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ČECH SYSTEM DOES NOT INDUCE APPROXIMATE SYSTEMS

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Dedicated to the memory of Professor Sibe Mardešić

Abstract. With every topological space X is associated its Čech system $\boldsymbol{C}(X) = (|N(\mathcal{U})|, [p_{\mathcal{U}\mathcal{V}}], Cov(X))$. It is well-known that the Čech system $\boldsymbol{C}(X)$ of X is an inverse system in the homotopy category $HPol$ whose objects are polyhedra and morphisms are homotopy classes of continuous maps between polyhedra. We consider the following question posed by S. Mardešić. For a given Čech system $(|N(\mathcal{U})|, [p_{\mathcal{U}\mathcal{V}}], Cov(X))$ of a space X, is it possible to select a member $q_{\mathcal{U}\mathcal{V}} \in [p_{\mathcal{U}\mathcal{V}}]$ in each homotopy class $[p_{\mathcal{U}\mathcal{V}}]$ in such a way that the obtained system $(|N(\mathcal{U})|, q_{\mathcal{U}\mathcal{V}}, Cov(X))$ is an approximate system? We answer the question in the negative by proving that for each Hausdorff arc-like continuum X any such system $(|N(\mathcal{U})|, q_{\mathcal{U}\mathcal{V}}, Cov(X))$ is not an approximate system.

1. Introduction and main result

Let K be a simplicial complex. Denote by $|K|$ the carrier of K (i.e. the union of all simplexes belonging to K) endowed with the CW-topology. By a *polyhedron* we mean a space X such that $X = |K|$ for some simplicial complex K. If $X = |K|$ and $Y = |L|$ are polyhedra, then every simplicial map $f : K \to L$ determines in a natural way a continuous map $X \to Y$ for which we use the same notation f.

Recall that an *inverse system* in a category $\mathcal{C}$ is a collection $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ which consists of an index set Λ, endowed with a directed preorder $\preceq$, of objects X_λ from $\mathcal{C}$, for $\lambda \in \Lambda$, and of morphisms $p_{\lambda\lambda'} : X_{\lambda'} \to X_\lambda$ from $\mathcal{C}$, for $\lambda \preceq \lambda'$. On morphisms $p_{\lambda\lambda'}$ one imposes the functorial requirement $p_{\lambda\lambda'}p_{\lambda'\lambda''} = p_{\lambda\lambda''}$, for $\lambda \preceq \lambda' \preceq \lambda''$, and $p_{\lambda\lambda} = id_{X_\lambda}$, for $\lambda \in \Lambda$.

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With every topological space X one associates an inverse system $\boldsymbol{C}(X) = (|N(\mathcal{U})|, [p_{\mathcal{U}\mathcal{V}}], Cov(X))$ in the homotopy category $HPol$ of polyhedra and homotopy classes of continuous maps called the *Čech system* of X. The index set $Cov(X)$ is the set of all normal coverings $\mathcal{U}$ of X. A normal covering of X is an open covering $\mathcal{U}$ which admits a partition of unity subordinated to $\mathcal{U}$. If X is a paracompact space, then $Cov(X)$ coincides with a set of all open coverings of X (see [4, App. 1, §3.1 Corollary 1]). The set $Cov(X)$ is preordered by the relation $\preceq$, where $\mathcal{U} \preceq \mathcal{V}$ means that $\mathcal{V}$ is a refinement of $\mathcal{U}$. For each $\mathcal{U} \in Cov(X)$, a simplicial complex $N(\mathcal{U})$ is the nerve of $\mathcal{U}$ and $[p_{\mathcal{U}\mathcal{V}}], \mathcal{U} \preceq \mathcal{V}$, is the unique homotopy class to which belong projections $p_{\mathcal{U}\mathcal{V}} : |N(\mathcal{V})| \to |N(\mathcal{U})|$. Recall that vertices of $N(\mathcal{U})$ are the elements $U \in \mathcal{U}$, and vertices $U_1, \dots, U_n \in \mathcal{U}$ span a simplex of $N(\mathcal{U})$ whenever $U_1 \cap \dots \cap U_n \neq \emptyset$. A projection $p_{\mathcal{U}\mathcal{V}} : |N(\mathcal{V})| \to |N(\mathcal{U})|, \mathcal{U} \preceq \mathcal{V}$, is a continuous map determined by a simplicial map $p_{\mathcal{U}\mathcal{V}} : N(\mathcal{V}) \to N(\mathcal{U})$ which sends a vertex V of $N(\mathcal{V})$ to a vertex U of $N(\mathcal{U})$ with $V \subseteq U$. Any two projections $p_{\mathcal{U}\mathcal{V}}, q_{\mathcal{U}\mathcal{V}} : |N(\mathcal{V})| \to |N(\mathcal{U})|, \mathcal{U} \preceq \mathcal{V}$, are contiguous and thus also homotopic. Hence, projections $p_{\mathcal{U}\mathcal{V}} : |N(\mathcal{V})| \to |N(\mathcal{U})|$ are not unique but they all belong to the same homotopy class. The Čech system is studied in detail in [7, App. 1, §3].

It was noticed long ago, in fifties of the past century, that studying compact Hausdorff non-metrizable spaces using inverse systems of polyhedra and their limits has some deficiencies. For instance, S. Mardešić proved that there exist 1-dimensional compact Hausdorff spaces which are not limits of inverse systems of 1-dimensional polyhedra ([2, Theorem 4]) and there exist chainable spaces which are not limits of inverse systems of arcs ([1, Theorem 6]). These results were among the reasons which led S. Mardešić and L.R. Rubin to introduce in 1989 a more flexible kind of inverse systems of metric compacta and continuous maps, called approximate inverse systems ([3]). S. Mardešić and T. Watanabe soon extended the notion to arbitrary topological spaces ([5]). The main idea was to abandon the rigid functorial requirement $p_{\lambda\lambda'}p_{\lambda'\lambda''} = p_{\lambda\lambda''}$, for $\lambda \preceq \lambda' \preceq \lambda''$, and allow the continuous maps $p_{\lambda\lambda'}p_{\lambda'\lambda''}$ and $p_{\lambda\lambda''}$ to differ. However, the difference should be arbitrarily small when λ' is sufficiently large. Precisely, an approximate inverse system (approximate system, for short) $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ consists of the same data as ordinary inverse system in the category Top of topological spaces and continuous maps. However, besides the requirement that $p_{\lambda\lambda}$ is the identity map, one imposes the following condition.

(A) For any $\lambda \in \Lambda$ and any normal covering $\mathcal{U}$ of X_λ, there exists an $\lambda' \succeq \lambda$ such that for any $\lambda_2 \succeq \lambda_1 \succeq \lambda'$ the maps $p_{\lambda\lambda_1}p_{\lambda_1\lambda_2}$ and $p_{\lambda\lambda_2}$ are $\mathcal{U}$-near, i.e. for each $x \in X_{\lambda_2}$ there exists a $U \in \mathcal{U}$ such that points $p_{\lambda\lambda_1}p_{\lambda_1\lambda_2}(x)$ and $p_{\lambda\lambda_2}(x)$ belong to U.

S. Mardešić asked the following related question:
Let $\boldsymbol{C}(X) = (|N(\mathcal{U})|, [p_{\mathcal{U}\mathcal{V}}], Cov(X))$ be the Čech system of a space X. Is it possible to select one projection $q_{\mathcal{U}\mathcal{V}}$ in each homotopy class $[p_{\mathcal{U}\mathcal{V}}], \mathcal{U} \preceq \mathcal{V}$, in such a way that the obtained system $(|N(\mathcal{U})|, q_{\mathcal{U}\mathcal{V}}, Cov(X))$ is an approximate system? In other words, does the Čech system of a space X induce approximate systems (associated with X)?

We answer the question in the negative by showing that the Čech system $\boldsymbol{C}(X)$ of an arbitrary Hausdorff arc-like space X does not induce any associated approximate system as it is proved in Theorem 2.13 in the next section.

2. Arc-like spaces and their Čech systems

Definition 2.1. *Let $\mathcal{P}$ be a non-empty class of compact polyhedra and let X be a T_1-space. We say that X is $\mathcal{P}$-like, if for each open covering $\mathcal{U}$ of X, there exist a polyhedron $P \in \mathcal{P}$, an open covering $\mathcal{V}$ of P and a surjective map $f : X \to P$ such that the open covering $f^{-1}\mathcal{V} = (f^{-1}(V), V \in \mathcal{V})$ of X refines $\mathcal{U}$.*

Proposition 2.2. *Let $\mathcal{P}$ be a non-empty class of compact polyhedra and let a T_1-space X be $\mathcal{P}$-like. Then X is a compact Hausdorff space. Furthermore, if each member of $\mathcal{P}$ is connected, then X is connected, too.*

Proof. Let $\mathcal{U}$ be an arbitrary open covering of X. Since X is $\mathcal{P}$-like there exist a compact polyhedron P, a surjective map $f : X \to P$ and a finite open covering $\mathcal{V}$ of P such that $f^{-1}\mathcal{V} \preceq \mathcal{U}$. Hence, $f^{-1}\mathcal{V}$ is a finite, open refinement of $\mathcal{U}$, which proves that X is compact. Let $x, y \in X$ be different points of X. Since X is a T_1-space, a collection $\mathcal{U} = (X \setminus \{x\}, X \setminus \{y\})$ is an open covering of X. Then, there exist a compact polyhedron P, a map $f : X \to P$ and an open covering $\mathcal{V}$ of P such that $f^{-1}\mathcal{V} \preceq \mathcal{U}$. Note that $f(x) \neq f(y)$. Indeed, assume the contrary, i.e.. $f(x) = f(y) = p \in P$ and take an open set $V \in \mathcal{V}$ which contains p. Then $f^{-1}V$ is contained in $X \setminus \{x\}$ or in $X \setminus \{y\}$. However, $f^{-1}V$ contains x and y and we get a contradiction in both cases. Hence, $f(x) \neq f(y)$. Polyhedra are Hausdorff spaces and we can find open disjoint sets $W_1, W_2 \subseteq P$ such that $f(x) \in W_1$ and $f(y) \in W_2$. Then $f^{-1}W_1$ and $f^{-1}W_2$ are required disjoint open neighborhoods of x and y respectively. Assume that each member of $\mathcal{P}$ is a connected polyhedron. We claim that X is connected. Assume the contrary. Since X is disconnected, there exist two non-empty disjoint open sets U_1, U_2 in X such that $X = U_1 \cup U_2$. Then $\mathcal{U} = (U_1, U_2)$ is an open covering of X and there exist a compact connected polyhedron $P \in \mathcal{P}$, a map $f : X \to P$ and an open covering $\mathcal{V}$ of P such that $f^{-1}\mathcal{V} \preceq \mathcal{U}$. Denote by $W_1 = \cup \{V \in \mathcal{V} : f^{-1}V \subseteq U_1\}$ and by $W_2 = \cup \{V \in \mathcal{V} : f^{-1}V \subseteq U_2\}$. First note that both open sets W_1 and W_2 are non-empty. Assume that $W_1 = \emptyset$ and choose $x \in U_1$. Take $V \in \mathcal{V}$ such that $f(x) \in V$. Since $W_1 = \emptyset$, it follows that $f^{-1}V \subseteq U_2$ and consequently $x \in U_2$ which is a contradiction. Also note that W_1 and W_2 are disjoint sets. Assume that there exist $p \in W_1 \cap W_2$. Since f is surjective, there exists $x \in X$ such that $f(x) = p$. Then there exists $V, V' \in \mathcal{V}$ such that $p \in V, V'$, $f^{-1}V \subseteq U_1$ and $f^{-1}V' \subseteq U_2$. This implies $x \in U_1 \cap U_2$ and we get a contradiction. So, W_1, W_2 are non-empty disjoint open sets and $P = W_1 \cup W_2$. We conclude that P is disconnected and get a contradiction. □

Definition 2.3. *A Hausdorff space X is said to be arc-like (or snake-like), if X is $\mathcal{P}$-like, where $\mathcal{P}$ consists only of the unit segment $I = [0,1] \subseteq \mathbb{R}$.*

Here the unit segment $I = [0,1] \subseteq \mathbb{R}$ is considered as the carrier of a simplicial complex K which has two vertices (points 0 and 1) and one 1-simplex. According to the previous proposition any arc-like space is a Hausdorff continuum, i.e. a compact connected Hausdorff space. Arc-like spaces can be characterized by a certain property of their open coverings. To show that firstly we define chainable coverings of a space.

Definition 2.4. *A finite open covering $\mathcal{U} = (U_i, i = 1, \dots, n)$ of a space X is called chainable provided $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leq 1, i, j \in \{1, \dots, n\}$.*

A polyhedron homeomorphic to the unit segment $I = [0,1] \subseteq \mathbb{R}$ is called an *arc.* Note that the nerve $|N(\mathcal{U})|$ of any chainable covering $\mathcal{U} = (U_i, i = 1, \dots, n)$, $n \geq 2$, is an arc. If $\mathcal{V}$ is a chainable covering of a space Y and $f : X \to Y$ is a surjective map, then $f^{-1}\mathcal{V}$ is a chainable covering of a space X.

Proposition 2.5. *A Haudsorff space X is arc-like if and only if each open covering $\mathcal{U}$ of X admits a chainable refinement.*

Proof. Assume that X is arc-like and take an arbitrary open covering $\mathcal{U}$ of X. Then there exist an open covering $\mathcal{V}$ of I and a surjection $f : X \to I$ such that $f^{-1}\mathcal{V}$ refines $\mathcal{U}$. Put $I = |K|$, where K is a simplicial complex having two vertices $0, 1$ and one 1-simplex. Then there exists a subdivision L of K such that a finite open covering $\mathcal{S}$ of I consisting of open stars $st(v, L)$ of the vertices v of L refines $\mathcal{V}$ (see Theorem 4 in [7, App. 1, §1.1]). Assume that the vertices $v_i, i = 1, \dots, n$, of L are indexed in such a way that $0 = v_1 < v_2 < \cdots < v_{n-1} < v_n = 1$ and put $W_i = st(v_i, L), i = 1, \dots, n$. Note that $W_i \cap W_j \neq \emptyset$ if and only if vertices v_i, v_j span a simplex of L. Consequently, $W_i \cap W_j \neq \emptyset$ if and only if $|i - j| \leq 1$, which shows that $\mathcal{S} = (W_i, i = 1, \dots, n)$ is a chainable covering of I. Then, $f^{-1}\mathcal{S} = (f^{-1}W_i, i = 1, \dots, n)$ is a chainable covering of X which refines $f^{-1}\mathcal{V}$ and then also $\mathcal{U}$.

Conversely, assume that each open covering of X admits a chainable refinement. First note that X is compact and connected. Compactness is obvious since chainable coverings are open and finite by definition. Assume that X is not connected. Then there exists an open covering $\mathcal{U} = (U_1, U_2)$ of

X consisting of two disjoint non-empty open sets. Let $\mathcal{V} = (V_i, i = 1, \dots, n)$ be a chainable refinement of $\mathcal{U}$. Since $V_i \cap V_{i+1} \neq \emptyset$, for each $i = 1, \dots, n-1$, it follows that all V_i are contained in U_1 or all V_i are contained in $U_{2.}$. This contradicts the fact that U_1 and U_2 are both non-empty sets. Let us show that X is arc-like. Take an open covering $\mathcal{U}$ of X consisting of open sets $U \neq X$. Let $\mathcal{V} = (V_i, i = 1, \dots, n)$ be a chainable refinement of $\mathcal{U}$ and consider a canonical map $p_{\mathcal{V}} : X \to |N(\mathcal{V})|$ for $\mathcal{V}$, i.e. a map having property that $p_{\mathcal{V}}^{-1}(st(V, N(\mathcal{V})) \subseteq V$, for each $V \in \mathcal{V}$ (see [7, page 326]). Since $|N(\mathcal{V})|$ is homeomorphic to I, $P = p_{\mathcal{V}}(X)$ is a compact and connected subset of $|N(\mathcal{V})|$ and P is not a singleton, it follows that P is homeomorphic to I as well. Then there exist $j, k, 1 \leq j \leq k \leq n$, such that $\mathcal{W} = (st(V_i, N(\mathcal{V})) \cap P, i = j, \dots, k)$ is an open covering of P consisting of non-empty sets. Note that $p_{\mathcal{V}}^{-1}(st(V_i, N(\mathcal{V})) \cap P) \subseteq p_{\mathcal{V}}^{-1}(st(V_i, N(\mathcal{V}))) \subseteq V_i, i = j, \dots, k$, which shows that $p_{\mathcal{V}}^{-1}(\mathcal{W})$ refines $\mathcal{V}$ and also $\mathcal{U}$. Let $h : P \to I$ be a homeomorphism. Then $hp_{\mathcal{V}} : X \to I$ is a surjection, $\mathcal{W}' = (h(W), W \in \mathcal{W})$ is an open covering of I and $(hp_{\mathcal{V}})^{-1}(\mathcal{W}')$ refines $\mathcal{U}$, which shows that X is arc-like. □

Definition 2.6. *A Hausdorrf space X is said to be chainable, if each open covering of X admits a chainable refinement.*

According to Proposition 2.5, a Hausdorff space X is arc-like if and only if X is chainable.

Definition 2.7. *A covering $(A_\lambda, \lambda \in \Lambda)$ of a set X is called irreducible if, for each $\lambda_0 \in \Lambda$, a family $(A_\lambda, \lambda \in \Lambda \setminus \{\lambda_0\})$ is not a covering of X. A covering $(A_\lambda, \lambda \in \Lambda)$ of a set X is called reducible if it is not irreducible.*

Put $I = |K|$,where K is a simplicial complex consisting of two vertices $0, 1$ and one 1-simplex. Let L be a subdivision of K with n vertices $\{v_1, \dots, v_n\}$ such that $0 = v_1 < v_2 < \cdots < v_n = 1$. Then an open covering $\mathcal{S} = (st(v_i, L), i = 1, \dots, n)$ of I consisting of open stars $st(v_i, L)$ of the vertices v_i of L is an irreducible chainable covering of I. If $\mathcal{V}$ is an irreducible covering of a set Y and $f : X \to Y$ is a surjective function, then $f^{-1}\mathcal{V}$ is an irreducible covering of a set X.

Lemma 2.8. *Let $(A_\lambda, \lambda \in \Lambda)$ be an irreducible covering of a set X. Then for each $\lambda \in \Lambda$ there is an element $x_\lambda \in X$ such that $x_\lambda \in A_\lambda$ and $x_\lambda \notin A_{\lambda'}$ for $\lambda' \neq \lambda$.*

Proof. Since $(A_\lambda, \lambda \in \Lambda)$ is an irreducible covering of X, for each $\lambda \in \Lambda$, a subset $X \setminus (\bigcup_{\lambda' \neq \lambda} A_{\lambda'})$ of X is non-empty and we can choose $x_\lambda \in X$ with the required properties. □

Lemma 2.9. *Let $\mathcal{U} = (U_i, i = 1, \dots, n), n \geq 2$, be a chainable covering of a connected space X. Then there exists an irreducible chainable subcovering $\mathcal{V}$ of $\mathcal{U}$.*

Proof. If $\mathcal{U}$ is irreducible, put $\mathcal{V} = \mathcal{U}$. Assume that $\mathcal{U}$ is reducible. If $n = 2$, then $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$. Putting $V_1 = U_2$ in the first case or $V_1 = U_1$ in the second case we get the desired subcovering $\mathcal{V}$. If $n > 2$, then there exists $i \in \{1, \ldots, n\}$ such that $X = \bigcup_{j \neq i} U_j$.and we claim that $i \notin \{2, \ldots, n-1\}$. Indeed, assume the contrary. Then $X = (U_1 \cup \cdots \cup U_{i-1}) \cup (U_{i+1} \cup \cdots \cup U_n)$ is the union of two non-empty disjoint open sets, which contradicts connectedness of X. Hence $i = 1$ or $i = n$ and the desired irreducible chainable subcovering $\mathcal{V}$ of $\mathcal{U}$ is one of the coverings $(U_i, i = 2, \ldots, n)$, $(U_i, i = 1, \ldots, n-1)$ or $(U_i, i = 2, \ldots, n-1)$, depending on i. ∎

Lemma 2.10. *Let a Hausdorff space X be arc-like. Then each open covering $\mathcal{U}$ of X admits an irreducible chainable refinement $\mathcal{V}$.*

Proof. The claim follows directly from the Proposition 2.5 and Lemma 2.9. ∎

Lemma 2.11. *Let a Hausdorff space X be arc-like and let $\{x_1, \ldots, x_m\}$ be a finite subset of X consisting of different points. Then each open covering $\mathcal{U}$ of X admits an irreducible chainable refinement $\mathcal{V}$ such that each point $x_i, i = 1, \ldots, m$, belongs to exactly one member of $\mathcal{V}$ and different points $x_i, x_j, 1 \leq i, j \leq m$, belong to different members of $\mathcal{V}$.*

Proof. Take an arbitrary open covering $\mathcal{U}$ of X. For each $i = 1, \ldots, m$, put $W_i := X \setminus \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m\}$. Then $\mathcal{W} = (W_i, i = 1, \ldots, m)$ is an open covering of X and, for each $i = 1, \ldots, m$, $W_i \cap \{x_1, \ldots, x_m\} = \{x_i\}$. Let $\mathcal{U}'$ be an open covering of X which refines both $\mathcal{U}$ and $\mathcal{W}$. Since X is an arc-like space there is a surjective map $f : X \to I$ and an open covering $\mathcal{V}'$ of I such that $f^{-1}\mathcal{V}'$ refines $\mathcal{U}'$. Note that $f(x_i) \neq f(x_j)$ for each $i \neq j$. Indeed, assume $f(x_i) = f(x_j)$, for some $i \neq j$. Take a $V' \in \mathcal{V}'$ such that $f(x_i) = f(x_j) \in V'$ and let W_k be an element of $\mathcal{W}$ with $f^{-1}V' \subseteq W_k$. Then $x_i, x_j \in X \setminus \{x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m\}$,which implies $x_i = x_j = x_k$, and we get a contradiction. Consider I as the carrier of a simplicial complex K having two vertices $0, 1$ and one 1-simplex. Then there is a subdivision L of K such that a finite open covering $\mathcal{S}$ of I consisting of open stars $st(v, L)$ of the vertices v of L refines $\mathcal{V}'$. Let L' be a subdivision of L such that the set of vertices of L' contains all vertices of L and all points $f(x_i), i = 1, \ldots, m$. Denote by $\{v_1, \ldots, v_n\}, n \geq m$, the set of vertices of L' such that $0 = v_1 < v_2 < \cdots < v_n = 1$. An open finite covering $\mathcal{S}'$ of I consisting of open stars $st(v_i, L')$ of the vertices v_i of L' refines $\mathcal{S}$ and also $\mathcal{V}'$. Then $\mathcal{V} = f^{-1}\mathcal{S}'$ is an irreducible chainable covering having required properties. ∎

Let $\preceq$ be a binary relation on a set Λ. A subset $\Lambda' \subseteq \Lambda$ is said to be *cofinal* in Λ, if, for each $\lambda \in \Lambda$, there exists some $\lambda' \in \Lambda'$ such that $\lambda \preceq \lambda'$. If $(\Lambda, \preceq)$ is a directed preordered set and $\Lambda' \subseteq \Lambda$ is cofinal in Λ, then $(\Lambda', \preceq)$ is a directed preordered set as well. Let X be an arc-like space X and denote by $\mathcal{IC}(X)$

a subset of $Cov(X)$ consisting of all irreducible chainable coverings of X. According to Lemma 2.10, $\mathcal{IC}(X)$ is cofinal in $(Cov(X), \preceq)$ and, consequently, $(\mathcal{IC}(X), \preceq)$ is a directed preordered set.

THEOREM 2.12. *Let a Hausdorff space X be arc-like and let $\mathcal{IC}(X)$ be a subset of $Cov(X)$ consisting of all irreducible chainable coverings of X. For each pair $\mathcal{U}, \mathcal{V} \in \mathcal{IC}(X)$ such that $\mathcal{U} \preceq \mathcal{V}$, select a projection $p_{\mathcal{UV}} : |N(\mathcal{V})| \to |N(\mathcal{U})|$. Then the obtained system $(|N(\mathcal{U})|, p_{\mathcal{UV}}, \mathcal{IC}(X))$ is not an approximate system.*

PROOF. Assume the contrary, i.e. $(|N(\mathcal{U})|, p_{\mathcal{UV}}, \mathcal{IC}(X))$ is an approximate system. Let $\mathcal{U} \in \mathcal{IC}(X)$ be an arbitrary index , i.e. $\mathcal{U}$ is an irreducible chainable covering of X and assume $\mathcal{U} = (U_1, \dots U_n), n \geq 2$. Choose an open covering $\mathcal{A}$ of $|N(\mathcal{U})|$ having property that each member of $\mathcal{A}$ contains at most one vertex $U \in |N(\mathcal{U})|$ (for instance, such $\mathcal{A}$ is an open covering $(st(U, N(\mathcal{U})), U \in \mathcal{U})$ consisting of open stars of vertices U of $N(\mathcal{U})$). Since the system $(|N(\mathcal{U})|, p_{\mathcal{UV}}, \mathcal{IC}(X))$ is approximate, there exists a covering, i.e. an index, $\mathcal{U}_0 = (U_1^0, \dots, U_m^0) \in \mathcal{IC}(X)$ which refines $\mathcal{U}$, and for each $\mathcal{V}, \mathcal{W} \in \mathcal{IC}(X)$ satisfying $\mathcal{U}_0 \preceq \mathcal{V} \preceq \mathcal{W}$, maps $p_{\mathcal{UV}}p_{\mathcal{VW}}$ and $p_{\mathcal{UW}}$ are $\mathcal{A}$-near. Take an arbitrary vertex $W \in |N(\mathcal{W})|$. Since the projections are determined by simplicial maps, points $p_{\mathcal{UV}}p_{\mathcal{VW}}(W)$ and $p_{\mathcal{UW}}(W)$ are vertices of the nerve $|N(\mathcal{U})|$. On the other hand, each member of $\mathcal{A}$ contains at most one vertex of $|N(\mathcal{U})|$, and we conclude that $p_{\mathcal{UV}}p_{\mathcal{VW}}(W) = p_{\mathcal{UW}}(W)$, for each vertex $W \in |N(\mathcal{W})|$. This implies $p_{\mathcal{UV}}p_{\mathcal{VW}}(y) = p_{\mathcal{UW}}(y)$, for each $y \in |N(\mathcal{W})|$. Thus, $p_{\mathcal{UV}}p_{\mathcal{VW}} = p_{\mathcal{UW}}$, for any $\mathcal{V}, \mathcal{W} \in \mathcal{IC}(X)$ such that $\mathcal{U}_0 \preceq \mathcal{V} \preceq \mathcal{W}$. According to Lemma 2.8, for each $i = 1, \dots, n$, we can choose a point $x_i \in X$ such that $x_i \in U_i \setminus (\bigcup\limits_{j \neq i} U_j)$. Let U_i^0 and U_j^0 be members of the covering $\mathcal{U}_0$ with $x_1 \in U_i^0$ and $x_2 \in U_j^0$. Note that $i \neq j$. Indeed, assume that $i = j$. Since U_1 is the only member of $\mathcal{U}$ which contains x_1 and U_2 is the only member of $\mathcal{U}$ which contains x_2, we get $U_i^0 \subseteq U_1$ and $U_i^0 = U_j^0 \subseteq U_2$. Hence, $x_1 \in U_1 \cap U_2$ and we get a contradiction. Hence, $i \neq j$. Note that $p_{\mathcal{UU}_0}(U_i^0) = U_1$ and $p_{\mathcal{UU}_0}(U_j^0) = U_2$. Without loss of generality assume $i < j$. We claim that there exists $k, i \leq k < j$, such that $p_{\mathcal{UU}_0}(U_k^0) = U_1$ and $p_{\mathcal{UU}_0}(U_{k+1}^0) = U_2$. To prove that consider a finite set $F = \left\{n : p_{\mathcal{UU}_0}(U_n^0) = U_1, i \leq n < j\right\}$. F is a non-empty set, since $i \in F$. Put $k := \max F$. If $k = j - 1$, the claim holds, because $p_{\mathcal{UU}_0}(U_j^0) = U_2$. Assume $k < j - 1$. Since $p_{\mathcal{UU}_0}(U_k^0) = U_1$, it follows $U_k^0 \subseteq U_1$. $\mathcal{U}_0$ is a chainable covering, so $U_k^0 \cap U_{k+1}^0 \neq \emptyset$. Let $U_l \in \mathcal{U}$ be a member of $\mathcal{U}$ such that $p_{\mathcal{UU}_0}(U_{k+1}^0) = U_l$. Then $\emptyset \neq U_k^0 \cap U_{k+1}^0 \subseteq U_1 \cap U_l$ and we conclude that $l \in \{1, 2\}$. Assume that $p_{\mathcal{UU}_0}(U_{k+1}^0) = U_1$. Since $i \leq k + 1 < j$, it follows that $k + 1 \in F$ which contradicts $k = \max F$. Therefore, $p_{\mathcal{UU}_0}(U_{k+1}^0) = U_2$ and the claim is proved. So, we get $p_{\mathcal{UU}_0}(U_k^0) = U_1$ and $p_{\mathcal{UU}_0}(U_{k+1}^0) = U_2$. Note that $U_k^0 \cap U_{k+1}^0$ is an infinite set. Indeed, assume that $U_k^0 \cap U_{k+1}^0$ is a finite set $U_k^0 \cap U_{k+1}^0 = \left\{y_1, y_2, \dots, y_{n(k)}\right\}$. Then we get $\{y_1\} = (U_k^0 \cap U_{k+1}^0) \setminus \{y_2, \ldots, y_{n(k)}\}$ which implies that X has an isolated point y_1 and that contradicts connectedness of X. Choose arbitrary points $x, x' \in U_k^0 \cap U_{k+1}^0$, $x \neq x'$. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be open coverings of X given by $\mathcal{V}_1 = (U_1^0, \ldots, U_{k-1}^0, U_k^0 \setminus \{x\}, U_{k+1}^0 \setminus \{x'\}, U_{k+2}^0, \ldots, U_m^0)$, $\mathcal{V}_2 = (U_1^0, \ldots, U_{k-1}^0, U_k^0 \setminus \{x'\}, U_{k+1}^0 \setminus \{x\}, U_{k+2}^0, \ldots, U_m^0)$. Note that $\mathcal{V}_1$ and $\mathcal{V}_2$ are irreducible chainable coverings of X which refine $\mathcal{U}_0$, i.e. $\mathcal{V}_1, \mathcal{V}_2 \in \mathcal{IC}(X)$, $\mathcal{U}_0 \preceq \mathcal{V}_1, \mathcal{V}_2$. Moreover, $p_{\mathcal{U}_0\mathcal{V}_1}(U_{k+1}^0 \setminus \{x'\}) = U_{k+1}^0$ and $p_{\mathcal{U}_0\mathcal{V}_2}(U_k^0 \setminus \{x'\}) = U_k^0$. According to Lemma 2.11, there exists an irreducible chainable covering $\mathcal{W}$ of X which refines both $\mathcal{V}_1$ and $\mathcal{V}_2$, each of the points x, x' belongs to exactly one member of $\mathcal{W}$ and x, x' belong to different members of $\mathcal{W}$. Hence $\mathcal{W} \in \mathcal{IC}(X)$ and $\mathcal{U}_0 \preceq \mathcal{V}_1, \mathcal{V}_2 \preceq \mathcal{W}$. Let $W \in \mathcal{W}$ be the only element of $\mathcal{W}$ which contains x. Note that $x' \notin W$ and $U_{k+1}^0 \setminus \{x'\}$ is the only element of $\mathcal{V}_1$ which contains x. This implies $p_{\mathcal{V}_1\mathcal{W}}(W) = U_{k+1}^0 \setminus \{x'\}$ and we get $p_{\mathcal{U}\mathcal{W}}(W) = p_{\mathcal{U}\mathcal{V}_1} p_{\mathcal{V}_1\mathcal{W}}(W) = p_{\mathcal{U}\mathcal{V}_1}(U_{k+1}^0 \setminus \{x'\}) = p_{\mathcal{U}\mathcal{U}_0} p_{\mathcal{U}_0\mathcal{V}_1}(U_{k+1}^0 \setminus \{x'\}) = p_{\mathcal{U}\mathcal{U}_0}(U_{k+1}^0) = U_2$. On the other hand, $U_k^0 \setminus \{x'\}$ is the only element of $\mathcal{V}_2$ which contains x and we get $p_{\mathcal{U}\mathcal{W}}(W) = p_{\mathcal{U}\mathcal{V}_2} p_{\mathcal{V}_2\mathcal{W}}(W) = p_{\mathcal{U}\mathcal{V}_2}(U_k^0 \setminus \{x'\}) = p_{\mathcal{U}\mathcal{U}_0} p_{\mathcal{U}_0\mathcal{V}_2}(U_k^0 \setminus \{x'\}) = p_{\mathcal{U}\mathcal{U}_0}(U_k^0) = U_1$. This yields a contradiction since U_1 and U_2 are different vertices of $|N(\mathcal{U})|$. ∎

THEOREM 2.13. *The Čech system* $\boldsymbol{C}(X) = (|N(\mathcal{U})|, [p_{\mathcal{U}\mathcal{V}}], Cov(X))$ *of a Hausdorff arc-like space* X *does not induce approximate systems, i.e. it is not possible to select one projection* $q_{\mathcal{U}\mathcal{V}}$ *in each homotopy class* $[p_{\mathcal{U}\mathcal{V}}]$, $\mathcal{U} \preceq \mathcal{V}$, *in such a way that* $(|N(\mathcal{U})|, q_{\mathcal{U}\mathcal{V}}, Cov(X))$ *becomes an approximate system.*

PROOF. Assume the contrary, i.e. there exists a selection of projections $q_{\mathcal{U}\mathcal{V}} \in [p_{\mathcal{U}\mathcal{V}}]$, for each pair of coverings $\mathcal{U} \preceq \mathcal{V}$, such that the obtained system $(|N(\mathcal{U})|, q_{\mathcal{U}\mathcal{V}}, Cov(X))$ is an approximate system. Then $(|N(\mathcal{U})|, q_{\mathcal{U}\mathcal{V}}, \mathcal{IC}(X))$ is an approximate system, too, which contradicts Theorem 2.12. ∎

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Čechov sustav ne inducira aproksimativne sustave

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Sažetak. Svakom topološkom prostoru X pridružen je njegov Čechov sustav $\boldsymbol{C}(X) = (|N(\mathcal{U})|, [p_{\mathcal{U}\mathcal{V}}], Cov(X))$. Dobro je poznato da je Čechov sustav $\boldsymbol{C}(X)$ od X inverzni sustav u homotopskoj kategoriji $HPol$ čiji su objekti poliedri, a morfizmi homotopske klase neprekidnih preslikavanja među poliedrima. Sibe Mardešić postavio je sljedeće pitanje: Za dani Čechov sustav $(|N(\mathcal{U})|, [p_{\mathcal{U}\mathcal{V}}], Cov(X))$ prostora X, je li moguće izabrati član $q_{\mathcal{U}\mathcal{V}} \in [p_{\mathcal{U}\mathcal{V}}]$ u svakoj homotopskoj klasi $[p_{\mathcal{U}\mathcal{V}}]$ tako da dobiveni sustav $(|N(\mathcal{U})|, q_{\mathcal{U}\mathcal{V}}, Cov(X))$ bude aproksimativni sustav? Ovdje pokazujemo da je odgovor na to pitanje negativan, budući da, za svaki Hausdorffov kontinuum X koji je poput luka, svaki takav sustav $(|N(\mathcal{U})|, q_{\mathcal{U}\mathcal{V}}, Cov(X))$ nije aproksimativni sustav.

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