$

for each $x \in \mathbb{N}^k$.

A number $x \in \mathbb{R}$ is said to be *computable* if there exists a computable function $f : \mathbb{N} \to \mathbb{Q}$ such that

$$|x - f(k)| < 2^{-k},$$

for each $k \in \mathbb{N}$.

A function $f : \mathbb{N}^k \to \mathbb{R}$ is said to be *computable* if there exists a computable function $F : \mathbb{N}^{k+1} \to \mathbb{Q}$ such that

$$|f(x) - F(x, i)| < 2^{-i},$$

for all $x \in \mathbb{N}^k$ and $i \in \mathbb{N}$.

Let (X, d) be a metric space and let $\alpha = (\alpha_i)$ be a dense sequence in (X, d) such that the function $\mathbb{N}^2 \to \mathbb{R}$,

$$(i, j) \mapsto d(\alpha_i, \alpha_j)$$

is computable. Then we say that the triple (X, d, α) is a *computable metric space.*

If (X, d, α) is a computable metric space, $i \in \mathbb{N}$ and $q \in \mathbb{Q}$, $q > 0$, then we will say that $B(\alpha_i, q)$ is a *rational open ball* in (X, d, α).

From now on, let $q : \mathbb{N} \to \mathbb{Q}$ be some fixed computable function whose image is the set of all positive rational numbers and let $\tau_1, \tau_2 : \mathbb{N} \to \mathbb{N}$ be some fixed computable functions such that $\{(\tau_1(i), \tau_2(i)) \mid i \in \mathbb{N}\} = \mathbb{N}^2$.

If (X, d, α) is a computable metric space and $i \in \mathbb{N}$, we define $\lambda_i = \alpha_{\tau_1(i)}$, $\rho_i = q_{\tau_2}$ and

$$I_i = B(\lambda_i, \rho_i).$$

Then $\{I_i \mid i \in \mathbb{N}\}$ is the family of all rational open balls in (X, d, α).

The finite union $B_1 \cup \cdots \cup B_k$ of rational open balls in a computable metric space (X, d, α) will be called a *rational open set* in (X, d, α).

Let $\sigma : \mathbb{N}^2 \to \mathbb{N}$ and $\eta : \mathbb{N} \to \mathbb{N}$ be some fixed computable functions such that $\{(\sigma(j, 0), \ldots, \sigma(j, \eta(j))) \mid j \in \mathbb{N}\}$ is the set of all finite nonempty sequences in $\mathbb{N}$. We use the following notation: $(j)_i$ instead of $\sigma(j, i)$ and $\overline{j}$ instead of $\eta(j)$. Hence

$$\{((j)_0, \ldots, (j)_{\overline{j}}) \mid j \in \mathbb{N}\}$$

is the set of all nonempty finite sequences in $\mathbb{N}$. For $j \in \mathbb{N}$ let

$$[j] = \{(j)_i \mid 0 \leq i \leq \overline{j}\}. \tag{2.1}$$

Then $\{[j] \mid j \in \mathbb{N}\}$ is the family of all nonempty finite subsets of $\mathbb{N}$.

Suppose (X, d, α) is a computable metric space. For $j \in \mathbb{N}$, we define

$$J_j = \bigcup_{i \in [j]} I_i.$$

Then $\{J_j \mid j \in \mathbb{N}\}$ is the family of all rational open sets in (X, d, α). For $i \in \mathbb{N}$ we define $\Lambda_i = \alpha([i])$, i.e.

$$\Lambda_i = \{\alpha_j \mid j \in [i]\}.$$

Then $\{\Lambda_i \mid i \in \mathbb{N}\}$ is the family of all finite nonempty subsets of $\{\alpha_i \mid i \in \mathbb{N}\}$.

Let (X, d, α) be a computable metric space and $x \in X$. Density of the sequence α implies that for each $k \in \mathbb{N}$ there exists $i \in \mathbb{N}$ such that $d(x, \alpha_i) < 2^{-k}$. We say that the point $x \in X$ is *computable* in (X, d, α) if there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that

$$d(x, \alpha_{f(k)}) < 2^{-k}$$

for each $k \in \mathbb{N}$.

Moreover, since a nonempty compact set S in (X, d) is totally bounded, it is easily seen that for each $k \in \mathbb{N}$ there exists a finite nonempty subset A of $\{\alpha_i \mid i \in \mathbb{N}\}$ such that $d_H(S, A) < 2^{-k}$, where d_H is the Hausdorff metric. We say that a compact set S is *computable* in (X, d, α) if $S = \emptyset$ or $S \neq \emptyset$ and there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that

$$d_H(S, \Lambda_{f(k)}) < 2^{-k}$$

for each $k \in \mathbb{N}$.

Let S be a compact set in (X, d). We say that S is *semicomputable* in (X, d, α) if

$$\{j \in \mathbb{N} \mid S \subseteq J_j\}$$

is a computably enumerable (c.e.) subset of $\mathbb{N}$.

It is not hard to check that the definitions of a computable and semicomputable set do not depend on the choice of functions q, τ_1, τ_2, σ and η.

2.1. *Computability at a point.* If (X,d) is metric space, $A,B \subseteq X$ and $\varepsilon > 0$ we write $A \prec_\varepsilon B$ if for each $a \in A$ there exists $b \in B$ such that $d(a,b) < \varepsilon$, i.e. if

$$A \subseteq \bigcup_{b \in B} B(b,\varepsilon).$$

Let (X,d,α) be a computable metric space and $A,B \subseteq X$, $A \subseteq B$. We say that A is *computable up to* B if there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that

$$A \prec_{2^{-k}} \Lambda_{f(k)} \text{ and } \Lambda_{f(k)} \prec_{2^{-k}} B$$

for each $k \in \mathbb{N}$. Note that a nonempty compact set S is computable if and only if S is computable up to S. Furthermore, if $A_1,\dots,A_k$ are sets computable up to S, then $A_1 \cup \dots \cup A_k$ is computable up to S (see [6]).

Let (X,d,α) be a computable metric space, $S \subseteq X$ and $x \in S$. We say that S is *computable at* x if there exists a neighbourhood N of x in S such that N is computable up to S.

Computability of a set can now be characterised in the following way. (see [5])

Proposition 2.1. *Let (X,d,α) be a computable metric space and let $S \subseteq X$ be a compact set. Then S is computable if and only if S is computable at x for each $x \in S$.*

2.2. *Formal disjointness and diameter.* Let (X,d,α) be a computable metric space. For $i,j \in \mathbb{N}$ we write $I_i \diamond I_j$ if

$$d(\lambda_i,\lambda_j) > \rho_i + \rho_j.$$

Note that this is a relation between the numbers i and j, not the sets I_i and I_j. Note that $I_i \diamond I_j$ implies $\mathrm{Cl}(I_i) \cap \mathrm{Cl}(I_j) = \emptyset$. Here we use $\mathrm{Cl}(A)$ to denote the closure of $A \subseteq X$.

Let $i,j \in \mathbb{N}$. We write $J_i \diamond J_j$ if $I_k \diamond I_l$ for all $k \in [i]$ and $l \in [j]$. Clearly, if $J_i \diamond J_j$, then $\mathrm{Cl}(J_i) \cap \mathrm{Cl}(J_j) = \emptyset$.

The main reason why we introduced this relation is the fact that

$$\left\{(i,j) \in \mathbb{N}^2 \mid J_i \cap J_j = \emptyset\right\},$$

need not be c.e., but it is easily shown (see [7]) that the set

$$\left\{(i,j) \in \mathbb{N}^2 \mid J_i \diamond J_j\right\}$$

is c.e.

Let $A \subseteq X$, $j \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, $\lambda > 0$. We write $A \subseteq_\lambda J_j$ if

$$A \subseteq J_j \text{ and } (I_i \cap A \neq \emptyset \text{ and } \rho_i < \lambda \text{ for each } i \in [j]).$$

Note that $A \subseteq_\lambda J_j$ and $\lambda \leq \lambda'$ implies $A \subseteq_{\lambda'} J_j$.

In general computable metric space the function $\mathbb{N} \to \mathbb{R}$, $j \mapsto \mathrm{diam}(J_j)$, need not be computable. This is a motivation for next definition.

Let (X,d) be a metric space, $x_0,\dots,x_k\in X$ and $r_0,\dots,r_k>0$ The *formal diameter* associated to the finite sequence $(x_0,r_0),\dots,(x_k,r_k)$ is the number

$$D=\max_{0\leq v,w\leq k} d(x_v,x_w)+2\max_{0\leq v\leq k} r_v.$$

Note that $\operatorname{diam}(B(x_0,r_0)\cup\cdots\cup B(x_k,r_k))\leq D$.

Let (X,d,α) be a computable metric space. We define the function $\operatorname{fdiam}:\mathbb{N}\to\mathbb{R}$ in the following way, for $j\in\mathbb{N}$, the number $\operatorname{fdiam}(j)$ is the formal diameter associated to the finite sequence

$$\left(\lambda_{(j)_0},\rho_{(j)_0}\right),\dots,\left(\lambda_{(j)_{\overline{j}}},\rho_{(j)_{\overline{j}}}\right).$$

Note that $\operatorname{diam}(J_j)\leq\operatorname{fdiam}(j)$ for each $j\in\mathbb{N}$.

Furthermore, for $l\in\mathbb{N}$ we define function $\operatorname{fmesh}:\mathbb{N}\to\mathbb{R}$,

$$\operatorname{fmesh}(l)=\max_{0\leq i\leq\overline{l}}\operatorname{fdiam}(J_{(l)_i}),\ \ l\in\mathbb{N}.$$

We have the following Lemma (see Lemmas 4.6, 4.7 and 4.8 in [3]).

Lemma 2.2. *Let (X,d,α) be a computable metric space.*

(1) If A and B are nonempty disjoint compact sets in (X,d), then there exists $\lambda>0$ such that for all $j_1,j_2\in\mathbb{N}$ and $A'\subseteq A$, $B'\subseteq B$ the following implication holds:

$$A'\subseteq_\lambda J_{j_1}\ \text{and}\ B'\subseteq_\lambda J_{j_2}\ \Rightarrow\ J_{j_1}\diamond J_{j_2}.$$

(2) If $A_1,\dots,A_n$ are compact nonempty sets in (X,d) and $r>0$, then there exist $j_1,\dots,j_n\in\mathbb{N}$ such that

$$A_1\subseteq_r J_{j_1}\ \dots,\ A_n\subseteq_r J_{j_n}$$

and such that for all $p,q\in\{1,\dots,n\}$ the following holds:

$$A_p\cap A_q=\emptyset\ \Rightarrow\ J_{j_p}\diamond J_{j_q}.$$

(3) If $A\subseteq X$, $j\in\mathbb{N}$ and $\lambda>0$ such that $A\subseteq_\lambda J_j$, then

$$\operatorname{fdiam}(j)<4\lambda+\operatorname{diam}A.$$

3. Starlike neighbourhoods

Let (X,d,α) be a computable metric space. For $l\in\mathbb{N}$ we define

$$\mathcal{H}_l=J_{(l)_0}\cup\cdots\cup J_{(l)_{\overline{l}}}.$$

We say that $\mathcal{H}_l$ is a *formal chain* if for all $i,j\in\{0,\dots,\overline{l}\}$ the following implication holds:

$$|i-j|>1\ \Rightarrow\ J_{(l)_i}\diamond J_{(l)_j}.$$

Let $n\in\mathbb{N}\setminus\{0\}$ and $(l^1,\dots,l^n)\in\mathbb{N}^n$ be such that:

(1) $\mathcal{H}_{l^1},\dots,\mathcal{H}_{l^n}$ are formal chains;

(2) if $u, v \in \{1, \dots, n\}$, $u \neq v$, then

$$J_{(l^u)_i} \diamond J_{(l^v)_j},$$

for all $i \in \{0, \dots, \overline{l^u}\}$, $j \in \{1, \dots, \overline{l^v}\}$.

Then we say that $(l^1, \dots, l^n)$ is a *formal star chain* in (X, d, α). The following Proposition can be proved by similar techinques as in [6]:

Proposition 3.1. *The set*

$$\{(l^1, \dots, l^n) \in \mathbb{N}^n \mid (l^1, \dots, l^n) \text{ is a formal star chain}\}$$

is c.e., for all $n \in \mathbb{N} \setminus \{0\}$.

Note that for $n \in \mathbb{N} \setminus \{0\}$ and $(l^1, \dots, l^n) \in \mathbb{N}^n$ obviously exists $k \in \mathbb{N}$ such that

$$\mathcal{H}_{l^1} \cup \dots \cup \mathcal{H}_{l^n} = J_k.$$

Similarly as in the proof of Lemma 33 in [7] we get that for each $n \in \mathbb{N} \setminus \{0\}$ there exists a computable function $h : \mathbb{N}^n \to \mathbb{N}$ such that for each $(l^1, \dots, l^n) \in \mathbb{N}^n$,

$$\mathcal{H}_{l^1} \cup \dots \cup \mathcal{H}_{l^n} = J_{h(l^1, \dots, l^n)}. \tag{3.1}$$

For $n \in \mathbb{N} \setminus \{0\}$ and $i \in \{1, \dots, n\}$ let

$$I_i^n = \{(t_1, t_2, \dots, t_n) \in \mathbb{R}^n \mid t_k = 0 \text{ for } k \neq i,\ t_i \in [0, 1]\},$$
$$T^n = I_1^n \cup I_2^n \cup \dots \cup I_n^n,$$
$$\mathring{I}_i^n = \{(t_1, t_2, \dots, t_n) \in \mathbb{R}^n \mid t_k = 0 \text{ for } k \neq i,\ t_i \in [0, 1\rangle\},$$
$$\mathring{T}^n = \mathring{I}_1^n \cup \mathring{I}_2^n \cup \dots \cup \mathring{I}_n^n.$$

If $S \subseteq \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, then

$$\alpha S = \{\alpha x \mid x \in S\}.$$

Theorem 3.2. *Let (X, d, α) be a computable metric space and S a semi-computable set in that space. If $x \in S$ has the property that there exists $n \in \mathbb{N}$, $n \geq 2$ and continuous injective function $f : T^n \to S$ such that $f(0) = x$ and $f(\mathring{T}^n)$ is open in S, then S is computable at x.*

Proof. Since $S \setminus f(\mathring{T}^n)$ is closed in a compact set S, $S \setminus f(\mathring{T}^n)$ is compact and obviously $f(\frac{1}{2}T^n) \cap (S \setminus f(\mathring{T}^n)) = \emptyset$. Hence there exists $m_0 \in \mathbb{N}$ such that

$$S \setminus f(\mathring{T}^n) \subseteq J_{m_0},$$
$$f(\tfrac{1}{2}T^n) \cap J_{m_0} = \emptyset.$$

Now it is easily seen that

$$f(\tfrac{1}{2}T^n) \subseteq S \setminus J_{m_0} \subseteq f(\mathring{T}^n).$$

To simplify notation, for $a, b \in [0, 1]$, $a < b$ and $i \in \{1, \dots, n\}$ we denote

$$K_i = f(I_i^n),$$

$$ {}_a^b K_i = f\left(\{(t_1,\dots,t_n)\in I_i^n \mid t_i\in[a,b]\}\right). $$

Since $\bigcup_{j\neq i} K_j \cup {}_0^{\frac14}K_i$ and ${}_{\frac13}^{\frac12}K_i$ are nonempty disjoint compact sets, by Lemma 2.2(1) for each $i\in\{1,\dots,n\}$, there exists $\lambda_i>0$ with the property that for all $j_1,j_2\in\mathbb{N}$ and $A'\subseteq \bigcup_{j\neq i} K_j\cup {}_0^{\frac14}K_i$, $B'\subseteq {}_{\frac13}^{\frac12}K_i$ the following implication holds:

$$ A'\subseteq_{\lambda_i} J_{j_1},\ B'\subseteq_{\lambda_i} J_{j_2}\ \Rightarrow\ J_{j_1}\diamond J_{j_2}. $$

Analogusly for each $i\in\{1,\dots,n\}$ there exists $\mu_i>0$ such that for all $j_1,j_2\in\mathbb{N}$ and $A'\subseteq\bigcup_{j\neq i}K_j\cup {}_0^{\frac16}K_i$, $B'\subseteq {}_{\frac14}^{1}K_i$ we have:

$$ A'\subseteq_{\mu_i} J_{j_1},\ B'\subseteq_{\mu_i} J_{j_2}\ \Rightarrow\ J_{j_1}\diamond J_{j_2}. $$

We put

$$ \lambda=\min\{\lambda_1,\dots,\lambda_n,\mu_1,\dots,\mu_n\}. $$

Let us define

$$ B_i={}_{\frac13}^{\frac12}K_i, $$

$$ P={}_0^{\frac16}K_1\cup\dots\cup{}_0^{\frac16}K_n. $$

If $b_1,\dots,b_n,p\in\mathbb{N}$ are such that

$$ P\subseteq_\lambda J_p \text{ and } B_i\subseteq_\lambda J_{b_i},\ i=1,\dots,n, $$

from Lemma 2.2(1) we conclude that

$$ J_{b_i}\diamond J_p,\ i=1,\dots,n \text{ and } J_{b_i}\diamond J_{b_j},\ i\neq j. $$

Since $f(\frac12\mathring{T}^n)$ is open in S and $P\subseteq f(\frac12\mathring{T}^n)$, we can aditionally assume that $S\cap J_p\subseteq f(\frac12\mathring{T}^n)$.

Let $k\in\mathbb{N}$. Since f is uniformly continuous (because T_n is a compact metric space) and each I_i^n is a line segment, it can be shown (Proposition 4.5 in [3]) that for each $i\in\{1,\dots,n\}$ we can find compact sets $F_0^i,\dots,F_{m_i}^i$ such that

$$ K_i=F_0^i\cup\dots\cup F_{m_i}^i, $$

$$ F_u^i\cap F_v^i\neq\emptyset\ \Leftrightarrow\ |u-v|\leq 1, $$

$$ \operatorname{diam} F_k^i<2^{-k-1},\ k=1,\dots,m_i, $$

and that for some $e_i\in\{1,\dots,m_i-1\}$

$$ {}_0^{\frac14}K_i=F_0^i\cup\dots\cup F_{e_i}^i, $$

$$ {}_{\frac14}^{1}K_i=F_{e_i+1}^i\cup\dots\cup F_{m_i}^i. $$

Let $\mu=\min\{\lambda,2^{-k-3}\}$. By Lemma 2.2(2), for each $i\in\{1,\dots,n\}$, there exist $j_1^i,\dots,j_{m_i}^i\in\mathbb{N}$ such that

$$ F_k^i\subseteq_\mu J_{j_k^i},\ k=0,\dots,m_i, $$

and such that for all $u, v \in \{1, \dots, m_i\}$ the following holds:

$$F_u^i \cap F_v^i = \emptyset \;\Rightarrow\; J_{j_u^i} \diamond J_{j_v^i}.$$

Also, for $i \in \{1, \dots, n\}$, $u \in \{1, \dots, m_i\}$, from Lemma 2.2(3) we have

$$\operatorname{fdiam}(J_u^i) < \underbrace{4\mu}_{<4\cdot 2^{-k-3}} + \underbrace{\operatorname{diam}(F_u^i)}_{<2^{-k-1}} < 2^{-k}. \tag{3.2}$$

Moreover, the way we have chosen λ implies that for $i \in \{1, \dots, n\}$ we have

$$J_0^i, \dots, J_{e_i}^i \diamond J_{b_i},$$

$$J_0^j, \dots, J_{m_j}^j \diamond J_{b_i},\; j \neq i,$$

$$J_{e_i+1}^i, \dots, J_{m_i}^i \diamond J_p.$$

Let $l^1, \dots, l^n \in \mathbb{N}$ be such that

$$\big((l^i)_0, \dots, (l^i)_{\overline{l^i}}\big) = \big(j_0^i, \dots, j_{m_i}^i\big),\; i = 1, \dots, n.$$

Then $(l^1, \dots, l^n)$ is a formal star chain which covers $f(T^n)$ (and $S \setminus J_{m_0}$ as well). From (3.2) we also get

$$\max_{1 \leq i \leq n} \operatorname{fmesh}(l^i) < 2^{-k}.$$

We have now shown that for each $k \in \mathbb{N}$ there exist $l^1, \dots, l^n, e_1, \dots, e_n \in \mathbb{N}$ with these properties:

(1) $(l^1, \dots, l^n)$ is a formal star chain;

(2) $S \setminus J_{m_0} \subseteq J_{h(l^1, \dots, l^n)}$;

(3) $\max_{1 \leq i \leq n} \operatorname{fmesh}(l^i) < 2^{-k}$;

(4) $0 \leq e^i < \overline{l^i}$, $i = 1, \dots, n$;

(5) $J_{(l^i)_u} \diamond J_{b_i}$, $u = 0, \dots, e_i$,
$J_{(l^j)_0}, \dots, J_{(l^j)_{\overline{l^j}}} \diamond J_{b_i},\; j \neq i$,
$J_{(l^i)_u} \diamond J_p$, $u = e_i + 1, \dots, \overline{l^i}$, $i = 1, \dots, n$.

Conditions (1)-(5) are computably enumerable (arguments for that are very similar to those in the proof of Theorem 5.6 in [6]), i.e. the set Ω of all $(k, l^1, \dots, l^n, e_1, \dots, e_n) \in \mathbb{N}^{2n+1}$ such that (1)-(5) hold is computably enumerable, so there exist computable functions $\tilde{l^1}, \dots, \tilde{l^n}, \tilde{e_1}, \dots, \tilde{e_n} : \mathbb{N} \to \mathbb{N}$ such that for each $k \in \mathbb{N}$ we have $(k, \tilde{l^1}(k), \dots, \tilde{l^n}(k), \tilde{e_1}(k), \dots, \tilde{e_n}(k)) \in \Omega$.

Let $(k, l^1, \dots, l^n, e_1, \dots, e_n) \in \Omega$. As

$$S \cap J_p \subseteq f(\tfrac{1}{2}\mathring{T}^n) \subseteq S \setminus J_{m_0},$$

it is clear that

$$S \cap J_p \subseteq \bigcup_{i=1}^{n} \left(J_{(l^i)_0} \cup \cdots \cup J_{(l^i)_{\overline{l^i}}} \right).$$

Furthermore, since for $i \in \{1, \ldots, n\}$, $u \in \{e_i + 1, \ldots, \overline{l^i}\}$

$$J_{(l^i)_u} \diamond J_p,$$

and therefore

$$J_{(l^i)_u} \cap (S \cap J_p) = \emptyset,$$

we have

$$S \cap J_p \subseteq \bigcup_{i=1}^{n} \left(J_{(l^i)_0} \cup \cdots \cup J_{(l^i)_{e_i}} \right). \tag{3.3}$$

We claim that

$$J_{(l^i)_u} \cap S \neq \emptyset,$$

for all $i \in \{1, \ldots, n\}$, $u \in \{0, \ldots, e_i\}$. Namely, if $J_{(l^i)_u} \cap S = \emptyset$, we define sets

$$U = \bigcup_{j \neq i} \left(J_{(l^j)_0} \cup \cdots \cup J_{(l^j)_{\overline{l^j}}} \right) \cup \bigcup_{t=0}^{u-1} J_{(l^i)_t},$$

$$V = \bigcup_{t=u+1}^{\overline{l^i}} J_{(l^i)_t}.$$

Since $(l^1, \ldots, l^n)$ is a formal star chain, it is easily seen that $U \cap V = \emptyset$. From condition (5) it follows that for $i \in \{1, \ldots, n\}$ we have

$$B_i \subseteq \bigcup_{j=e_i+1}^{\overline{l^i}} J_{(l^i)_j},$$

so $B_i \subseteq V$ and $B_j \subseteq U$, for some $j \neq i$. Hence

$$U \cap f(\tfrac{1}{2}T^n) \neq \emptyset,$$

$$V \cap f(\tfrac{1}{2}T^n) \neq \emptyset,$$

$$f(\tfrac{1}{2}T^n) \subseteq S \setminus J_{m_0} \subseteq U \cup V,$$

so (U, V) is a separation of $f(\frac{1}{2}T^n)$ in (X, d), what is a contradiction because $f(\frac{1}{2}T^n)$ is connected. So we conclude that

$$J_{(l^i)_u} \cap S \neq \emptyset, \tag{3.4}$$

for all $i \in \{1, \ldots, n\}$, $u \in \{0, \ldots, e_i\}$.

Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a computable function such that $\alpha_{\varphi(j)} \in J_j$, for each $j \in \mathbb{N}$. Let $g : \mathbb{N} \to \mathbb{N}$ be any computable function with the property that for $k \in \mathbb{N}$,

$$[g(k)] = \bigcup_{i=1}^{n} \left\{ \varphi\left((\tilde{l^i}(k))_0 \right), \dots, \varphi\left((\tilde{l^i}(k))_{\tilde{e_i}(k)} \right) \right\}$$

(function g can be obtained similarly like in the proof of Theorem 5.6 in [6]). Since $\operatorname{diam}(J_{(\tilde{l^i}(k))_u}) < 2^{-k}$, for all $i \in \{1, \dots, n\}$, $u \in \{0, \dots, \tilde{e_i}(k)\}$, $k \in \mathbb{N}$, from (3.3) we get

$$S \cap J_p \prec_{2^{-k}} \Lambda_{g(k)},$$

and from (3.4)

$$\Lambda_{g(k)} \prec_{2^{-k}} S.$$

Hence $S \cap J_p$ is a neighbourhood of x in S which is computable up to S and we conclude that S is computable at x. ∎

4. Wedge of circles

Let S be a Hausdorff space that is union of the subspaces $S_1, \dots, S_n$ each of which is homeomorphic to the unit circle S^1 and such that $S_i \cap S_j = \{p\}$ whenever $i \neq j$. Then S is called the *wedge of circles* ([12]).

Note that S can be embedded in the plane. If C_i denotes the circle of radius i with center at $(i, 0)$ in $\mathbb{R}^2$, then there exists a homeomorphism $g : C_1 \cup \dots \cup C_n \to S$ such that $g(C_i) = g(S_i)$, $i = 1, \dots, n$ and $g(0) = p$.

Let (X, d, α) be a computable metric space. To prove that the implication

$$S \text{ semicomputable } \Rightarrow S \text{ computable}$$

holds if S is a wedge of circles, we will use the following result (Theorem 5.6 in [6]):

THEOREM 4.1. *Suppose (X, d, α) is a computable metric space, S a semicomputable set in this space and $x \in S$ a point which has a neighbourhood in S homeomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N} \setminus \{0\}$. Then S is computable at x.*

THEOREM 4.2. *Let (X, d, α) be a computable metric space and $S \subseteq X$ a wedge of circles. If S is semicomputable, then S is computable.*

PROOF. Let $S = S_1 \cup \dots \cup S_n$, $S_i \cong S^1$, $i = 1, \dots, n$, $S_i \cap S_j = \{p\}$, whenever $i \neq j$ and $g : C_1 \cup \dots \cup C_n \to S$ homeomorphism such that $g(C_i) = S_i$, where $C_i = B((i, 0), i)$ in $\mathbb{R}^2$, $i = 1, \dots, n$ and $g(0) = p$.

Suppose $x \in S \setminus \{p\}$. Then there is a unique $i \in \{1, \dots, n\}$ such that $x \in S_i$. Since $C_i \setminus \{0\}$ is obviously open in $C_1 \cup \dots \cup C_n$ and $g(C_i \setminus \{0\}) = S_i \setminus \{p\}$, we conclude that $S_i \setminus \{p\}$ is an open set in S. Since $S_i \cong S^1$, for some $p' \in S^1$ we have

$$S_i \setminus \{p\} \cong S^1 \setminus \{p'\} \cong \mathbb{R}.$$

Hence every point $x \in S \setminus \{p\}$ has a neighbourhood in S homeomorphic to $\mathbb{R}$, so by Theorem 4.1, S is computable at x, for each $x \in S \setminus \{p\}$.

Now we are going to show that S is computable at p. Note that if $\varepsilon \in \langle 0, 2\rangle$ the set

$$S' = (C_1 \cup \cdots \cup C_n) \cap \overline{B}(0, \varepsilon)$$

is homeomorphic to T^{2n}. Namely, S' is a union od $2n$ arcs each of which has 0 for one endpoint and for any two distinct arcs, 0 is the only point that lies in their intersection. Since T^{2n} is also a union of $2n$ arcs with the same properties, it is easily seen that there is a homeomorphism $h : T^{2n} \to S'$ such that $h(0) = 0$ and

$$h(\mathring{T}^n) = (C_1 \cup \cdots \cup C_n) \cap B(0, \varepsilon).$$

Now it follows that the function $f = g \circ h$ has the properties needed in Theorem 3.2: f is obviously continuous injective function,

$$f(0) = g(h(0)) = g(0) = p,$$

$$f(\mathring{T}^n) = g(h(\mathring{T}^n))) = g((C_1 \cup \cdots \cup C_n) \cap B(0, \varepsilon)).$$

As g is a homeomorphism, we conclude that $f(\mathring{T}^n)$ is open in S. Therefore, from Theorem 3.2 we get that S is computable at p.

Hence we have shown that S is computable at every point, so by Proposition 2.1, S is a computable set. ∎

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Izračunljivost klina kružnica

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Sažetak. Proučavamo uvjete uz koje je poluizračunljiv skup u izračunljivom metričkom prostoru izračunljiv. Posebno, fokusiramo se na klinove kružnica i dokazujemo da je svaki poluizračunljiv klin kružnica izračunljiv.

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Received: 28.2.2017.