

## SOME RESULTS ON THE SCHWARTZ SPACE OF $\Gamma \backslash G$

GORAN MUIĆ

*to Sibe Mardešić, in memoriam*

ABSTRACT. Let  $G$  be a connected semisimple Lie group with finite center. Let  $\Gamma \subset G$  be a discrete subgroup. We study closed admissible irreducible subrepresentations of the space of distributions  $\mathcal{S}(\Gamma \backslash G)'$  defined by Casselman, and their relations to automorphic forms on  $\Gamma \backslash G$  when  $\Gamma$  is a congruence subgroup.

### 1. INTRODUCTION

Let  $G$  be a connected semisimple Lie group with finite center. Let  $K$  be the maximal compact subgroup of  $G$ , and  $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$  the center of the universal enveloping algebra of the complexification of the Lie algebra  $\mathfrak{g}$  of  $G$ . Let  $\Gamma \subset G$  be a discrete subgroup. For example, it could be a trivial group. But the main example is given by the following

ASSUMPTIONS 1.1. *We assume that  $G$  is a group of  $\mathbb{R}$ -points of a semisimple algebraic group  $\mathcal{G}$  defined over  $\mathbb{Q}$ . Assume that  $G$  is connected and not compact. Let  $\Gamma \subset G$  be a congruence subgroup with respect to the arithmetic structure given by the fact that  $\mathcal{G}$  defined over  $\mathbb{Q}$  (see [6]).*

In [8], Casselman has defined the Schwartz space  $\mathcal{S}(\Gamma \backslash G)$  (see Section 3 for definition). It is obvious that  $G$  acts on the right. The corresponding representation is a smooth representation of moderate growth ([7], [25]). The main object of the interest is the strong topological dual space  $\mathcal{S}(\Gamma \backslash G)'$ . This is the space of all continuous linear functionals on  $\mathcal{S}(\Gamma \backslash G)$  equipped with the strong topology. By general theory of topological vector spaces, the space  $\mathcal{S}(\Gamma \backslash G)'$  is a complete locally convex vector space. The natural action of  $G$  on  $\mathcal{S}(\Gamma \backslash G)'$  is continuous. The usual representation-theoretic arguments are valid there ([12], Section 2).

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The main interest in the space  $\mathcal{S}(\Gamma \backslash G)'$  is that its Garding space can be identified with the space of functions of uniform moderate growth  $\mathcal{A}_{umg}(\Gamma \backslash G)$  (see (UMG-1) and (UMG-2) in Section 3 for the definition). Under Assumption 1.1,  $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ -finite  $\mathcal{A}_{umg}(\Gamma \backslash G)$  are smooth automorphic forms on  $G$  for  $\Gamma$ . Also,  $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ -finite and  $K$ -finite on the right in  $\mathcal{A}_{umg}(\Gamma \backslash G)$  are equal to the space usual space  $\mathcal{A}(\Gamma \backslash G)$  of  $K$ -finite automorphic forms for  $\Gamma$  [6].

Now, we describe the content of the paper and main results proved in the paper. In Section 2, under Assumption 1.1, we recall the notion of smooth and  $K$ -finite automorphic forms. In Section 3, we describe the results of Casselman [8] used in the paper. In Section 4 we prove some main results in the paper. This section is strongly motivated by a lecture of Wallach [26]. Some of the results here are probably well-known, and we present our way of understanding them. We let  $(\pi, \mathcal{H})$  be an irreducible admissible representation of  $G$  acting on the Hilbert space  $\mathcal{H}$ . The space of  $\mathcal{H}^{\infty}$  vectors in  $\mathcal{H}$  is a representation of moderate growth. The main results of Section 4 gives the description of closed irreducible admissible subrepresentations of  $\mathcal{S}(\Gamma \backslash G)'$  in terms of continuous  $\Gamma$ -invariant functionals on  $\mathcal{H}^{\infty}$  (see Proposition 4.3, Theorem 4.7). The proofs use deep results of Casselman and Wallach ([7], [25]) on smooth globalization of representations at the critical points. Examples of subrepresentation can explicitly be constructed using Eisenstein series [13], or be shown to exists using Poincaré series ([17], [18], [19], [20]), or the trace formula ([1], [2]). In Theorem 4.8, we prove that the trivial representation is the only finite-dimensional subrepresentation of  $\mathcal{S}(\Gamma \backslash G)'$  under Assumption 1.1 and assuming that  $G$  has no compact components. In Section 5, we study realization inside  $\mathcal{S}(\Gamma \backslash G)'$  of irreducible subrepresentations  $\mathcal{H}$  of  $L^2(\Gamma \backslash G)$  (see Theorem 5.6). In this case,  $\mathcal{H}^{\infty} \subset \mathcal{A}_{umg}(\Gamma \backslash G)$ . The proof of Theorem 5.6 contains the proof of the fact that smooth cuspidal automorphic forms are rapidly decreasing. This is proved using methods of Casselman and Wallach. Different proof is contained in [16]. In Theorem 4.6, we relate various topologies on  $\mathcal{H}^{\infty}$  for an irreducible subspace  $\mathcal{H} \subset L^2(\Gamma \backslash G)$ . For example, we prove that if the sequence of elements in  $\mathcal{H}^{\infty}$ ,  $(\varphi_n)_{n \geq 1}$ , converges to  $\varphi \in \mathcal{H}^{\infty}$  in the standard topology on  $\mathcal{H}^{\infty}$ , then it converges to  $\varphi$  in usual topology on  $C^{\infty}(G)$  (see the description before the statement of Theorem 5.7. In Section 6, we study  $\Gamma$ -invariants in  $\mathcal{S}'(G)$  and their relation to the space  $(\mathcal{S}(G)')^{\Gamma}$  (see Proposition 6.3). In Proposition 6.4 we give the interpretation of the classical construction of automorphic via Poincaré series (see for example [20]) in terms of  $\Gamma$ -invariants in  $\mathcal{S}'(G)$ .

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## 2. PRELIMINARIES

In this section we assume that  $G$  is a connected semisimple Lie group with finite center, and recall the notion of the norm on  $G$ . It is essential for all what follows.

We fix a minimal parabolic subgroup  $P = MAN$  of  $G$  in the usual way (see [24], Section 2). We have the Iwasawa decomposition  $G = NAK$ .

We recall the notion of a norm on the group following [24], 2.A.2. A norm  $\|\cdot\|$  is a continuous function  $G \rightarrow [1, \infty[$  satisfying the following properties:

- (1)  $\|x^{-1}\| = \|x\|$ , for all  $x \in G$ ;
- (2)  $\|x \cdot y\| \leq \|x\| \cdot \|y\|$ , for all  $x, y \in G$ ;
- (3) the sets  $\{x \in G; \|x\| \leq r\}$  are compact for all  $r \geq 1$ ;
- (4)  $\|k_1 \exp(tX)k_2\| = \|\exp(X)\|^t$ , for all  $k_1, k_2 \in K, X \in \mathfrak{p}, t \geq 0$ .

Any two norms  $\|\cdot\|_i, i = 1, 2$ , are equivalent: there exist  $C, r > 0$  such that  $\|x\|_1 \leq C\|x\|_2^r$ , for all  $x \in G$ .

We recall the following lemma:

LEMMA 2.1. *There exists a real number  $d_0 > 0$  such that  $\int_G \|g\|^{-d} dg < \infty$  for  $d \geq d_0$ . Since  $\|g\| \geq 1$  for all  $g \in G$ , the lemma follows.*

PROOF. The existence of  $d_0 > 0$  such that  $\int_G \|g\|^{-d} dg < \infty$  is proved in [24, Lemma 2.A.2.4].  $\square$

In the remainder of this section, we assume the following:

ASSUMPTIONS 2.2. *We assume that  $G$  is a group of  $\mathbb{R}$ -points of a semisimple algebraic group  $\mathcal{G}$  defined over  $\mathbb{Q}$ . Assume that  $G$  is connected and not compact. Let  $\Gamma \subset G$  be a congruence subgroup with respect to the arithmetic structure given by the fact that  $\mathcal{G}$  defined over  $\mathbb{Q}$  (see [6]).*

The group satisfying the Assumption 2.2 is a connected semisimple Lie group with finite center. Also,  $\Gamma$  is a discrete subgroup of  $G$  and it has a finite covolume.

An automorphic form (or a  $K$ -finite automorphic form; see [10]) for  $\Gamma$  is a function  $f \in C^\infty(G)$  satisfying the following three conditions ([26] or [6]):

- (A-1)  $f$  is  $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ -finite and  $K$ -finite on the right;
- (A-2)  $f$  is left-invariant under  $\Gamma$  i.e.,  $f(\gamma x) = f(x)$  for all  $\gamma \in \Gamma, x \in G$ ;
- (A-3) there exists  $r \in \mathbb{R}, r > 0$  such that for each  $u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  there exists a constant  $C_u > 0$  such that  $|u.f(x)| \leq C_u \cdot \|x\|^r$ , for all  $x \in G$ .

A smooth automorphic form (see [8], [10]) for  $\Gamma$  is a function  $f \in C^\infty(G)$  satisfying (A1)–(A3) except possibly  $K$ -finiteness. We discuss smooth automorphic forms in more detail the next section.

We write  $\mathcal{A}(\Gamma \backslash G)$  (resp.,  $\mathcal{A}^\infty(\Gamma \backslash G)$ ) for the vector space of all automorphic forms (resp., smooth automorphic forms). Obviously,  $\mathcal{A}(\Gamma \backslash G) \subset$

$\mathcal{A}^\infty(\Gamma \backslash G)$ . It is easy to see that  $\mathcal{A}(\Gamma \backslash G)$  is a  $(\mathfrak{g}, K)$ -module (using [12], Theorem 1), and since  $G$  is connected, the space  $\mathcal{A}^\infty(\Gamma \backslash G)$  is  $G$ -invariant. An automorphic form  $f \in \mathcal{A}^\infty(\Gamma \backslash G)$  is a  $\Gamma$ -cuspidal automorphic form if for every proper  $\mathbb{Q}$ -proper parabolic subgroup  $\mathcal{P} \subset \mathcal{G}$  we have

$$\int_{U \cap \Gamma \backslash U} f(ux) dx = 0, \quad x \in G,$$

where  $U$  is the group of  $\mathbb{R}$ -points of the unipotent radical of  $\mathcal{P}$ . We remark that the quotient  $U \cap \Gamma \backslash U$  is compact. We use normalized  $U$ -invariant measure on  $U \cap \Gamma \backslash U$ . The space of all  $\Gamma$ -cuspidal automorphic forms (resp.,  $\Gamma$ -cuspidal smooth automorphic forms) for  $\Gamma$  is denoted by  $\mathcal{A}_{\text{cusp}}(\Gamma \backslash G)$  (resp.,  $\mathcal{A}_{\text{cusp}}^\infty(\Gamma \backslash G)$ ). The space  $\mathcal{A}_{\text{cusp}}(\Gamma \backslash G)$  is a  $(\mathfrak{g}, K)$ -submodule of  $\mathcal{A}(\Gamma \backslash G)$ . The space  $\mathcal{A}_{\text{cusp}}^\infty(\Gamma \backslash G)$  is  $G$ -invariant.

Following Casselman [8], we define

$$\|g\|_{\Gamma \backslash G} = \inf_{\gamma \in \Gamma} \|\gamma g\|, \quad g \in G.$$

It is obvious that  $\|\cdot\|_{\Gamma \backslash G}$  is  $\Gamma$ -invariant on the right, and that  $\|g\|_{\Gamma \backslash G} \leq \|g\|$  for all  $g \in G$ . The condition (A-3) is equivalent to

(A-3') there exists  $r \in \mathbb{R}$ ,  $r > 0$  such that for each  $u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  there exists a constant  $C_u > 0$  such that  $|u.f(x)| \leq C_u \cdot \|x\|_{\Gamma \backslash G}^r$ , for all  $x \in G$ .

We recall the following standard result:

LEMMA 2.3. *Under above assumptions, we have the following:*

- (a) *If  $f \in C^\infty(G)$  satisfies (A-1), (A-2), and there exists  $p \geq 1$  such that  $f \in L^p(\Gamma \backslash G)$ , then  $f$  satisfies (A-3), and it is therefore an automorphic form. We speak about  $p$ -integrable automorphic form, for  $p = 1$  (resp.,  $p = 2$ ) we speak about integrable (resp., square-integrable) automorphic form.*
- (b) *Let  $p \geq 1$ . Every  $p$ -integrable automorphic form is integrable.*
- (c) *Bounded integrable automorphic form is square-integrable.*
- (d) *If  $f$  is square integrable automorphic form, then the minimal  $G$ -invariant closed subspace of  $L^2(\Gamma \backslash G)$  is a direct sum of finitely many irreducible unitary representations.*
- (e) *Every  $\Gamma$ -cuspidal automorphic form is square-integrable.*

PROOF. For the claims (a) and (e) we refer to [6] and reference there. Since the volume of  $\Gamma \backslash G$  is finite, the claim (b) follows from Hölder inequality (as in [17], Section 3). The claim (c) is obvious. The claim (d) follows from ([24], Corollary 3.4.7 and Theorem 4.2.1).  $\square$

In [20, Proposition 4.7] we give a simple proof of Lemma 2.3 (a) using results of Casselman [8] recalled in the next section.

### 3. SOME RESULTS OF CASSELMAN

In this section we assume that  $G$  is a semisimple connected Lie group with finite center. We assume that  $\Gamma$  is a discrete subgroup of  $G$ . For example,  $\Gamma$  could be a congruence subgroup or just a trivial group.

We recall the definition of the Schwartz space  $\mathcal{S}(\Gamma \backslash G)$  defined by Casselman ([8], page 292). It consists of all functions  $f \in C^\infty(G)$  satisfying the following conditions:

- (CS-1)  $f$  is left-invariant under  $\Gamma$  i.e.,  $f(\gamma x) = f(x)$  for all  $\gamma \in \Gamma, x \in G$ ;
- (CS-2)  $\|f\|_{u, -n} < \infty$  for all  $u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ , and all natural numbers  $n \geq 1$ .

In above definition, for  $u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ , and a real number  $s$ , we let

$$\|f\|_{u, s} \stackrel{def}{=} \sup_{x \in G} \|x\|_{\Gamma \backslash G}^{-s} |u.f(x)|.$$

Since  $\|x\|_{\Gamma \backslash G} \geq 1$ , we have

$$\|f\|_{u, s'} \leq \|f\|_{u, s},$$

for  $s' > s$ .

We recall the following result (see [8], 1.8 Proposition):

PROPOSITION 3.1. *Using above notation, we have the following:*

- (i) *The Schwartz space  $\mathcal{S}(\Gamma \backslash G)$  is a Fréchet space under the seminorms:  $\| \cdot \|_{u, -n}, u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}}), n \in \mathbb{Z}_{\geq 1}$ .*
- (ii) *The right regular representation of  $G$  on  $\mathcal{S}(\Gamma \backslash G)$  is a smooth Fréchet representation of moderate growth.*

We recall the definition of representation of moderate growth. Let  $(\pi, V)$  be a continuous representation on the Fréchet space  $V$ . We say that  $(\pi, V)$  is of moderate growth if it is smooth and if for any continuous semi-norm  $\rho$  there exists an integer  $n$ , a constant  $C > 0$ , and another continuous semi-norm  $\nu$  such that

$$\|\pi(g)v\|_{\rho} \leq C \|g\|^n \|v\|_{\nu}, \quad g \in G, v \in V.$$

We recall that the semi-norms on a locally convex vector space (for example, a Fréchet space)  $V$  are constructed via Minkowski functionals.

The following definition is from [8, page 295].

DEFINITION 3.2. *The space  $\mathcal{S}(\Gamma \backslash G)'$  of tempered distributions or distributions of moderate growth on  $\Gamma \backslash G$  is the strong topological dual of  $\mathcal{S}(\Gamma \backslash G)$ .*

For convenience of the reader, we recall the definition of a strong topological dual in our particular case. By general theory, the subset  $B \subset \mathcal{S}(\Gamma \backslash G)$  is bounded if for every neighborhood  $V$  of 0 there exists  $s > 0$  such that  $B \subset tV$ , for  $t > s$ . This definition is not very practical to use. Again from the general

theory (and easy to see directly),  $B \subset \mathcal{S}(\Gamma \backslash G)$  is bounded if and only if it is bounded in every semi-norm defining topology on  $\mathcal{S}(\Gamma \backslash G)$  i.e.,

$$\sup_{f \in B} \|f\|_{u, -n} < \infty, \quad u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}}), \quad n \in \mathbb{Z}_{\geq 1}.$$

The strong topological dual  $\mathcal{S}(\Gamma \backslash G)'$  of  $\mathcal{S}(\Gamma \backslash G)$  is the space of continuous functionals on  $X$  equipped with strong topology i.e. topology of uniform convergence on bounded sets in  $\mathcal{S}(\Gamma \backslash G)$  i.e. topology given by semi-norms

$$\|\alpha\|_B = \sup_{f \in B} |\alpha(f)|, \quad \text{where } B \text{ ranges over bounded sets of } \mathcal{S}(\Gamma \backslash G).$$

By general theory of topological vector spaces, the space  $\mathcal{S}(\Gamma \backslash G)'$  is a complete locally convex (defined by above semi-norms) vector space.

The natural action of  $G$  on  $\mathcal{S}(\Gamma \backslash G)'$  is continuous. The usual representation-theoretic arguments are valid there ([12], Section 2).

Following Casselman, we consider the two spaces of functions: the functions of moderate growth  $\mathcal{A}_{mg}(\Gamma \backslash G)$ , and the functions of uniform moderate growth  $\mathcal{A}_{umg}(\Gamma \backslash G)$ . The space  $\mathcal{A}_{mg}(\Gamma \backslash G)$  consists of the functions  $f \in C^\infty(G)$  satisfying the following conditions:

- (MG-1)  $f$  is left-invariant under  $\Gamma$  i.e.,  $f(\gamma x) = f(x)$  for all  $\gamma \in \Gamma, x \in G$ ;
- (MG-2) for each  $u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  there exists a constant  $C_u > 0, r_u \in \mathbb{R}, r_u > 0$  such that  $|u.f(x)| \leq C_u \cdot \|x\|^{r_u}$ , for all  $x \in G$ .

The space  $\mathcal{A}_{umg}(\Gamma \backslash G)$  consists of the functions  $f \in C^\infty(G)$  satisfying the following conditions:

- (UMG-1)  $f$  is left-invariant under  $\Gamma$  i.e.,  $f(\gamma x) = f(x)$  for all  $\gamma \in \Gamma, x \in G$ ;
- (UMG-2) there exists  $r \in \mathbb{R}, r > 0$  such that for each  $u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  there exists a constant  $C_u > 0$  such that  $|u.f(x)| \leq C_u \cdot \|x\|^r$ , for all  $x \in G$ .

We note that in the second definition  $r$  is independent of  $u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ .

LEMMA 3.3. *We maintain the assumptions of the first paragraph of Section 2. Then, the spaces of functions which are  $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ -finite and  $K$ -finite on the right in  $\mathcal{A}_{mg}(\Gamma \backslash G)$ , and in  $\mathcal{A}_{umg}(\Gamma \backslash G)$  coincide, and are equal to the space  $\mathcal{A}(\Gamma \backslash G)$  of automorphic forms for  $\Gamma$ . Next, the space of smooth automorphic forms  $\mathcal{A}^\infty(\Gamma \backslash G)$  is a subspace of  $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ -finite functions in  $\mathcal{A}_{umg}(\Gamma \backslash G)$ . Furthermore, we have*

$$\mathcal{A}(\Gamma \backslash G) \subset \mathcal{A}^\infty(\Gamma \backslash G) \subset \mathcal{A}_{umg}(\Gamma \backslash G) \subset \mathcal{A}_{mg}(\Gamma \backslash G).$$

PROOF. This is a simple observation made in [20, Lemma 4.4].  $\square$

LEMMA 3.4. *The Garding space in  $\mathcal{S}(\Gamma \backslash G)'$  is equal to the space  $\mathcal{A}_{umg}(\Gamma \backslash G)$ .*

PROOF. This is [8, Theorem 1.16].  $\square$

We remark that  $\mathcal{S}(\Gamma \backslash G)'$  is not a Fréchet space so [11] can not be applied to prove that the space of smooth vectors is the same as the Garding space. Therefore, for example, in the settings of Lemma 3.3,  $\mathcal{A}^\infty(\Gamma \backslash G)$  is just subspace of the space of all  $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ -finite vectors in  $\mathcal{S}(\Gamma \backslash G)'$ .

Regarding smooth vectors in  $\mathcal{S}(\Gamma \backslash G)'$ , the following lemma will be used later (see [20], Lemma 4.6):

LEMMA 3.5. *Assume that  $f \in L^p(\Gamma \backslash G)$ , for some  $p \geq 1$ , and  $\alpha \in C_c^\infty(G)$ . Then,  $f \star \alpha$  is equal almost everywhere to a function in  $\mathcal{A}_{umg}(\Gamma \backslash G)$ .*

The referee has pointed out that Lemma 3.5 can be also proved using [21, Lemma I.2.5] under the Assumption 2.2.

#### 4. SOME RESULTS ON THE SPACES $\mathcal{S}(\Gamma \backslash G)'$

This section is strongly motivated by a lecture of Wallach [26]. Some of the results here are probably well-known, and we present our way of understanding them. We also give a complete description of irreducible closed subrepresentations  $\mathcal{S}(\Gamma \backslash G)'$ . We prove that under proper assumptions on  $G$  and  $\Gamma$  only finite dimensional subrepresentation of  $\mathcal{S}(\Gamma \backslash G)'$  is trivial representation.

In this section, we let  $(\pi, \mathcal{H})$  be an irreducible admissible representation of  $G$  acting on the Hilbert space  $\mathcal{H}$ . We write  $\langle \cdot, \cdot \rangle$  for the inner product on  $\mathcal{H}$ . We denote by  $\mathcal{H}^\infty$  the subspace of smooth vectors in  $\mathcal{H}$ . It is a complete Fréchet space under the family of semi-norms:

$$\|h\|_u = \|\pi(u)h\|, \quad u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}}),$$

where  $\|\cdot\|$  is the norm on  $\mathcal{H}$  derived from  $\langle \cdot, \cdot \rangle$ . It is a smooth Fréchet representation of moderate growth ([25], Lemma 11.5.1). In particular, if  $\lambda$  is a continuous functional on  $\mathcal{H}^\infty$ , then there exists  $d \in \mathbb{R}$ , and a continuous semi-norm  $\kappa$  such that

$$(4.1) \quad |\lambda(\pi(g)h)| \leq \|g\|^d \kappa(h), \quad g \in G, \quad h \in \mathcal{H}^\infty.$$

The reader can easily check that if (4.1) holds for any  $d = d_0$ , then it holds for all  $d \geq d_0$ . We make the following definition (see also [19], (3-4)):

DEFINITION 4.1. *Let  $d_{\mathcal{H}, \lambda} = d_{\pi, \lambda} \geq -\infty$  be the infimum of all  $d \in \mathbb{R}$  such that (4.1) holds for some continuous semi-norm  $\kappa = \kappa_d$ .*

LEMMA 4.2. *The Fréchet representation  $G$  on  $\mathcal{H}^\infty$  is irreducible in the category of Fréchet representations.*

PROOF. This representation is a canonical globalization (see [25], Chapter 11, or [7]) of a  $(\mathfrak{g}, K)$ -module  $\mathcal{H}_K$ . Hence, the lemma. It is also easy to give a direct proof. Let  $\mathcal{V} \subset \mathcal{H}^\infty$  be a closed subrepresentation different than  $\{0\}$ .

Pick any  $v \in \mathcal{V}$ ,  $v \neq 0$ . Then since  $\mathcal{H}^\infty$  is a smooth representation, the Fourier expansion converges absolutely ([12], Lemma 5):

$$v = \sum_{\delta \in \hat{K}} E_\delta(v),$$

where we fix the normalized Haar measure  $dk$  on  $K$ , and let

$$E_\delta(v) = \int_K d(\delta) \overline{\xi_\delta(k)} \pi(k)v \, dk.$$

Here, as usual  $\hat{K}$  is the set of equivalence of irreducible representations of  $K$ . Also, for  $\delta \in \hat{K}$ , we write  $d(\delta)$  and  $\xi_\delta$  for the degree and character of  $\delta$ , respectively. The vector  $E_\delta(v)$  belongs to the  $\delta$ -isotypic component  $\mathcal{V}(\delta)$  of  $\mathcal{V}$ . This shows that  $\mathcal{H}_K \cap \mathcal{V}$  is dense in  $\mathcal{V}$ . In particular,  $\mathcal{H}_K \cap \mathcal{V}$  is non-zero ( $\mathfrak{g}, K$ )-submodule of  $\mathcal{H}_K$ . Hence,  $\mathcal{H}_K \subset \mathcal{V}$  since  $\mathcal{H}_K$  is irreducible. But because of the same reason  $\mathcal{H}_K$  is dense in  $\mathcal{H}^\infty$ . This implies that  $\mathcal{V} = \mathcal{H}^\infty$ .  $\square$

**PROPOSITION 4.3.** *Let  $\Gamma \subset G$  be a discrete subgroup. Let  $\lambda$  be a continuous functional on  $\mathcal{H}^\infty$  which is  $\Gamma$ -invariant. Then, we have the following:*

- (i) *The pairing  $\mathcal{H}^\infty \times \mathcal{S}(\Gamma \backslash G) \rightarrow \mathbb{C}$  given by  $(h, f) \mapsto \int_{\Gamma \backslash G} \lambda(\pi(g)h) f(g) dg$  is well-defined, continuous in appropriate locally convex topologies, and  $G$ -equivariant.*
- (ii) *The map  $\mathcal{H}^\infty \rightarrow \mathcal{S}(\Gamma \backslash G)'$  which maps  $h \mapsto \alpha_{\lambda, \Gamma}(h)$  where*

$$\alpha_{\lambda, \Gamma}(h)(f) = \int_{\Gamma \backslash G} \lambda(\pi(g)h) f(g) dg, \quad f \in \mathcal{S}(\Gamma \backslash G),$$

*is a continuous map of locally convex representations of  $G$ . The image is contained in  $\mathcal{A}_{\text{umg}}(\Gamma \backslash G)$ .*

- (iii) *If  $\lambda \neq 0$ , then  $\alpha_{\lambda, \Gamma}$  is an embedding. The closure  $Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))$  is a closed irreducible admissible subrepresentation of  $\mathcal{S}(\Gamma \backslash G)'$ .*

**PROOF.** We prove (i). First, we may assume that  $d > 0$  in (4.1). Then,  $\Gamma$ -invariance implies that

$$|\lambda(\pi(g)h)| = |\lambda(\pi(\gamma g)h)| \leq \|\gamma g\|^d \kappa(h),$$

for all  $\gamma \in \Gamma$ ,  $g \in G$ , and  $h \in \mathcal{H}^\infty$ . Hence

$$|\lambda(\pi(g)h)| \leq \|g\|_{\Gamma \backslash G}^d \kappa(h),$$

$g \in G$ , and  $h \in \mathcal{H}^\infty$ .

Next,  $\int_G \|g\|^{-d_0} dg < \infty$  for all sufficiently large  $d_0 > 0$ . Then, [8, Proposition 1.9] implies that  $\int_{\Gamma \backslash G} \|g\|_{\Gamma \backslash G}^{-d_0} dg < \infty$  for all sufficiently large  $d_0 > 0$ .



Hence

$$(4.2) \quad \left| \int_{\Gamma \backslash G} \lambda(\pi(g)h)f(g)dg \right| \leq \int_{\Gamma \backslash G} |\lambda(\pi(g)h)f(g)| dg \leq \kappa(h) \|f\|_{1, -d_0} \cdot \int_{\Gamma \backslash G} \frac{1}{\|g\|_{\Gamma \backslash G}^{-d+d_0}} dg.$$

Consequently, the pairing is well-defined and continuous. It is clearly  $G$ -equivariant. This proves (i).

Now, we prove (ii). The continuity of  $\alpha_{\lambda, \Gamma}$  is obvious from above inequality since if  $B \subset \mathcal{S}(\Gamma \backslash G)$  is bounded, and if we let

$$M_B = \sup_{f \in B} \|f\|_{1, -d_0} < \infty,$$

then we have

$$\|\alpha_{\lambda, \Gamma}(h)\|_B = \sup_{f \in B} |\alpha_{\lambda, \Gamma}(h)(f)| \leq M_B \cdot \left( \int_{\Gamma \backslash G} \frac{1}{\|g\|_{\Gamma \backslash G}^{-d+d_0}} dg \right) \kappa(h), \quad h \in \mathcal{H}^\infty.$$

Next, the first paragraph of the proof shows that the function  $g \mapsto \lambda(\pi(g)h)$  belongs to  $\mathcal{A}_{umg}(\Gamma \backslash G)$ . This completes the proof of (ii).

A different argument is based on results of Casselman (see Lemma 3.4). Indeed, because of the Dixmier–Malliavin, each  $h \in \mathcal{H}^\infty$  can be written in the form

$$h = \sum_{i=1}^l \pi(\beta_i)h_i,$$

for some  $\beta_i \in C_c^\infty(G)$  and  $h_i \in \mathcal{H}^\infty$ . Hence, we have

$$\alpha_{\lambda, \Gamma}(h) = \sum_{i=1}^l r'(\beta_i)\alpha(h_i)$$

which implies that  $\alpha_{\lambda, \Gamma}(h) \in \mathcal{A}_{umg}(\Gamma \backslash G)$ .

Now, we prove (iii). Let  $f \in C_c^\infty(G)$ . Then,  $P_\Gamma(f)(x) \stackrel{def}{=} \sum_{\gamma \in \Gamma} f(\gamma x)$  for  $x \in G$ , defines an element of  $\mathcal{S}(\Gamma \backslash G)$  which is compactly supported modulo  $\Gamma$ . For  $h \in \mathcal{H}^\infty$ , we have

$$\alpha_{\lambda, \Gamma}(h)(P_\Gamma(f)) = \int_{\Gamma \backslash G} \lambda(\pi(g)h)P_\Gamma(f)(g)dg = \int_G \lambda(\pi(g)h)f(g)dg.$$

Letting  $f \in C_c^\infty(G)$  vary, we see that there exists at least one  $h \in \mathcal{H}^\infty$  such that  $\alpha_{\lambda, \Gamma}(h) \neq 0$  provided that  $\lambda \neq 0$ . In view of Lemma 4.2, this implies that  $\alpha_{\lambda, \Gamma}$  is an embedding. Next, as in the proof of Lemma 4.2, we define projectors

$$E_\delta(\alpha) = \int_K d(\delta)\overline{\xi_\delta(k)} r'(k)\alpha dk, \quad \alpha \in \mathcal{S}(\Gamma \backslash G)'$$

for  $\delta \in \hat{K}$ . Since  $\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty)$  is obviously dense in  $Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))$ , we have that

$$E_\delta(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))$$

is dense in

$$E_\delta(Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))).$$

But

$$E_\delta(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty)) = \alpha_{\lambda, \Gamma}(E_\delta(\mathcal{H}^\infty)) = \alpha_{\lambda, \Gamma}(\mathcal{H}^\infty(\delta)) = \alpha_{\lambda, \Gamma}(\mathcal{H}_K(\delta))$$

is a finite-dimensional space. Hence, it is closed. Thus, we have that

$$Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))(\delta) \stackrel{\text{def}}{=} E_\delta(Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))) = \alpha_{\lambda, \Gamma}(\mathcal{H}_K(\delta))$$

is finite-dimensional. This proves that  $Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))$  is admissible. We show that  $Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))$  is irreducible i.e., only closed  $G$ -invariant subspaces of  $Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))$  are  $\{0\}$  and  $Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))$ . We use smooth vectors.

Using the argument from [24, Lemma 1.6.4], the subspace of smooth vectors  $Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))^\infty$  in  $Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))$  is a complete locally convex representation of  $G$  where topology is defined by the semi-norms:

$$\alpha \mapsto \|r'(u)\alpha\|_B,$$

where  $u \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$  and  $B \subset \mathcal{S}(\Gamma \backslash G)$  is bounded. The key thing is that each smooth vector has a Fourier expansion analogous to the one in the proof of Lemma 4.2. Then, as in the proof of Lemma 4.2 we see that  $Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))^\infty$  is irreducible meaning that only  $G$ -invariant subspaces are trivial and everything.

Now, if  $W \subset Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))$  is closed  $G$ -invariant subspace. Assume  $W \neq 0$ . Then

$$W^\infty \subset Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))^\infty$$

is closed  $G$ -invariant subspace in appropriate topology. It is dense in  $W$  (see [12], Corollary 1), and therefore non-zero. But then we must have

$$W^\infty = Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))^\infty.$$

Again because the smooth vectors are dense ([12], Corollary 1), this implies

$$W = Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty)).$$

□

The Garding space  $\mathcal{A}_{umg}(\Gamma \backslash G)$  has a natural filtration by the smooth Frechét representations:

$$\begin{aligned} \mathcal{S}(\Gamma \backslash G) &\subset \cdots \subset \mathcal{A}_{umg, -1}(\Gamma \backslash G) \subset \mathcal{A}_{umg, 0}(\Gamma \backslash G) \subset \\ &\subset \mathcal{A}_{umg, 1}(\Gamma \backslash G) \subset \mathcal{A}_{umg, 2}(\Gamma \backslash G) \subset \cdots, \end{aligned}$$

where for an integer  $n$  we let

$$\mathcal{A}_{umg, n}(\Gamma \backslash G) = \{\varphi \in \mathcal{A}_{umg}(\Gamma \backslash G); \|\varphi\|_{u, n} < \infty, u \in \mathcal{U}(\mathfrak{g}_\mathbb{C})\}.$$

We remark that all embeddings are continuous, and that

$$\mathcal{S}(\Gamma \backslash G) = \bigcap_{n \in \mathbb{Z}} \mathcal{A}_{umg,n}(\Gamma \backslash G).$$

We may therefore let

$$\mathcal{A}_{umg,-\infty}(\Gamma \backslash G) = \mathcal{S}(\Gamma \backslash G).$$

LEMMA 4.4. *Let  $n \geq -\infty$ . Then, the representation of  $G$  on  $\mathcal{A}_{umg,n}(\Gamma \backslash G)$  is of moderate growth.*

PROOF. The proof is similar to the proof of [25, Lemma 11.5.1]. We remarked above that representation is smooth. Let  $\rho$  be the continuous seminorm on  $\mathcal{A}_{umg,n}(\Gamma \backslash G)$ . Then, since  $\rho$  is continuous, there exists a constant  $C > 0$ , and  $u_1, \dots, u_l \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  such that

$$\|\varphi\|_{\rho} \leq C \cdot (\|\varphi\|_{u_1,n} + \dots + \|\varphi\|_{u_l,n}), \quad \varphi \in \mathcal{A}_{umg,n}(\Gamma \backslash G).$$

This is for the case  $n > -\infty$ . But when  $n = -\infty$ , the above is true for convenient integer (again denoted by)  $n$ . In this case, we fix such  $n$ .

Next, we consider the standard filtration of  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  by finite  $G$ -invariant subspaces:

$$\mathcal{U}^0(\mathfrak{g}_{\mathbb{C}}) = \mathbb{C} \subset \mathcal{U}^1(\mathfrak{g}_{\mathbb{C}}) \subset \mathcal{U}^2(\mathfrak{g}_{\mathbb{C}}) \subset \dots$$

Let  $k \geq 0$ . Let  $v_1, \dots, v_k$  be the basis of  $\mathcal{U}^k(\mathfrak{g}_{\mathbb{C}})$ . Then, there exists smooth functions  $\eta_{i,j}$  such that

$$Ad(g)v_i = \sum_{j=1}^k \eta_{ij}(g)v_j.$$

Clearly,  $\eta_{i,j}$  are matrix coefficients of the representation on  $\mathcal{U}^k(\mathfrak{g}_{\mathbb{C}})$ . By the construction of the norm, there exists  $D, r > 0$  such that

$$|\eta_{ij}(g)| \leq D \cdot \|g\|^r, \quad g \in G,$$

for all  $i, j$ .

We assume that  $k$  is large enough so that  $u_1, \dots, u_l \in \mathcal{U}^k(\mathfrak{g}_{\mathbb{C}})$ . Then, we can write

$$Ad(g)u_i = \sum_{j=1}^k \nu_{ij}(g)v_j.$$

The functions  $\nu_{ij}$  are linear combinations of functions  $\eta_{ij}$ . Therefore, there exists  $D_1 > 0$  such that

$$|\nu_{ij}(g)| \leq D_1 \cdot \|g\|^r, \quad g \in G,$$

for all  $i, j$ .

Now, for  $\varphi \in \mathcal{A}_{umg,n}(\Gamma \backslash G)$ , and  $g \in G$ , using properties of the norm, we have

$$\begin{aligned}
\|r(g)\varphi\|_\rho &\leq C \cdot \sum_{i=1}^l \|\varphi\|_{u_i,n} = C \cdot \sum_{i=1}^l \sup_{x \in G} \|x\|^{-n} |u_i \cdot r(g)\varphi(x)| \\
&= C \cdot \sum_{i=1}^l \sup_{x \in G} \|x\|^{-n} |r(g)Ad(g^{-1}u_i) \cdot \varphi(x)| \\
&= C \cdot \sum_{i=1}^l \sup_{x \in G} \|x\|^{-n} |(Ad(g^{-1}u_i) \cdot \varphi(xg))| \\
&\leq C \cdot \sum_{i=1}^l \sum_{j=1}^k |\nu_{ij}(g^{-1})| \sup_{x \in G} \|x\|^{-n} |v_j \cdot \varphi(xg)| \\
&= C \cdot \sum_{i=1}^l \sum_{j=1}^k |\nu_{ij}(g^{-1})| \sup_{x \in G} \|xg^{-1}\|^{-n} |v_j \cdot \varphi(x)| \\
&\leq CD_1 \cdot \sum_{i=1}^l \sum_{j=1}^k \|g^{-1}\|^r \sup_{x \in G} \|xg^{-1}\|^{-n} |v_j \cdot \varphi(x)| \\
&\leq CD_1 \cdot \sum_{i=1}^l \sum_{j=1}^k \|g\|^{n+r} \sup_{x \in G} \|x\|^{-n} |v_j \cdot \varphi(x)| \\
&= lCD_1 \|g\|^{n+r} \sum_{j=1}^k \|\varphi\|
\end{aligned}$$

□

LEMMA 4.5. *Let  $n \geq -\infty$ . Then, the linear functional  $\varphi \mapsto \varphi(1)$  is continuous on  $\mathcal{A}_{umg,n}(\Gamma \backslash G)$ .*

PROOF. Assume first that  $n > -\infty$ . Then, we have

$$|\varphi(1)| = \|1\|_{\Gamma \backslash G}^{-n} |\varphi(1)| \leq \|\varphi\|_{1,-n},$$

for  $\varphi \in \mathcal{A}_{umg,n}(\Gamma \backslash G)$ . The case  $n = -\infty$  is a consequence of above inequalities. □

LEMMA 4.6. *Let  $\Gamma \subset G$  be a discrete subgroup. Let  $\lambda$  be a continuous functional on  $\mathcal{H}^\infty$  which is  $\Gamma$ -invariant. For any integer  $n$  such that  $n > d_{\pi,\lambda}$ , the map, which assigns to  $h \in \mathcal{H}^\infty$  a function  $g \mapsto \lambda(\pi(g)h)$  in  $\mathcal{A}_{umg,n}(\Gamma \backslash G)$ , is  $G$ -equivariant, continuous in appropriate locally convex topologies, and if  $\lambda \neq 0$ , then it is an embedding. Moreover, the same holds if  $n = d_{\pi,\lambda} = -\infty$ .*

PROOF. By definition of  $d_{\pi, \lambda}$  (see Definition 4.1) and the fact that  $\|x\| \geq 1$  for all  $x \in G$ , we have (see (4.1))

$$|\lambda(\pi(g)h)| \leq \|g\|^d \kappa(h), \quad g \in G, \quad h \in \mathcal{H}^\infty.$$

The semi-norm  $h \mapsto \kappa(\pi(u)h)$  is again continuous, for  $u \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ , and we have as a consequence of above inequality

$$|\lambda(\pi(g)\pi(u)h)| \leq \|g\|^d \kappa(\pi(u)h), \quad g \in G, \quad h \in \mathcal{H}^\infty.$$

For  $\gamma \in \Gamma$ , the  $\Gamma$ -invariance of  $\lambda$  implies that

$$|\lambda(\pi(g)\pi(u)h)| = |\lambda(\pi(\gamma g)\pi(u)h)| \leq \|\gamma g\|^d \kappa(\pi(u)h).$$

Since the norm is continuous and  $\Gamma$  discrete, for fixed  $x \in G$ , there exists  $\gamma_0 \in \Gamma$  such that

$$\|\gamma_0 x\| = \|x\|_{\Gamma \backslash G} = \inf_{\gamma \in \Gamma} \|\gamma x\|.$$

Thus above inequality implies

$$|\lambda(\pi(g)\pi(u)h)| \leq \|g\|_{\Gamma \backslash G}^d \kappa(\pi(u)h).$$

This implies that

$$\sup_{g \in G} \|g\|_{\Gamma \backslash G}^{-d} |\lambda(\pi(g)\pi(u)h)| \leq \kappa(\pi(u)h).$$

Now, the lemma easily follows.  $\square$

Now, we prove the main result of the present section.

**THEOREM 4.7.** *Let  $\mathcal{V} \subset \mathcal{S}(\Gamma \backslash G)'$  be a closed irreducible admissible sub-representation of  $\mathcal{S}(\Gamma \backslash G)'$ . Then, there exists an irreducible admissible representation of  $G$  acting on a Hilbert space  $\mathcal{H}$ , and a non-zero  $\Gamma$ -invariant continuous functional on  $\mathcal{H}^\infty$  such that*

$$\mathcal{V} = Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty)).$$

PROOF. By [12, Lemma 4],  $\mathcal{V}^\infty \cap \mathcal{V}_K$  is dense in  $\mathcal{V}$ . Since  $\mathcal{V}$  is admissible, we see that  $\mathcal{V}_K \subset \mathcal{V}^\infty$ . It is easy to check that  $\mathcal{V}_K$  is an irreducible  $(\mathfrak{g}, K)$ -module. In particular, every vector in  $\mathcal{V}_K$  is  $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ -finite. Therefore, by [12, Theorem 1], for each  $\varphi \in \mathcal{V}_K$  there exists  $\alpha \in C_c^\infty(G)$  such that  $r'(\alpha)\varphi = \varphi$ . Hence,  $\mathcal{V}_K$  belongs to the Garding space of  $\mathcal{V}$ , and consequently to the Garding space of  $\mathcal{S}(\Gamma \backslash G)'$  which is  $\mathcal{A}_{umg}(\Gamma \backslash G)$ . By means of the Casselman subrepresentation theorem, we can find an infinitesimal embedding of  $\mathcal{V}_K$  into a principal series of  $G$ . In this way, we obtain a globalization of  $\mathcal{V}_K$  i.e., there exists an irreducible admissible representation  $(\pi, \mathcal{H})$  on the Hilbert space  $\mathcal{H}$  infinitesimally equivalent to  $\mathcal{V}_K$ . Let us fix an isomorphism  $\eta : \mathcal{H}_K \rightarrow \mathcal{V}_K$ .

We recall the filtration of  $\mathcal{A}_{umg}(\Gamma \backslash G)$  by the representations of moderate growth (see Lemma 4.4) :

$$\mathcal{A}_{umg,1}(\Gamma \backslash G) \subset \mathcal{A}_{umg,2}(\Gamma \backslash G) \subset \mathcal{A}_{umg,3}(\Gamma \backslash G) \subset \dots$$

This is also filtration of  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ -modules. Since  $\mathcal{V}_K$  is irreducible, there exists  $n \geq 1$  such that

$$\mathcal{V}_K \subset \mathcal{A}_{umg,n}(\Gamma \backslash G).$$

Let  $V_n$  be the closure of  $\mathcal{V}_K$  in  $\mathcal{A}_{umg,n}(\Gamma \backslash G)$ . It is obvious that a  $(\mathfrak{g}, K)$ -module on the space of  $K$ -finite vectors in  $W_n$  is  $\mathcal{V}_K$ . Therefore,  $V_n$  is irreducible. We remark that  $V_n$  being a closed subrepresentation of a representation of moderate growth  $\mathcal{A}_{umg,n}(\Gamma \backslash G)$  is also a representation of moderate growth (see Lemma 4.4; [25], Lemma 11.5.2). But  $\mathcal{H}^\infty$  is also a representation of moderate growth ([25], Lemma 11.5.1) and irreducible (see Lemma 4.2). So, the isomorphism  $\eta : \mathcal{H}_K \rightarrow (W_n)_K = \mathcal{V}_K$ , extends to a continuous isomorphism of  $G$ -representations  $\eta : \mathcal{H}^\infty \rightarrow V_n$  applying [25, Theorem 11.5.1].

Now, the required linear functional is

$$\lambda(h) \stackrel{def}{=} \eta(h)(1).$$

Indeed, it is obviously continuous (see Lemma 4.5). Next, it is  $\Gamma$ -invariant since

$$\lambda(\pi(\gamma)h) = \eta(\pi(\gamma)h)(1) = r(\gamma)\eta(h)(1) = \eta(h)(\gamma) = \eta(h)(1), \quad h \in \mathcal{H}^\infty, \quad \gamma \in \Gamma.$$

Now, using the notation introduced in Proposition 4.3, we compute

$$\lambda(\pi(g)h) = \eta(\pi(g)h)(1) = r(g)\eta(h)(1) = \eta(h)(g), \quad h \in \mathcal{H}^\infty, \quad g \in G.$$

For  $h \in \mathcal{H}_K$ , above computation and Proposition 4.3 (ii) implies that

$$\alpha_{\lambda,\Gamma}(\mathcal{H}_K) = \mathcal{V}_K.$$

The proof of Proposition 4.3 shows that the space of  $K$ -finite vectors of  $Cl(\alpha_{\lambda,\Gamma}(\mathcal{H}^\infty))$  is  $\alpha_{\lambda,\Gamma}(\mathcal{H}_K)$  and it is dense in  $Cl(\alpha_{\lambda,\Gamma}(\mathcal{H}^\infty))$ . Since  $\mathcal{V}_K$  is dense in  $\mathcal{V}_K$ , the theorem follows.  $\square$

Examples of subrepresentation can explicitly be constructed using Eisenstein series [13], or be shown to exist using Poincaré series [20], or the trace formula ([1], [2]). Now, we show that there are no finite-dimensional representations except the trivial representation in  $\mathcal{S}(\Gamma \backslash G)'$  under appropriate assumptions.

**THEOREM 4.8.** *We maintain Assumption 2.2. Then, if  $G$  has no compact components, then the trivial representation is the only finite-dimensional subrepresentation of  $\mathcal{S}(\Gamma \backslash G)'$ .*

**PROOF.** Let  $\mathcal{V} \subset \mathcal{S}(\Gamma \backslash G)'$  be a finite-dimensional subrepresentation. Then, by Theorem 4.7, there exists finite dimensional representation  $\mathcal{H}$  (satisfying the assumption of the first paragraph of this section) such that

$$\mathcal{V} = Cl(\alpha_{\lambda,\Gamma}(\mathcal{H}^\infty)) = \alpha_{\lambda,\Gamma}(\mathcal{H}^\infty).$$

Since  $\mathcal{H}$  is finite-dimensional, we have  $\mathcal{H}^\infty = \mathcal{H}$ , and there is a non-zero  $\Gamma$ -invariant functional on  $\mathcal{H}$ . So, the algebraic dual  $\mathcal{H}'$  is smooth irreducible representation of  $G$  having a non-zero  $\Gamma$ -invariant vector. By general theory,  $\mathcal{H}'$  is a restriction of an algebraic (holomorphic) representation of  $\mathcal{G}(\mathbb{C})$  to  $G$ . But the Borel density theorem [3] implies that  $\Gamma$  is Zariski dense in  $\mathcal{G}(\mathbb{C})$ . Because of that a  $\Gamma$ -invariant vector is also  $\mathcal{G}(\mathbb{C})$ -invariant. In particular, it is  $G$ -invariant. But  $\mathcal{H}'$  is an irreducible representation of  $G$ . Hence,  $\mathcal{H}'$  is one-dimensional and  $G$  acts trivially. Thus, the same holds for  $\mathcal{H}$  and consequently for  $\mathcal{V}$ .  $\square$

The most important consequence of Theorem 4.8 is the following corollary. In fact, as the referee observed, the theory of Schwartz spaces is not necessary here, the Borel density theorem [3] can be used as in the previous proof.

**COROLLARY 4.9.** *We maintain Assumption 2.2. Then, if  $G$  has no compact components, then the trivial representation is the only finite-dimensional subrepresentation of a  $(\mathfrak{g}, K)$ -module  $\mathcal{A}(\Gamma \backslash G)$  (defined in Section 2).*

## 5. RESULTS ON $L^2(\Gamma \backslash G)$

In this section we continue with the assumptions of previous Section 4. The reader should review the second paragraph of Section 4.

We consider the usual embedding  $\mathcal{A}_{umg}(\Gamma \backslash G) \hookrightarrow \mathcal{S}(\Gamma \backslash G)'$ , given by  $\varphi \mapsto \beta_\varphi$  where  $\beta_\varphi$  is defined by  $\beta_\varphi(f) = \int_{\Gamma \backslash G} \varphi(x) f(x) dx$ , for  $f \in \mathcal{S}(\Gamma \backslash G)$ .

**LEMMA 5.1.** *We equip the space of smooth vectors  $(\mathcal{S}(\Gamma \backslash G)')^\infty$  with the usual topology (described in the proof below). Let  $n \geq -\infty$ . Then, the embedding  $\mathcal{A}_{umg,n}(\Gamma \backslash G) \hookrightarrow (\mathcal{S}(\Gamma \backslash G)')^\infty$ , given by  $\varphi \mapsto \beta_\varphi$ , is  $G$ -equivariant and continuous.*

**PROOF.** Recall that the space of smooth vectors  $(\mathcal{S}(\Gamma \backslash G)')^\infty$  in  $\mathcal{S}(\Gamma \backslash G)'$  is a complete locally convex representation of  $G$  where topology is defined by the semi-norms:

$$\alpha \mapsto \|r'(u)\alpha\|_B,$$

where  $u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  and  $B \subset \mathcal{S}(\Gamma \backslash G)$  is bounded.

Let  $u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  and let  $B \subset \mathcal{S}(\Gamma \backslash G)$  be a bounded set. Let  $\varphi \in \mathcal{A}_{umg,n}(\Gamma \backslash G)$ . Then, assuming that in the computation below  $n$  means any integer if originally we have  $n = -\infty$ ,

$$\sup_{f \in B} |r'(u)\beta_\varphi(f)| = \sup_{f \in B} \left| \int_{\Gamma \backslash G} u \cdot \varphi(x) f(x) dx \right|.$$

We note again, [8, Proposition 1.9] implies that  $\int_{\Gamma \backslash G} \|g\|_{\Gamma \backslash G}^{-d_0} dg < \infty$  for all sufficiently large  $d_0 > 0$ . Let  $M_B = \sup_{f \in B} \|f\|_{1, -d_0} < \infty$ . Hence

$$\sup_{f \in B} |r'(u)\beta_\varphi(f)| \leq \left( M_B \cdot \int_{\Gamma \backslash G} \|g\|_{\Gamma \backslash G}^{n-d_0} dg \right) \|u \cdot \varphi\|_{u, n}.$$

The continuity of the map easily follows. The map is obviously  $G$ -equivariant.  $\square$

We recall the classical and well-known argument in our settings. Let  $X \in \mathfrak{g}$ . Then, for  $F \in L^1(\Gamma \backslash G) \cap C^\infty(G)$  and  $f \in \mathcal{S}(\Gamma \backslash G)$ , we have

$$\begin{aligned} & \int_{\Gamma \backslash G} X.F(x)f(x)dx = \\ &= \int_{\Gamma \backslash G} \frac{d}{dt} \Big|_{t=0} F(x \exp(tX))f(x)dx \\ &= \int_{\Gamma \backslash G} \frac{d}{dt} \Big|_{t=0} (F(x \exp(tX))f(x \exp(tX))) dx \\ &\quad - \int_{\Gamma \backslash G} F(x) \frac{d}{dt} \Big|_{t=0} f(x \exp(tX))dx \\ &= \int_{\Gamma \backslash G} \frac{d}{dt} \Big|_{t=0} (F(x)f(x)) dx - \int_{\Gamma \backslash G} F(x) \frac{d}{dt} \Big|_{t=0} f(x \exp(tX))dx \\ &= - \int_{\Gamma \backslash G} F(x) \frac{d}{dt} \Big|_{t=0} f(x \exp(tX))dx. \end{aligned}$$

The map  $\mathfrak{g} \rightarrow \mathfrak{g}$ , given by  $X \mapsto -X$ . This extends to a  $\mathbb{C}$ -linear anti-automorphism  $u \mapsto u^\#$  of  $\mathcal{U}(\mathfrak{g}_\mathbb{C})$  which satisfies

$$(5.1) \quad \int_{\Gamma \backslash G} u.F(x) f(x)dx = \int_{\Gamma \backslash G} F(x) u^\#.f(x)dx$$

Since  $\mathcal{S}(\Gamma \backslash G)$  is a smooth representation, for each  $u \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ , the map  $f \mapsto u.f$  is continuous. So, if  $\beta \in \mathcal{S}(\Gamma \backslash G)'$ , then  $f \mapsto \beta(u.f)$  is a continuous linear functional. Hence,  $\mathcal{S}(\Gamma \backslash G)'$  becomes  $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ -module:

$$u.\beta(f) = \beta(u^\#.f), \quad f \in \mathcal{S}(\Gamma \backslash G).$$

We consider the embedding of  $L^2(\Gamma \backslash G) \hookrightarrow \mathcal{S}(\Gamma \backslash G)'$ , given by  $\varphi \mapsto \beta_\varphi$  where  $\beta_\varphi$  is defined by  $\beta_\varphi(f) = \int_{\Gamma \backslash G} \varphi(x)f(x)dx$ , for  $f \in \mathcal{S}(\Gamma \backslash G)$ . It is proved in [8, Proposition 1.17] that the map is continuous. We sketch the argument. Let  $d > 0$  be an integer such that  $\int_{\Gamma \backslash G} \|x\|_{\Gamma \backslash G}^{-2d} dx < \infty$ . Let  $B \subset \mathcal{S}(\Gamma \backslash G)$  be a bounded set. Then, we have the following:



(5.2)

$$\begin{aligned} \|\beta_\varphi\|_B &= \sup_{f \in B} \left| \int_{\Gamma \backslash G} \varphi(x) f(x) dx \right| \leq \sup_{f \in B} \int_{\Gamma \backslash G} |\varphi(x)| |f(x)| dx \\ &= \left( \int_{\Gamma \backslash G} \|x\|_{\Gamma \backslash G}^{-2d} dx \right)^{1/2} \left( \sup_{f \in B} \|f\|_{1, -d} \right) \cdot \left( \int_{\Gamma \backslash G} |\varphi(x)|^2 dx \right)^{1/2} \end{aligned}$$

which clearly proves the continuity. It is by the general theory that we have continuous map of smooth representations  $(L^2(\Gamma \backslash G))^\infty \hookrightarrow (\mathcal{S}(\Gamma \backslash G)')^\infty$ , the image is actually in  $\mathcal{A}_{umg}(\Gamma \backslash G)$  (see Lemma 3.5). But even more is true

LEMMA 5.2. *If the sequence  $(\varphi_n)_{n \geq 1}$  in  $(L^2(\Gamma \backslash G))^\infty$  converges to  $\varphi \in L^2(\Gamma \backslash G)$ , then for each  $u \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$  the sequence  $(\beta_{r(u)\varphi_n})_{n \geq 1}$  converges to  $u \cdot \beta_\varphi$  in the topology of  $\mathcal{S}(\Gamma \backslash G)'$ .*

PROOF. Arguing as in (5.2) and using (5.1), we have

$$\begin{aligned} \|\beta_{u \cdot \varphi_n} - u \cdot \beta_\varphi\|_B &\leq \\ &\left( \int_{\Gamma \backslash G} \|x\|_{\Gamma \backslash G}^{-2d} dx \right)^{1/2} \left( \sup_{f \in B} \|f\|_{u^\#, -d} \right) \cdot \left( \int_{\Gamma \backslash G} |\varphi_n(x) - \varphi(x)|^2 dx \right)^{1/2}, \end{aligned}$$

for all bounded sets  $B \subset \mathcal{S}(\Gamma \backslash G)$  and  $u \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ .  $\square$

COROLLARY 5.3.  $\beta_\varphi$  is a smooth vector in  $\mathcal{S}(\Gamma \backslash G)'$  for all  $\varphi \in L^2(\Gamma \backslash G)$ .

PROOF. We recall that the space of smooth vectors  $(\mathcal{S}(\Gamma \backslash G)')^\infty$  in  $\mathcal{S}(\Gamma \backslash G)'$  is a complete locally convex representation of  $G$  where topology is defined by the semi-norms:

$$\alpha \mapsto \|r'(u)\alpha\|_B,$$

where  $u \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$  and  $B \subset \mathcal{S}(\Gamma \backslash G)$  is bounded. Now, since by the general theory, the image of  $(L^2(\Gamma \backslash G))^\infty$  belongs to  $(\mathcal{S}(\Gamma \backslash G)')^\infty$ , we may apply Lemma 5.2 to complete the proof.  $\square$

LEMMA 5.4. *Let  $\mathcal{H}$  be a closed irreducible  $G$ -invariant subspace of  $L^2(\Gamma \backslash G)$ . Then, we have the following commutative diagram:*

$$\begin{array}{ccccc} \mathcal{H} & \xrightarrow{\subset} & L^2(\Gamma \backslash G) & \xrightarrow{\varphi \mapsto \beta_\varphi} & \mathcal{S}(\Gamma \backslash G)' \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{H}^\infty & \xrightarrow{\subset} & (L^2(\Gamma \backslash G))^\infty & \xrightarrow{\varphi \mapsto \beta_\varphi} & \mathcal{A}_{umg}(\Gamma \backslash G), \end{array}$$

where in the first row are continuous maps, and the second row is also continuous if we equip  $\mathcal{A}_{umg}(\Gamma \backslash G)$  with the topology inherited from  $(\mathcal{S}(\Gamma \backslash G)')^\infty$ .

PROOF. Above discussions imply that the first row consists of continuous maps. Next, Lemma 3.4 and Dixmier–Malliavin theorem [11] assure that the image of  $(L^2(\Gamma \backslash G))^\infty$  in  $\mathcal{A}_{umg}(\Gamma \backslash G)$ . Finally, the commutativity of the diagram is a consequence of general facts about smooth vectors.  $\square$

The following result uses deep results about globalization due to Casselman [7] and Wallach [25].

LEMMA 5.5. *Let  $\mathcal{H}$  be a closed irreducible  $G$ -invariant subspace of  $L^2(\Gamma \backslash G)$ . Then, there exists  $n_0 \in \mathbb{Z}$  such that for  $n \geq n_0$ , the map  $\varphi \mapsto \beta_\varphi$  maps  $\mathcal{H}^\infty$  equipped with its natural topology into  $\mathcal{A}_{umg,n}(\Gamma \backslash G)$  (considered as a subspace of  $\mathcal{S}(\Gamma \backslash G)'$  but equipped with its standard topology) isomorphically onto its image which is closed in  $\mathcal{A}_{umg,n}(\Gamma \backslash G)$ .*

PROOF. By Lemma 5.4, the map  $\mathcal{H}^\infty \rightarrow \mathcal{A}_{umg}(\Gamma \backslash G)$ , given by  $\varphi \mapsto \beta_\varphi$ , is continuous if we equip  $\mathcal{A}_{umg}(\Gamma \backslash G)$  with the topology inherited from  $(\mathcal{S}(\Gamma \backslash G)')^\infty$ . It is also  $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ -equivariant. Select any non-zero  $\varphi \in \mathcal{H}_K$ . Then, there exists  $n_0 \in \mathbb{Z}$  such that

$$\varphi \in \mathcal{A}_{umg,n_0}(\Gamma \backslash G) \subset \mathcal{A}_{umg,n_0+1}(\Gamma \backslash G) \subset \mathcal{A}_{umg,n_0+2}(\Gamma \backslash G) \subset \dots$$

But since  $\mathcal{A}_{umg,n}(\Gamma \backslash G)$  are smooth representations, and  $\mathcal{H}_K$  is an irreducible  $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ -module, we see that the image of  $\mathcal{H}_K$  is contained in  $\mathcal{A}_{umg,n}(\Gamma \backslash G)$  for  $n \geq n_0$ . Let  $W_n$  be the closure of the image in  $\mathcal{A}_{umg,n}(\Gamma \backslash G)$  for  $n \geq n_0$ . It is obvious that a  $(\mathfrak{g}, K)$ -module on the space of  $K$ -finite vectors in  $W_n$  is the image of  $\mathcal{H}_K$ . Therefore,  $W_n$  is irreducible. We remark that  $W_n$  being a closed subrepresentation of a representation of moderate growth  $\mathcal{A}_{umg,n}(\Gamma \backslash G)$  is also a representation of moderate growth (see Lemma 4.4; [25], Lemma 11.5.2). But  $\mathcal{H}^\infty$  is also a representation of moderate growth ([25], Lemma 11.5.1) and irreducible (see Lemma 4.2). So, the map  $\mathcal{H}_K \rightarrow (W_n)_K$ ,  $\varphi \rightarrow \beta_\varphi$ , which is an isomorphism of  $(\mathfrak{g}, K)$ -modules, extends to a continuous isomorphism of  $G$ -representations  $\mathcal{H}^\infty$  and  $W_n$  applying [25, Theorem 11.5.1]. Let us finally determine this map. This is easy since by the composition with the continuous inclusion  $\mathcal{A}_{umg,n}(\Gamma \backslash G) \hookrightarrow (\mathcal{S}(\Gamma \backslash G)')^\infty$  (see Lemma 5.1), we obtained the map that coincides on  $\mathcal{H}_K$  with the continuous map given by the second row of the diagram in Lemma 5.4. Hence, the map is  $\varphi \rightarrow \beta_\varphi$ .  $\square$

The first main result of this section is the following theorem. The reader should review the statement of Proposition 4.3.

THEOREM 5.6. *Let  $\mathcal{H}$  be a closed irreducible  $G$ -invariant subspace of  $L^2(\Gamma \backslash G)$ . Then, we have the following:*

(i) The continuous inclusion  $\mathcal{H} \hookrightarrow L^2(\Gamma \backslash G)$  gives rise to a continuous linear functional  $\lambda$  such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\varphi \mapsto \beta_\varphi} & \mathcal{S}(\Gamma \backslash G)' \\ \subset \uparrow & & \uparrow = \\ \mathcal{H}^\infty & \xrightarrow{\alpha_{\lambda, \Gamma}} & \mathcal{S}(\Gamma \backslash G)' \end{array}$$

Furthermore,  $\mathcal{H}$  is embedded into the smooth vectors of the closure  $Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))$ .

(ii) In addition, assume that Assumption 2.2 holds. Then, if  $\mathcal{H}$  is tempered, then  $d_{\mathcal{H}, \lambda} = -\infty$ .

PROOF. We prove (i). Lemma 5.4 implies that  $\mathcal{H}^\infty \subset \mathcal{A}_{umg}(\Gamma \backslash G)$ . Next, by Lemma 5.5, there exists an integer  $n \geq 1$  such that  $\mathcal{H}^\infty \subset \mathcal{A}_{umg, n}(\Gamma \backslash G)$ . This inclusion is continuous in appropriate topologies. Hence, by Lemma 4.5,  $\varphi \mapsto \varphi(1)$  is a continuous functional on  $\mathcal{H}^\infty$ . If we denote this functional by  $\lambda$ , then the commutativity of the diagram follows. The last part of (i) follows from Corollary 5.3.

We prove (ii). Because of the Assumption 2.2, we may consider the space of the closed subspace  $L^2_{cusp}(\Gamma \backslash G)$  of cuspidal functions in  $L^2(\Gamma \backslash G)$ . It is a  $G$ -subrepresentation. By a result of Wallach [23], since  $\mathcal{H}$  is a tempered closed subrepresentation of  $L^2(\Gamma \backslash G)$ ,  $\mathcal{H}$  is a closed subrepresentation of  $L^2_{cusp}(\Gamma \backslash G)$ . Then, using the notation of Section 2,  $\mathcal{H}_K \subset \mathcal{A}_{cusp}(\Gamma \backslash G)$ , and in fact

$$\mathcal{H}_K \subset \mathcal{A}_{cusp}(\Gamma \backslash G) \cap \mathcal{S}(\Gamma \backslash G),$$

since  $K$ -finite cuspidal automorphic forms are rapidly decreasing [6]. Now, arguing as in the proof of Lemma 5.5, we see that

$$\mathcal{H}^\infty \subset \mathcal{A}_{cusp}^\infty(\Gamma \backslash G) \cap \mathcal{S}(\Gamma \backslash G).$$

This implies (ii). It also shows that smooth cuspidal automorphic forms are rapidly decreasing. Which gives a different proof of the fact proved also in [16].  $\square$

We maintain the Assumption 2.2, and assume that  $G$  posses representations in discrete series ([15], [12]). Then, if  $(\pi, \mathcal{H})$  is a representation in discrete series, then there exists infinitely many congruence subgroups  $\Gamma$  such that we can embed it in  $L^2_{cusp}(\Gamma \backslash G)$  ([22], [9]). Therefore, it posses a non-zero  $\Gamma$  invariant functional such that  $d_{\pi, \lambda} = -\infty$ . On the other hand, by counting tempered representation, most of them do not appear as subrepresentations of  $L^2(\Gamma \backslash G)$  for a congruence subgroup  $\Gamma$ .

Following Harish-Chandra ([12], Section 5), we introduce the topology on  $C^\infty(G)$  by means of seminorms

$$\nu_{\Omega, u}, \quad \Omega \subset G \text{ is compact and } u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$$

defined by

$$\nu_{\Omega, u}(F) = \sup_{x \in \Omega} \|uF(x)\|.$$

We end this section with the following theorem:

**THEOREM 5.7.** *Let  $\mathcal{H}$  be a closed irreducible  $G$ -invariant subspace of  $L^2(\Gamma \backslash G)$ . Assume that the sequence of elements in  $\mathcal{H}^\infty$ ,  $(\varphi_n)_{n \geq 1}$ , converges to  $\varphi \in \mathcal{H}^\infty$  in the standard topology on  $\mathcal{H}^\infty$ . Then, it converges to  $\varphi$  in above described topology on  $C^\infty(G)$ . In addition, assume that Assumption 2.2 holds. Then, if  $\mathcal{H} \subset L^2_{cusp}(\Gamma \backslash G)$ , then  $u \cdot \varphi_n \mapsto u \cdot \varphi$  uniformly on  $G$  for all  $u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ .*

**PROOF.** The first part follows from Lemma 5.5. In addition, for the second part, we need the fact that

$$\mathcal{H}^\infty \subset \mathcal{A}_{cusp}^\infty(\Gamma \backslash G) \cap \mathcal{S}(\Gamma \backslash G)$$

established in the proof of Theorem 4.9.  $\square$

## 6. ON $\Gamma$ -INVARIANTS IN $\mathcal{S}'(G)$

Let  $\Gamma \subset G$  be a discrete subgroup. Then, the canonical map  $\mathcal{S}(G) \rightarrow \mathcal{S}(\Gamma \backslash G)$ , given by  $P_\Gamma(f)(x) = \sum_{\gamma \in \Gamma} f(\gamma x)$ , is a continuous ([8], Proposition 1.110). We sketch the argument since the details of the argument will be useful later. Let  $u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ . Let  $n \geq 1$  be an integer. Then, we have

$$\|P_\Gamma(f)\|_{u, -n} = \sup_{x \in G} \|x\|_{\Gamma \backslash G}^n |u \cdot P_\Gamma(f)(x)|.$$

Since  $u \cdot P_\Gamma(f) = P_\Gamma(u \cdot f)$  and  $\|x\|_{\Gamma \backslash G} \leq \|\gamma x\|$ , we obtain

$$\begin{aligned} \|x\|_{\Gamma \backslash G}^n |u \cdot P_\Gamma(f)(x)| &= \|x\|_{\Gamma \backslash G}^n |P_\Gamma(u \cdot f)(x)| \\ &\leq \sum_{\gamma \in \Gamma} \|\gamma x\|^n \cdot |u \cdot f(\gamma x)| \leq \|f\|_{u, -d-n} \left( \sum_{\gamma \in \Gamma} \|\gamma x\|^{-d} \right), \end{aligned}$$

where  $d > 0$  is large enough such that  $\int_G \|x\|^{-d} dx < \infty$ . But, by [8, Lemma 1.10], we have

$$(6.1) \quad M_d \stackrel{\text{def}}{=} \sup_{x \in G} \sum_{\gamma \in \Gamma} \|\gamma x\|^{-d} < \infty.$$

Thus, we obtain

$$(6.2) \quad \|P_\Gamma(f)\|_{u, -n} \leq M_d \|f\|_{u, -d-n}.$$

The group  $\Gamma$  acts on the left on  $\mathcal{S}(G)$ :  $l(\gamma)f(x) = f(\gamma^{-1}x)$ . By duality  $\Gamma$  acts on  $\mathcal{S}(G)'$ :  $l'(\gamma)\alpha(f) = \alpha(l(\gamma^{-1})f)$ .

**LEMMA 6.1.** *Let  $\gamma \in \Gamma$ . Then, the linear operator  $l(\gamma)$  (resp.,  $l'(\gamma)$ ) is continuous in the topology on  $\mathcal{S}(G)$  (resp.,  $\mathcal{S}(G)'$ ).*

PROOF. Indeed, for  $u \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ , and for an integer  $n \geq 1$ , we have the following:

$$\|l(\gamma)f\|_{u,-n} = \sup_{x \in G} \|x\|^n |u.f(\gamma^{-1}x)| = \sup_{x \in G} \|\gamma x\|^n |u.f(x)| \leq \|\gamma\|^n \|f\|_{u,-n}.$$

This proves that  $l(\gamma)$  is continuous. Next, we have

$$\|l'(\gamma)\alpha\|_B = \sup_{f \in B} |\alpha(l(\gamma^{-1})f)| = \sup_{f \in l(\gamma^{-1})B} |\alpha(f)| = \|\alpha\|_{l(\gamma^{-1})B}.$$

We remark that since  $l(\gamma^{-1})$  is continuous, the set  $l(\gamma^{-1})B$  is bounded. This proves that  $l'(\gamma)$  is also continuous.  $\square$

LEMMA 6.2. *Let  $(\mathcal{S}(G)')^\Gamma$  be the space of all  $\alpha \in \mathcal{S}(G)'$  such that  $l'(\gamma)\alpha = \alpha$  for all  $\gamma \in \Gamma$ . Then,  $(\mathcal{S}(G)')^\Gamma$  is a closed subrepresentation of  $\mathcal{S}(G)$  (where  $G$  acts by right translations).*

PROOF. Indeed, if  $(\alpha_\lambda)_{\lambda \in \Lambda}$  is a net in  $(\mathcal{S}(G)')^\Gamma$  which converges to  $\alpha \in \mathcal{S}(G)'$  i.e., the nets  $\|\alpha_\lambda - \alpha\|_B$ , where  $B \subset \mathcal{S}(G)$  is bounded, converge to zero. Then, since for  $\gamma \in \Gamma$ , the operator  $l'(\gamma)$  is continuous, we have that the net  $l'(\gamma)\alpha_\lambda$  converges to  $l'(\gamma)\alpha$ . This implies  $l'(\gamma)\alpha = \alpha$ . Hence,  $\alpha \in (\mathcal{S}(G)')^\Gamma$ .  $\square$

PROPOSITION 6.3. *We maintain Assumption 2.2. Then, the canonical map  $\mathcal{S}(\Gamma \backslash G)' \rightarrow \mathcal{S}(G)'$  is a continuous embedding with the image dense in the closed subrepresentation  $(\mathcal{S}(G)')^\Gamma$ . The space  $\mathcal{A}_{umg}(\Gamma \backslash G)$  gets identified with the Garding space of the subrepresentation  $(\mathcal{S}(G)')^\Gamma$ .*

PROOF. Since Assumption 2.2 holds, the canonical map  $\mathcal{S}(G) \rightarrow \mathcal{S}(\Gamma \backslash G)$ , given by  $P_\Gamma(f)(x) = \sum_{\gamma \in \Gamma} f(\gamma x)$ , is a continuous epimorphism ([8], Proposition 1.11, Theorem 2.2).

Next, the map  $\mathcal{S}(\Gamma \backslash G)' \rightarrow \mathcal{S}(G)'$  is an embedding. It is also obvious that its image is contained in  $(\mathcal{S}(G)')^\Gamma$ . Let us that it is continuous. Let  $B \subset \mathcal{S}(G)$  be a bounded set. Then, since  $P_\Gamma$  is continuous,  $P_\Gamma(B) \subset \mathcal{S}(\Gamma \backslash G)$  is a bounded set. Then, we have

$$\|\alpha \circ P_\Gamma\|_B = \sup_{f \in B} |\alpha(P_\Gamma(f))| = \|\alpha\|_{P_\Gamma(B)}.$$

This proves the continuity of the map.

The space  $\mathcal{A}_{umg}(\Gamma \backslash G)$  is the Garding space of  $\mathcal{S}(\Gamma \backslash G)'$ . Thus its image is contained in the Garding space of the subrepresentation  $(\mathcal{S}(G)')^\Gamma$ . But the Garding space of  $(\mathcal{S}(G)')^\Gamma$  is contained in the Garding space of  $\mathcal{S}(G)'$ . This space is  $\mathcal{A}_{umg}(G)$  (see Lemma 3.4). So let  $\alpha$  belong to the Garding space of  $(\mathcal{S}(G)')^\Gamma$ . Then, by what we have just said,  $\alpha$  is represented by a function  $\varphi \in \mathcal{A}_{umg}(G)$ :

$$\alpha(f) = \int_G \varphi(x)f(x)dx, \quad f \in \mathcal{S}(G).$$

Since  $\alpha$  is  $\Gamma$ -invariant, we have that  $\varphi(\gamma x) = \varphi(x)$ ,  $\gamma \in \Gamma$ ,  $x \in G$ . Now,  $\varphi \in \mathcal{A}_{umg}(\Gamma \backslash G)$ .

Finally, since  $\mathcal{A}_{umg}(\Gamma \backslash G)$  maps onto the Garding space of  $(\mathcal{S}(G)')^\Gamma$ , the space  $\mathcal{S}(\Gamma \backslash G)' \rightarrow \mathcal{S}(G)'$  maps onto a dense subspace of  $(\mathcal{S}(G)')^\Gamma$ .  $\square$

In the following proposition we give the most general construction of classical Poincaré series. In part, it generalizes [20, Theorem 6.4].

**PROPOSITION 6.4.** *Assume that  $\Gamma \subset G$  is a discrete subgroup. Let  $\varphi \in L^1(G)$ . Then the series  $\sum_{\gamma \in \Gamma} l(\gamma)\varphi$  converges absolutely in  $\mathcal{S}(G)'$  to an element of  $(\mathcal{S}(G)')^\Gamma$  (which is in the image of  $\mathcal{S}(\Gamma \backslash G)'$ ). Moreover, if  $\varphi$  is a smooth vector in the Banach representation  $L^1(G)$  under right-translations, then  $\sum_{\gamma \in \Gamma} l(\gamma)\varphi \in \mathcal{A}_{umg}(\Gamma \backslash G)$ .*

**PROOF.** Let  $B \subset \mathcal{S}(G)$  be a bounded set. We need to show that

$$\sum_{\gamma \in \Gamma} \|l(\gamma)\varphi\|_B < \infty.$$

Since  $\mathcal{S}(\Gamma \backslash G)'$  is complete, this proves the absolute convergence.

By definition, we have

$$\begin{aligned} \|l(\gamma)\varphi\|_B &= \sup_{f \in B} \left| \int_G \varphi(\gamma^{-1}x)f(x)dx \right| \leq \sup_{f \in B} \int_G |\varphi(\gamma^{-1}x)| |f(x)| dx \\ &= \sup_{f \in B} \int_G |\varphi(x)| |f(\gamma x)| dx \\ &\leq \left( \sup_{f \in B} \|f\|_{1,-d} \right) \cdot \int_G |\varphi(x)| \|\gamma x\|^{-d} dx. \end{aligned}$$

So, the series is

$$\leq \left( \sup_{f \in B} \|f\|_{1,-d} \right) \cdot M_d \int_G |\varphi(x)| dx < \infty,$$

where the number  $M_d$  is defined by (6.1).

The distribution in question is in fact the integration against the classical Poincaré series  $P_\Gamma(\varphi) \in L^1(\Gamma \backslash G)$ :

$$\begin{aligned} \int_G P_\Gamma(\varphi)(x)f(x)dx &= \sum_{\gamma \in \Gamma} \int_G \varphi(\gamma x)f(x)dx \\ &= \sum_{\gamma \in \Gamma} \int_G \varphi(\gamma^{-1}x)f(x)dx = \sum_{\gamma \in \Gamma} \int_G l(\gamma)\varphi(x)f(x)dx, \end{aligned}$$

for  $f \in \mathcal{S}(G)$ .

The space of smooth vectors in  $L^1(G)$ , where  $G$  acts by right translations  $r$ , is a Fréchet space under seminorms ([25], Lemma 11.5.1):

$$\|r(u)f\|_1 = \int_G |r(u)f(x)| dx, \quad u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}}).$$

Then, by Dixmier–Malliavin theorem [11], for smooth vector  $\varphi$  there exists, smooth vectors  $\varphi, \dots, \varphi_l$ , and  $\alpha_1, \dots, \alpha_l \in C_c^\infty(G)$  such that

$$\varphi = \sum_{i=1}^l r(\alpha_i)\varphi_i = \sum_{i=1}^l \varphi_i \star \alpha_i^\vee,$$

where as usual  $\alpha_i^\vee(x) = \alpha_i(x^{-1})$ . By the standard measure–theoretic arguments, we have

$$P_\Gamma(\varphi) = \sum_{i=1}^l P_\Gamma(\varphi_i) \star \alpha_i^\vee.$$

Now, we apply Lemma 3.5. □

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## Neki rezultati o Schwartzovom prostoru za $\Gamma \backslash G$

*Goran Muić*

SAŽETAK. Neka je  $G$  povezana poluprosta Liejeva grupa s konačnim centrom. Neka je  $\Gamma \subset G$  diskretna podgrupa. U ovom radu proučavamo dopustive zatvorene podreprezentacije prostora distribucija  $\mathcal{S}(\Gamma \backslash G)'$ , koji je definirao Casselman, te njihov odnos s prostorom automorfnihi formi na  $\Gamma \backslash G$  kada je  $\Gamma$  kongruencijska podgrupa.

Department of Mathematics  
 Faculty of Science  
 University of Zagreb  
 Bijenička 30  
 10000 Zagreb, Croatia  
*E-mail:* [gmuic@math.hr](mailto:gmuic@math.hr)

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