

**A NOTE ON THE AFFINE VERTEX ALGEBRA  
ASSOCIATED TO  $\mathfrak{gl}(1|1)$  AT THE CRITICAL LEVEL AND  
ITS GENERALIZATIONS**

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*Dedicated to the memory of Sibe Mardešić*

ABSTRACT. In this note we present an explicit realization of the affine vertex algebra  $V^{cri}(\mathfrak{gl}(1|1))$  inside of the tensor product  $F \otimes M$  where  $F$  is a fermionic vertex algebra and  $M$  is a commutative vertex algebra. This immediately gives an alternative description of the center of  $V^{cri}(\mathfrak{gl}(1|1))$  as a subalgebra  $M_0$  of  $M$ . We reconstruct the Molev-Mukhin formula for the Hilbert-Poincaré series of the center of  $V^{cri}(\mathfrak{gl}(1|1))$ . Moreover, we construct a family of irreducible  $V^{cri}(\mathfrak{gl}(1|1))$ -modules realized on  $F$  and parameterized by  $\chi^+, \chi^- \in \mathbb{C}((z))$ . We propose a generalization of  $V^{cri}(\mathfrak{gl}(1|1))$  as a critical level version of the super  $\mathcal{W}_{1+\infty}$  vertex algebra.

1. INTRODUCTION

Let  $\mathfrak{g}$  be the Lie superalgebra, and  $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}K$  the associated affine Lie superalgebra. The representation theory of affine Lie superalgebras at certain level  $k$  is closely related with the representation theory of the universal affine vertex algebra  $V^k(\mathfrak{g})$  associated to  $\mathfrak{g}$ .

For a given vertex algebra  $V$ , it is important problem to describe the structure of the center of  $V$ . The center is defined as the following vertex subalgebra of  $V$ :  $\mathfrak{Z}(V) = \{v \in V \mid [Y(v, z), Y(w, z)] = 0 \ \forall w \in V\}$ .

In the case of affine vertex algebras, the center is non-trivial only in the case of critical level. We denote the universal affine vertex algebra at the critical level by  $V^{cri}(\mathfrak{g})$  and its center by  $\mathfrak{Z}(\hat{\mathfrak{g}})$ . When  $\mathfrak{g}$  is a simple Lie algebra, the center of  $V^{cri}(\mathfrak{g})$ , called the Feigin-Frenkel center, is finitely-generated commutative vertex algebra (see [8], [9], [6]).

But in the case of Lie superalgebras,  $\mathfrak{Z}(\hat{\mathfrak{g}})$  can be infinitely-generated and it is not completely understood yet.

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In the recent paper [15], A. Molev and E. Mukhin determined the structure of  $\mathfrak{Z}(\mathfrak{g})$  in the case  $\mathfrak{g} = \mathfrak{gl}(1|1)$ , and presented conjectures on the structure of the center for  $\mathfrak{g} = \mathfrak{gl}(m|n)$ . In the case  $\mathfrak{g} = \mathfrak{gl}(1|1)$ , they constructed a family of central, Sugawara elements (see also [16]) and proved that these central elements generate  $\mathfrak{Z}(\mathfrak{g})$  (see [15, Theorem 2.1]).

The aim of this note is to present an alternative construction of the center of  $V^{cri}(\mathfrak{g})$  in the case  $\mathfrak{g} = \mathfrak{gl}(1|1)$  by using an explicit free-field realization of  $V^{cri}(\mathfrak{g})$ . Let us describe our result. We consider the commutative vertex algebra

$$M = \mathbb{C}[a^\pm(m + 1/2) \mid m \in \mathbb{Z}_{<0}]$$

which is uniquely determined by the following commutative fields:

$$a^\pm(z) = \sum_{m < 0} a^\pm(m + 1/2) z^{-m-1}.$$

We prove:

**THEOREM 1.1.**  *$\mathfrak{Z}(\widehat{\mathfrak{g}})$  is isomorphic to a vertex subalgebra of  $M$  generated by the following fields*

$$a^+(z) \partial^k a^-(z), \quad k \geq 0.$$

*In particular, the Hilbert Poincaré series of  $\mathfrak{Z}(\widehat{\mathfrak{g}})$  coincides with the  $q$ -character of the simple vertex algebra  $\mathcal{W}_{1+\infty, c}$  with  $c = -1$ .*

In Section 4, we explicitly construct a large family of the irreducible  $V^{cri}(\mathfrak{gl}(1|1))$  modules parametrized by  $\chi^\pm(z) \in \mathbb{C}((z))$ . It is interesting that these modules are realized on the Clifford vertex algebra  $F$ .

It will be interesting to study generalization of our description of  $\mathfrak{Z}(\widehat{\mathfrak{g}})$  for  $\mathfrak{g} = \mathfrak{gl}(1|1)$  in various directions. Of course, the most important open question is the determination of the center in the case  $\mathfrak{g} = \mathfrak{gl}(m|n)$ . It seems that the general case is much more complicated. But in the present paper we propose one different generalization. In Section 5 we introduce a vertex algebra  $V_n$  having large center, such that  $V_1 = V^{cri}(\mathfrak{gl}(1|1))$ . The construction of  $V_n$  is a critical level version of the super  $\mathcal{W}_{1+\infty}$ -algebra, constructed and analysed by T. Creutzig and A. Linshaw in [5].

In our forthcoming papers we shall study these generalizations in more details.

## 2. CERTAIN VERTEX ALGEBRAS

In the paper we assume that the reader is familiar with the basic theory of vertex algebras. In this section we briefly review results on vertex algebras which we use in the paper.

2.1. *The affine vertex algebra  $V^{cri}(\mathfrak{gl}(m|n))$ .* The vertex algebras associated to affine Lie superalgebras are most important examples of vertex algebras (cf. [9], [11], [13]).

We recall the definition of universal affine vertex algebra  $V^{cri}(\mathfrak{gl}(m|n))$  following notations in [15].

For  $1 \leq j \leq m+n$  we define  $\bar{j} \in \mathbb{Z}_2$ :

$$\bar{j} = \bar{0} \quad \text{for } 1 \leq j \leq m, \quad \bar{j} = 1 \quad \text{for } m+1 \leq j \leq m+n.$$

Let  $\mathfrak{g}$  be Lie superalgebra  $\mathfrak{gl}(m|n)$  with standard basis  $E_{i,j}$ ,  $1 \leq i, j \leq m+n$ . The affine Lie superalgebra  $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}K$  with commutation relations

$$(2.1) \quad [E_{i,j}(r), E_{k,\ell}(s)] = \delta_{k,j} E_{i,\ell}(r+s) - \delta_{i,\ell} E_{k,j}(r+s) (-1)^{(\bar{i}+\bar{j})(\bar{k}+\bar{\ell})} \\ + K \left( (n-m) \delta_{k,j} \delta_{i,\ell} (-1)^{\bar{i}} + \delta_{i,j} \delta_{k,\ell} (-1)^{\bar{i}+\bar{k}} \right) r \delta_{r+s,0}$$

where  $K$  is even central element, and  $x(n) = x \otimes t^n$  for  $x \in \mathfrak{g}$ ,  $r, s \in \mathbb{Z}$ .

Let  $\mathbb{C}v$  be the 1-dimensional  $P = \mathfrak{g} \otimes \mathbb{C}[t] + \mathbb{C}K$ -module such that

$$Kv = v, \quad (\mathfrak{g} \otimes \mathbb{C}[t])v = 0.$$

Then the  $\hat{\mathfrak{g}}$ -module

$$V^{cri}(\mathfrak{g}) = U(\hat{\mathfrak{g}}) \otimes_{U(P)} \mathbb{C}v$$

has the vertex algebra structure which is uniquely generated by the fields

$$x(z) = \sum_{r \in \mathbb{Z}} x(r) z^{-r-1} \quad (x \in \mathfrak{g}).$$

The vertex algebra  $V^{cri}(\mathfrak{g})$  is called the universal affine vertex algebra at the critical level.

The center of  $V^{cri}(\mathfrak{g})$  has the following description

$$\mathfrak{Z}(\hat{\mathfrak{g}}) = \{w \in V^{cri}(\mathfrak{g}) \mid (\mathfrak{g} \otimes \mathbb{C}[t]).w = 0\}.$$

REMARK 2.1. The standard choice of central element in  $\hat{\mathfrak{g}}$  is  $C = (n-m)K$  and it acts on  $V^{cri}(\mathfrak{g})$  as  $-h^\vee \text{Id}$ . Moreover, if  $m \neq n$ , then the standard definition of the affine vertex algebra  $V^{-h^\vee}(\mathfrak{g})$  at the critical level coincides with  $V^{cri}(\mathfrak{g})$ . But the case  $m = n$  is different. It was observed in [15] and [16] that then one needs to take central element  $K$  acting as identity on  $V^{cri}(\mathfrak{g})$ .

2.2. *Clifford vertex algebras.* The Clifford algebra  $CL_n$  is a complex associative algebra generated by

$$\Psi_i^\pm(r), \quad r \in \frac{1}{2} + \mathbb{Z}, \quad 1 \leq i \leq n,$$

and relations

$$\{\Psi_i^\pm(r), \Psi_j^\mp(s)\} = \delta_{i,j} \delta_{r+s,0}; \quad \{\Psi_i^\pm(r), \Psi_j^\pm(s)\} = 0,$$

where  $r, s \in \frac{1}{2} + \mathbb{Z}$ .

Let  $F^{(n)}$  be the irreducible  $CL$ -module generated by the cyclic vector  $\mathbf{1}$  such that

$$\Psi_i^\pm(r)\mathbf{1} = 0 \quad \text{for } r > 0.$$

As a vector space,  $F^{(n)}$  is isomorphic to the exterior algebra

$$F^{(n)} \cong \bigwedge (\Psi_i^\pm(-m - \frac{1}{2}) \mid m \in \mathbb{Z}_{\geq 0}, i = 1, \dots, n).$$

Define the following fields on  $F^{(n)}$

$$\Psi_i^+(z) = \sum_{m \in \mathbb{Z}} \Psi_i^+(m + \frac{1}{2})z^{-m-1}, \quad \Psi_i^-(z) = \sum_{m \in \mathbb{Z}} \Psi_i^-(m + \frac{1}{2})z^{-m-1}.$$

The fields  $\Psi_i^+(z)$  and  $\Psi_i^-(z)$ ,  $i = 1, \dots, n$  generate on  $F^{(n)}$  the unique structure of a simple vertex algebra (cf. [9], [11]).

It is well-known that the vertex subalgebra of  $F^{(n)}$  generated by the vectors

$$\{e_{i,j} = \Psi_i^+(-\frac{1}{2})\Psi_j^-(-\frac{1}{2})\mathbf{1} \mid i, j = 1, \dots, n\}$$

is isomorphic to the simple affine vertex algebra  $V_1(\mathfrak{gl}_n)$  at level 1. This implies that the Lie algebra  $\mathfrak{gl}_n$  acts on  $F^{(n)}$  by derivations. The fixed point subalgebra

$$(F^{(n)})^{\mathfrak{gl}_n}$$

is isomorphic to the simple vertex algebra  $\mathcal{W}_{1+\infty}$  at central charge  $c = n$  (see [10] for details on this construction).

Let us consider the case  $n = 1$ . Set  $F := F^{(1)}$ ,  $\Psi^\pm(z) = \Psi_1^\pm$ . A basis of  $F$  is given by

$$(2.2) \quad \Psi^+(-n_1 - \frac{1}{2}) \cdots \Psi^+(-n_r - \frac{1}{2})\Psi^-(-k_1 - \frac{1}{2}) \cdots \Psi^-(-k_s - \frac{1}{2})\mathbf{1},$$

where  $n_i, k_i \in \mathbb{Z}_{\geq 0}$ ,  $n_1 > n_2 > \cdots > n_r$ ,  $k_1 > k_2 > \cdots > k_s$ .

Let  $\alpha := \Psi^+(-1/2)\Psi^-(-1/2)\mathbf{1}$ . Then the operator  $\alpha(0)$  defines on  $F$  the following  $\mathbb{Z}$ -gradation

$$F = \bigoplus_{\ell \in \mathbb{Z}} F_\ell,$$

where

$$F_\ell = \{v \in F \mid \alpha(0)v = \ell v\}.$$

A basis of  $F_\ell$  is given by vectors (2.2) such that  $\ell = r - s$ .

Recall here that by using a boson-fermion correspondence, the fermionic vertex algebra  $F$  can be realized as the lattice vertex algebra  $V_L = M(1) \otimes \mathbb{C}[L]$  where

$$L = \mathbb{Z}\alpha, \quad \langle \alpha, \alpha \rangle = 1,$$

$M(1)$  is the Heisenberg vertex algebra generated by  $\alpha$  and  $\mathbb{C}[L]$  is the group algebra of  $L$ . In particular, as  $M(1)$ -modules  $F_\ell \cong M(1).e^{\ell\alpha}$ .

2.3. *Weyl vertex algebra*  $W^{(n)}$ . The Weyl vertex algebra  $W^{(n)}$  is generated by the fields

$$\gamma_i^\pm(z) = \sum_{m \in \mathbb{Z}} \gamma_i^\pm(m + \frac{1}{2}) z^{-m-1},$$

whose components satisfy the commutation relation for Weyl algebra

$$[\gamma_i^+(r), \gamma_j^-(s)] = \delta_{i,j} \delta_{r+s,0}, \quad [\gamma_i^\pm(r), \gamma_j^\pm(s)] = 0 \quad (r, s \in \frac{1}{2} + \mathbb{Z}, i, j = 1, \dots, n).$$

Choose the following Virasoro vector of central charge  $c = -n$ :

$$\omega = \frac{1}{2} \sum_{i=1}^n (\gamma_i^-(-\frac{3}{2}) \gamma_i^+(-\frac{1}{2}) - \gamma_i^+(-\frac{3}{2}) \gamma_i^-(-\frac{1}{2})) \mathbf{1}.$$

The associated Virasoro field is given by

$$L(z) = \frac{1}{2} \sum_i^n (:\partial \gamma_i^-(z) \gamma_i^+(z): - :\partial \gamma_i^+(z) \gamma_i^-(z):) = \sum_{m=-\infty}^{\infty} L(m) z^{-m-2}.$$

The vertex subalgebra of  $W^{(n)}$  generated by the vectors

$$\{e_{i,j} = -\gamma_i^+(-\frac{1}{2}) \gamma_j^-(-\frac{1}{2}) \mathbf{1} \mid i, j = 1, \dots, n\}$$

is isomorphic to the simple affine vertex algebra  $V_{-1}(\mathfrak{gl}_n)$  at level  $-1$ . This realization was found by A. Feingold and I. Frenkel in [7], and the simplicity of the realization (for  $n \geq 3$ ) was proved by the author and O. Perše in [3].

This again implies that the Lie algebra  $\mathfrak{gl}_n$  acts on  $W^{(n)}$  by derivations. The fixed point subalgebra

$$(W^{(n)})^{\mathfrak{gl}_n}$$

is isomorphic to the simple vertex algebra  $\mathcal{W}_{1+\infty}$  at central charge  $c = -n$  (for details see [12]). We denote this vertex algebra by  $\mathcal{W}_{1+\infty, -n}$ .

The case  $n = 1$  was studied by W. Wang in [17]. It was proved in [17, Section 5] that the  $q$ -character (i.e., the Hilbert–Poincaré series) of  $\mathcal{W}_{1+\infty, -1}$  with respect to  $L(0)$  is given by

$$\begin{aligned} \text{ch}[\mathcal{W}_{1+\infty, -1}](q) &= \text{tr } q^{L(0)}|_{\mathcal{W}_{1+\infty, -1}} \\ (2.3) \quad &= \text{Res}_z z^{-1} \frac{1}{\prod_{k \geq 1} (1 - q^{k-1/2} z)(1 - q^{k-1/2} z^{-1})} \\ &= \frac{1}{(q)_\infty^2} \sum_{n \in \mathbb{Z}} \text{sign}(n) q^{2n^2+n} = \frac{1}{(q)_\infty^2} \sum_{k=0}^{\infty} (-1)^k q^{\frac{k^2+k}{2}}. \end{aligned}$$

By using [15, Proposition 3.3] or [4, Example 7.1] we see that (2.3) can be written as

$$(2.4) \quad \frac{1}{(q)_\infty^2} \sum_{n \in \mathbb{Z}} \text{sign}(n) q^{2n^2+n} = \frac{1}{(q)_\infty} \sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q)_k^2}.$$

In the above formulas we use standard notations:

$$(q)_\infty = \prod_{i \geq 1} (1 - q^i), \quad (q)_k = \prod_{i=1}^k (1 - q^i).$$

REMARK 2.2. We should mention that the identity (2.4) was first proved by Ramanujan. Let  $\mathcal{W}(2, 2p - 1)$  denotes the singlet vertex algebra with central charge  $c_{1,p} = 1 - 6(p-1)^2/p$  (cf. [2]). Note that  $\mathcal{W}_{1+\infty, -1}$  is isomorphic to the tensor product of a rank-one Heisenberg vertex algebra and the singlet vertex algebra  $\mathcal{W}(2, 2p - 1)$  for  $p = 2$ . So the combinatorial identity (2.4) is essentially the  $q$ -series identity of the vacuum character for  $\mathcal{W}(2, 2p - 1)$  for  $p = 2$ .

An interesting generalization of the identity (2.4) for  $\mathcal{W}(2, 2p - 1)$ -characters for  $p \geq 3$  was proved by K. Bringmann and A. Milas in [4, Proposition 7.2].

2.4. *Commutative vertex algebra  $M^{(n)}$ .* Let

$$M^{(n)} = \mathbb{C}[a_i^+(m + \frac{1}{2}), a_i^-(m + \frac{1}{2}) \mid m \in \mathbb{Z}_{<0}, i = 1, \dots, n]$$

be the commutative vertex algebra generated by the fields

$$a_i^\pm(z) = \sum_{m \in \mathbb{Z}_{<0}} a_i^\pm(m + \frac{1}{2}) z^{-m-1}.$$

The  $\frac{1}{2}\mathbb{Z}_{\geq 0}$ -grading on  $M^{(n)}$  is given by the operator  $d \in \text{End}(M^{(n)})$  which is uniquely determined by the following formula:

$$[d, a_i^\pm(m + \frac{1}{2})] = -(m + \frac{1}{2})a_i^\pm(m + \frac{1}{2}).$$

The Lie algebra  $\mathfrak{gl}_n$  acts on the vertex algebra  $M^{(n)}$  by derivations. This action is uniquely determined by the following formula:

$$e_{i,j} \cdot a_k^+(r) = \delta_{j,k} a_i^+(r), \quad e_{i,j} \cdot a_k^-(r) = -\delta_{i,k} a_j^-(r),$$

for  $1 \leq i, j \leq n$ ,  $r \in \frac{1}{2} + \mathbb{Z}$ .

Moreover  $(M^{(n)})^{\mathfrak{gl}_n}$  is a subalgebra of  $M^{(n)}$ .  $(M^{(n)})^{\mathfrak{gl}_n}$  is  $d$ -invariant, and the operator  $d$  defines  $\mathbb{Z}_{\geq 0}$ -grading in  $(M^{(n)})^{\mathfrak{gl}_n}$ .

In the case  $n = 1$  we set  $a^\pm(z) := a_1^\pm(z)$ . The vertex algebra  $M = M^{(1)}$  has the following  $\mathbb{Z}$ -gradation

$$M = \bigoplus_{\ell \in \mathbb{Z}} M_\ell,$$

where  $M_\ell$  is a linear span of vectors

$$(2.5) \quad a^+(-n_1 - \frac{1}{2}) \cdots a^+(-n_r - \frac{1}{2}) a^-(-k_1 - \frac{1}{2}) \cdots a^-(-k_s - \frac{1}{2}) \mathbf{1}$$

such that  $n_i, k_i \in \mathbb{Z}_{\geq 0}$ ,  $n_1 \geq n_2 \geq \dots \geq n_r$ ,  $k_1 \geq k_2 \geq \dots \geq k_s$  and  $\ell = r - s$ .

It is easy to see that  $M_0$  is a vertex subalgebra of  $M$  and that  $M_0$  is strongly generated by vectors

$$(2.6) \quad a^+(-1/2)a^-(-m-1/2)\mathbf{1} \quad (m \in \mathbb{Z}_{\geq 0}).$$

REMARK 2.3. By using construction from [14] one can show a more general result that  $(M^{(n)})^{\mathfrak{gl}_n}$  is strongly generated by vectors

$$\sum_{i=1}^n a_i^+(-1/2)a_i^-(-m-1/2)\mathbf{1} \quad (m \in \mathbb{Z}_{\geq 0}).$$

$M_0$  is a graded commutative vertex algebra with the following  $\mathbb{Z}_{\geq 0}$ -gradation:

$$M_0 = \bigoplus_{m=0}^{\infty} M_0(m) \quad M_0(m) = \{v \in M_0(m) \mid dv = mv\}.$$

The Hilbert–Poincaré series of  $M_0$  is

$$\begin{aligned} \sum_{m=0}^{\infty} \dim M_0(m)q^m &= \operatorname{Res}_z z^{-1} \frac{1}{\prod_{k \geq 1} (1 - q^{k-1/2}z)(1 - q^{k-1/2}z^{-1})} \\ &= \frac{1}{(q)_{\infty}} \sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q)_k^2}. \end{aligned}$$

Therefore  $M_0$  is a graded commutative vertex algebra whose Hilbert–Poincaré series is given by (2.3).

Let  $\chi^{\pm}(z) = \sum_{n \in \mathbb{Z}} \chi_n^{\pm} z^{-n-1} \in \mathbb{C}((z))$ . Let  $M(\chi^+, \chi^-)$  denotes the 1-dimensional irreducible  $M$ -module with the property that every element  $a^{\pm}(n)$  acts on  $M(\chi^+, \chi^-)$  as multiplication by  $\chi_n^{\pm} \in \mathbb{C}$ .

### 3. THE MAIN RESULT

In this section we shall present an explicit realization of the vertex algebra  $V^{cri}(\mathfrak{gl}(1|1))$ . Our construction is similar to the realization of affine  $\mathfrak{gl}(1|1)$  from [11]. Main difference is that we replace Weyl vertex algebra (also called the symplectic boson vertex algebra) by the commutative vertex algebra  $M$ .

We consider the following subalgebra of  $F \otimes M$ :

$$(3.1) \quad V = (F \otimes M)_0 = \bigoplus_{\ell \in \mathbb{Z}} F_{\ell} \otimes M_{-\ell}.$$

Let us denote by  $\mathfrak{Z}(V)$  the center of the vertex algebra  $V$ .

LEMMA 3.1.  $\mathfrak{Z}(V)$  is isomorphic to the vertex algebra  $M_0$ .

PROOF. First we notice that  $M_0 \subset \mathfrak{Z}(V)$ . Since  $\alpha \in V$ , we have that the Heisenberg vertex algebra  $M(1)$  is a subalgebra of  $V$ . By using the fact that each  $F_\ell$  is irreducible  $M(1)$ -module with highest weight vector  $e^{\ell\alpha}$  we see that  $\mathfrak{Z}(V)$  is contained in  $M_0$ . The proof follows.  $\square$

We have the following result:

THEOREM 3.2.

(1) The vertex algebra  $V$  is strongly generated by

$$\{E_{i,j} \mid i, j = 1, 2\}$$

where

$$(3.2) \quad E_{1,1} = \alpha$$

$$(3.3) \quad E_{1,2} = \Psi^+(-\frac{1}{2})a^-(-\frac{1}{2})\mathbf{1}$$

$$(3.4) \quad E_{2,1} = \Psi^-(-\frac{1}{2})a^+(-\frac{1}{2})\mathbf{1}$$

$$(3.5) \quad E_{2,2} = a^+(-\frac{1}{2})a^-(-\frac{1}{2})\mathbf{1} - \alpha$$

(2) The vertex algebra  $V$  is isomorphic to  $V^{cri}(\mathfrak{gl}(1|1))$ .

(3) The center of  $V$  is isomorphic to the commutative vertex algebra  $M_0$ . Its Hilbert-Poincare series is given by (2.4).

PROOF. The proof of assertion (1) is similar to [11]. Since  $n \geq 0$  we have

$$E_{1,1}(n)E_{1,2} = \delta_{n,0}E_{1,2}, \quad E_{1,1}(n)E_{2,1} = -\delta_{n,0}E_{2,1}, \quad E_{1,1}(n)E_{2,2} = -\delta_{n,1},$$

$$E_{2,2}(n)E_{1,2} = -\delta_{n,0}E_{1,2}, \quad E_{2,2}(n)E_{2,1} = \delta_{n,0}E_{2,1},$$

$$E_{1,2}(n)E_{2,1} = \delta_{n,0}(E_{1,1} + E_{2,2})$$

the commutator formula for vertex algebras directly implies that the components of the fields

$$E_{i,j}(z) = Y(E_{i,j}, z) = \sum_{n \in \mathbb{Z}} E_{i,j}(n)z^{-n-1} \quad i, j \in \{1, 2\}$$

satisfy the commutation relations (2.1) for the affine  $\mathfrak{gl}(1|1)$  at the critical level. This gives a vertex algebra homomorphism  $\Phi : V^{cri}(\mathfrak{gl}(1|1)) \rightarrow V$ .

Let  $U = \text{Im}(\Phi) = V^{cri}(\mathfrak{gl}(1|1)) \cdot \mathbf{1}$ , i.e.,  $U$  is the vertex subalgebra of  $V$  generated by elements (3.2)–(3.5). Consider

$$\widehat{U} = \bigoplus_{s \in \mathbb{Z}} V^{cri}(\mathfrak{gl}(1|1)) \cdot e^{s\alpha}.$$

As in [11] we see that each  $V^{cri}(\mathfrak{gl}(1|1)) \cdot e^{s\alpha}$  is obtained from  $U$  by applying a spectral flow automorphism for  $\widehat{\mathfrak{gl}(1|1)}$ , and therefore

$$(V^{cri}(\mathfrak{gl}(1|1)) \cdot e^{s\alpha}) \cdot (V^{cri}(\mathfrak{gl}(1|1)) \cdot e^{r\alpha}) \subset V^{cri}(\mathfrak{gl}(1|1)) \cdot e^{(r+s)\alpha} \quad (r, s \in \mathbb{Z}).$$

This implies that  $\widehat{U}$  is a vertex subalgebra of  $F \otimes M$ . Since

$$a^+(-\frac{1}{2})\mathbf{1} = E_{2,1}(0)\Psi^+(-\frac{1}{2})\mathbf{1}, \quad a^-(-\frac{1}{2})\mathbf{1} = E_{1,2}(0)\Psi^-(-\frac{1}{2})\mathbf{1}$$

we get

$$e^{\pm\alpha}, a^{\pm}(-\frac{1}{2})\mathbf{1} \in V^{cri}(\mathfrak{gl}(1|1)). e^{\pm\alpha} \subset \widehat{U}.$$

So all generators of  $F \otimes M$  belong to the vertex subalgebra  $\widehat{U}$ . Therefore  $F \otimes M = \widehat{U}$ , which proves that  $U = V$ . This proves (1).

This gives a surjective vertex algebra homomorphism

$$\Phi : V^{cri}(\mathfrak{gl}(1|1)) \rightarrow V.$$

The injectivity can be proved easily by using the PBW basis of  $V^{cri}(\mathfrak{gl}(1|1))$ , which proves (2). The assertion (3) follows from Lemma 3.1. The proof follows.  $\square$

REMARK 3.3. It is interesting that the Hilbert-Poincare series of the center of  $V^{cri}(\mathfrak{gl}(1|1))$  coincides with  $\text{ch}[\mathcal{W}_{1+\infty, -1}](q)$ . It is a natural question to see if there is a similar interpretation of the Hilbert-Poincare series for the center of  $V^{cri}(\mathfrak{gl}(n|n))$  for general  $n$ . A different type of generalization will be proposed in Section 5.

#### 4. MODULES FOR $V^{crit}(\mathfrak{gl}(1|1))$ OF THE WHITTAKER TYPE

Affine vertex algebras at critical levels also contain modules of the Whittaker type. In the case of the affine Lie algebra  $A_1^{(1)}$  such modules were constructed by the author, R. Lu and K. Zhao in [1]. In this section we show that our realization from Section 3 can be applied in a construction of the Whittaker type of modules for  $V^{cri}(\mathfrak{gl}(1|1))$ .

THEOREM 4.1. *For every  $\chi^+, \chi^- \in \mathbb{C}((z))$ ,  $\chi^{\pm} \neq 0$ ,  $F(\chi^+, \chi^-) := F \otimes M(\chi^+, \chi^-)$  is an irreducible  $V^{cri}(\mathfrak{gl}(1|1))$ -module.*

PROOF. Since  $M(\chi^+, \chi^-)$  is a  $M$ -module, we have that  $F \otimes M(\chi^+, \chi^-)$  is a  $F \otimes M$ -module, and therefore  $F(\chi^+, \chi^-)$  is a  $(F \otimes M)_0 = V^{cri}(\mathfrak{gl}(1|1))$ -module. Since  $M(\chi^+, \chi^-)$  is 1-dimensional, it follows that as a vector space  $F(\chi^+, \chi^-)$  is isomorphic to  $F$ .

We first prove that  $F(\chi^+, \chi^-)$  is a cyclic  $V^{cri}(\mathfrak{gl}(1|1))$ -module and  $F(\chi^+, \chi^-) = V^{cri}(\mathfrak{gl}(1|1)).\mathbf{1}$ . Since  $\chi^{\pm} \neq 0$ , there are  $p^{\pm} \in \mathbb{Z}$  such that

$$\chi^{\pm}(z) = \sum_{j \geq -p^{\pm}} \chi_{-j}^{\pm} z^{j-1} \quad \chi_{p^{\pm}}^{\pm} \neq 0.$$

The action of  $E_{1,2}(n)$  and  $E_{2,1}(n)$  are given by the following formulas:

$$(4.1) \quad E_{1,2}(n) = \sum_{j \geq -p^-} \chi_{-j}^- \Psi^+(n - 1/2 + j),$$

$$(4.2) \quad E_{2,1}(n) = \sum_{j \geq -p^+} \chi_{-j}^+ \Psi^-(n - 1/2 + j).$$

Relations (4.1) -(4.2) imply that for  $m \in \mathbb{Z}_{>0}$

$$\begin{aligned} E_{1,2}(p^- - m + 1) \cdots E_{1,2}(p^-) \mathbf{1} &= (\chi_{p^-}^-)^m \Psi^+(-m - \frac{1}{2}) \cdots \Psi^+(-\frac{1}{2}) \mathbf{1} \\ &= \nu_1 e^{m\alpha} \quad (\nu_1 \neq 0), \\ E_{2,1}(p^+ - m + 1) \cdots E_{2,1}(p^+) \mathbf{1} &= (\chi_{p^+}^+)^m \Psi^-(-m - \frac{1}{2}) \cdots \Psi^-(-\frac{1}{2}) \mathbf{1} \\ &= \nu_2 e^{-m\alpha} \quad (\nu_2 \neq 0). \end{aligned}$$

This proves that  $e^{\ell\alpha} \in V^{cri}(\mathfrak{gl}(1|1)).\mathbf{1}$  for every  $\ell \in \mathbb{Z}$ . Since the Heisenberg vertex algebra  $M(1)$  is a subalgebra of  $V^{cri}(\mathfrak{gl}(1|1))$ , we get that  $F_\ell \subset V^{cri}(\mathfrak{gl}(1|1)).\mathbf{1}$  for each  $\ell$ . Therefore  $F(\chi^+, \chi^-)$  is a cyclic module.

Assume that  $U$  is a non-trivial submodule in  $F(\chi^+, \chi^-)$ . Using the action of the Heisenberg vertex algebra  $M(1)$  we get that there is  $\ell_0 \in \mathbb{Z}$  such that  $e^{\ell_0\alpha} \in U$ . By using the actions of elements  $E_{1,2}(n)$  and  $E_{2,1}(n)$  and formulas (4.1) -(4.2), we easily get that  $e^{\ell\alpha} \in U$  for every  $\ell \in \mathbb{Z}$ . Finally, by applying the action of the Heisenberg vertex algebra  $M(1)$  on vectors  $e^{\ell\alpha}$  we get that  $F_\ell \subset U$  for every  $\ell \in \mathbb{Z}$ . The proof follows.  $\square$

## 5. A GENERALIZATION

We shall briefly discuss a possible generalization of our construction. We omit some technical details, since a more detailed analysis will be presented in our forthcoming papers.

Let  $n \in \mathbb{Z}_{>0}$ , and consider the vertex algebras  $F^{(n)}$  and  $M^{(n)}$ . These vertex algebras admit a natural action of the Lie algebra  $\mathfrak{gl}_n$  (see previous sections). So  $\mathfrak{gl}_n$  acts on  $F^{(n)} \otimes M^{(n)}$ . Define the vertex algebra  $V_n$  as the fixed point subalgebra of this action:

$$V_n = (F^{(n)} \otimes M^{(n)})^{\mathfrak{gl}_n}.$$

Let  $\mathfrak{Z}(V_n)$  denotes the center of the vertex algebra  $V_n$ .

PROPOSITION 5.1.  $\mathfrak{Z}(V_n)$  is isomorphic to the vertex algebra  $(M^{(n)})^{\mathfrak{gl}_n}$ .

PROOF. Clearly  $(M^{(n)})^{\mathfrak{gl}_n} \subset \mathfrak{Z}(V_n)$ . Next we notice that  $F^{(n)}$  is a completely reducible  $W_{1+\infty, n} \times \mathfrak{gl}_n$ -module. By using the explicit decomposition from [10] and [12] we see that

$$\{v \in F^{(n)} \mid u_m v = 0 \ \forall u \in W_{1+\infty, n}, \ m \geq 0\} = \mathbf{C}\mathbf{1}.$$

Since  $W_{1+\infty, n}$  is a vertex subalgebra of  $V_n$ , we get that  $\mathfrak{Z}(V_n) \subset M \cap V_n = (M^{(n)})^{\mathfrak{gl}_n}$ . The claim follows.  $\square$

PROPOSITION 5.2. *The vertex algebra  $V_n$  is strongly generated by the following vectors*

$$(5.1) \quad j^{0,k} = - \sum_{i=1}^n \Psi_i^+(-1/2) \Psi_i^-(-k-1/2) \mathbf{1},$$

$$(5.2) \quad j^{1,k} = \sum_{i=1}^n a_i^+(-1/2) a_i^-(-k-1/2) \mathbf{1},$$

$$(5.3) \quad j^{+,k} = - \sum_{i=1}^n \Psi_i^+(-1/2) a_i^-(-k-1/2) \mathbf{1},$$

$$(5.4) \quad j^{-,k} = \sum_{i=1}^n a_i^+(-1/2) \Psi_i^-(-k-1/2) \mathbf{1},$$

where  $0 \leq k \leq n-1$ .

PROOF. The proof is essentially the same as the proof of [5, Theorem 7.1]. Here is the sketch of the proof with the explanation of some basic steps.

- The vertex algebra  $F^{(n)} \otimes M^{(n)}$  admits the filtration

$$\mathcal{SM}_0 \subset \mathcal{SM}_1 \subset \cdots, \quad F^{(n)} \otimes M^{(n)} = \cup_{k \geq 0} \mathcal{SM}_k,$$

where  $\mathcal{SM}_k$  is spanned by the products of generators  $\Psi_i^\pm(-n-1/2)$ ,  $a_i^\pm(-n-1/2)$  of length at most  $k$ . Since the action of  $\mathfrak{gl}_n$  preserves the above filtration on  $F^{(n)} \otimes M^{(n)}$ , we have the isomorphism of the associated graded (super)algebras:

$$\text{gr}(V_n) = \left( \text{gr}(F^{(n)} \otimes M^{(n)}) \right)^{\mathfrak{gl}_n} = \left( \text{gr}(F^{(n)} \otimes W^{(n)}) \right)^{\mathfrak{gl}_n}.$$

The generators of the above  $\mathbb{Z}_{\geq 0}$ -graded (super) algebras are determined in [5, Section 5]. As a consequence we conclude that  $V_n$  is strongly generated by elements  $j^{0,k}, j^{1,k}, j^{\pm,k}, k \geq 0$ .

- By using the fact that  $V_n$  contains the vertex subalgebra isomorphic to  $W_{1+\infty, n}$ , which is strongly generated by  $j^{0,k}, 1 \leq k \leq n$ , we get relation

$$j^{0,m} = P_m(j^{0,0}, \dots, j^{0,n-1}) \quad m \geq n$$

(cf. relation (7.1) of [5]). Then by applying the  $\mathfrak{gl}(1|1)$  action on the decoupling relation above, we get that  $V_n$  is strongly generated by the generators described in the statement.  $\square$

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**O afinjoj verteks-algebri kritičnog nivoa pridruženoj Liejevoj superalgebri  $\mathfrak{gl}(1|1)$  i njenim generalizacijama***Dražen Adamović*

SAŽETAK. U članku konstruiramo afnu verteks-algebru  $V^{cri}(\mathfrak{gl}(1|1))$  na tenzorskom produktu  $F \otimes M$ , fermionske verteks-algebre  $F$  i komutativne verteks algebre  $M$ . Na taj način karakteriziramo centar verteks-algebre  $V^{cri}(\mathfrak{gl}(1|1))$  kao podalgebru  $M_0$  komutativne verteks-algebre  $M$ . Dajemo novi dokaz Molev-Mukhinove formule za Hilbert-Poincarov formalni red centra verteks-algebre  $V^{cri}(\mathfrak{gl}(1|1))$ . Konstruiramo ireducibilne  $V^{cri}(\mathfrak{gl}(1|1))$ -module koji su realizirani na fermionskoj verteks-algebri  $F$  i parametrizirani s  $\chi^+, \chi^- \in \mathbb{C}((z))$ . Diskutiramo generalizaciju verteks-algebre  $V^{cri}(\mathfrak{gl}(1|1))$  pomoću super  $\mathcal{W}_{1+\infty}$  verteks-algebre na kritičnom nivou.

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