Existence and behaviour of some radial solutions of a semilinear elliptic equation with a gradient-term*

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Abstract. *In this paper we study the existence, behaviour and approximation of some positive radial solutions* $u(|x|)$ *of the equation* $\Delta u + \frac{1}{2}x\nabla u + u^p - \lambda u^q = 0$, $x \in \mathbb{R}^n$ $\forall |x| \geq a > 0$. *The errors of the approximations for solution* u *and the first derivative* u' *are defined by the functions which can be sufficiently small* $\forall |x| \geq a$.

Key words: *semilinear elliptic equation, radial solutions, existence, approximation*

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1. Introduction

Let us consider the equation

$$\Delta u + \frac{1}{2}x\nabla u + u^p - \lambda u^q = 0, \quad x \in \mathbb{R}^n, \tag{1}$$

where $\lambda \in \mathbb{R}$, $p > 1$ and $n \geq 2$. Since we are interested in radial solutions $u = u(r)$, $r = |x|$, we shall study the ordinary differential equation ($' = d/dr$)

$$u'' + \left(\frac{n-1}{r} + \frac{r}{2}\right)u' + u^p - \lambda u^q = 0, \quad r \geq a > 0. \tag{2}$$

Many authors have studied the equation (2). For example, in [2] the equation (2) is considered and the existence of a positive radial solution satisfying the condition

$$u(r) = O\left(r^{-n}\exp\left(-\frac{r^2}{4}\right)\right) \qquad \text{as } r \to \infty \tag{3}$$

is proved for

$$\lambda = 0, \quad 1 < p < \frac{n+2}{n-2} \quad \text{and} \quad \text{n} \geq 3.$$

In this paper we shall establish the existence of some positive solutions of (2) satisfying condition (3) and their approximation for every $r \geq a$.

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2. The main results

Theorem 1. *Let* $\alpha, \eta, s \in \mathbb{R}_+ = (0, \infty)$, $\lambda = 0$,

$$p \geq 1 + \frac{s}{n}, \quad 0 < s \leq 2, \tag{4}$$

$$\frac{1}{s}\left[2\,(2\alpha)^p + 4n\alpha a^{s-2}\right] \leq \eta \leq \alpha a^s, \tag{5}$$

$$a > 1 \quad \text{and} \quad s a^s \geq 4n a^{s-2} + 4\,(2\alpha)^{p-1}. \tag{6}$$

Then the equation (2) $(\lambda = 0)$ *has at least one positive solution* $u(r)$ *satisfying the conditions*

$$\left| u(r) - \alpha r^{-n} \exp\left(-\frac{r^2}{4}\right)\right| < \eta r^{-n-s} \exp\left(-\frac{r^2}{4}\right), \tag{7}$$

$$\left| u'(r) + \alpha\left(\frac{1}{2} + \frac{n}{r^2}\right) r^{-n+1} \exp\left(-\frac{r^2}{4}\right)\right| < \eta\left(\frac{1}{2} + \frac{n+s}{r^2}\right) r^{-n-s+1} \exp\left(-\frac{r^2}{4}\right) \tag{8}$$

$\forall r \geq a$.

Theorem 2. *Let* (4), (5) *and* (6) *hold true,* $\alpha, \eta, s \in \mathbb{R}_+$,

$$q \geq 1 + \frac{s}{n}, \quad 0 < \lambda \leq \frac{s\eta}{2}\,(2\alpha)^q.$$

Then the equation (2) *has at least one positive solution* $u(r)$ *satisfying the conditions* (7) *and* (8) $\forall r \geq a$.

Theorems 1 and *2* can be generalised. Let $\zeta \in C^2(\mathbb{R}_+, \mathbb{R}_+)$ and

$$H(\zeta) := \zeta'' + \left(\frac{n-1}{r} + \frac{r}{2}\right)\zeta'.$$

Theorem 3. *Let* $\lambda = 0$, $\varphi, \rho \in C^2(\mathbb{R}_+, \mathbb{R}_+)$ *such that*

$$\varphi(r) \geq \rho(r), \quad \rho'(r) < 0 \quad \forall r \geq a, \tag{9}$$

$$\varphi(r) \to 0 \quad \text{and} \quad \rho(\mathrm{r}) \to 0 \quad \text{as} \quad \mathrm{r} \to \infty. \tag{10}$$

If the functions φ *and* ρ *satisfy the conditions*

$$H(\varphi) + H(\rho) + (\varphi - \rho)^p > 0 \quad \text{and}$$

$$-H(\varphi) + H(\rho) - (\varphi + \rho)^p > 0 \quad \forall r > a,$$

then the equation (2) $(\lambda = 0)$ *has at least one positive solution* $u(r)$ *satisfying the conditions*

$$|u(r) - \varphi(r)| < \rho(r) \quad \text{and} \tag{11}$$

$$|u'(r) - \varphi'(r)| < -\rho'(r) \quad \forall r \geq a. \tag{12}$$

Theorem 4. *Let* $\varphi, \rho \in C^2(\mathbb{R}_+, \mathbb{R}_+)$, *such that the conditions* (9) *and* (10) *hold true. If the functions φ and ρ satisfy the conditions*

$$H(\varphi) + H(\rho) + (\varphi - \rho)^p - \lambda(\varphi + \rho)^q > 0 \quad \text{and}$$

$$-H(\varphi) + H(\rho) - (\varphi + \rho)^p + \lambda(\varphi - \rho)^q > 0 \quad \forall r > a,$$

then the equation (2) *has at least one positive solution* $u(r)$ *satisfying the conditions* (11) *and* (12).

3. Proofs of the theorems

We shall study the equation (2) by means of the equivalent system

$$\begin{aligned} u'(r) &= v, \\ v'(r) &= -\left(\frac{n-1}{r} + \frac{r}{2}\right) v - u^p + \lambda u^q, \\ r' &= 1, \quad r \geq a. \end{aligned} \tag{13}$$

According to known theorems, the Cauchy problem for the system (13) has the unique solution in $\Omega = \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$.

Let us consider the behaviour of integral curves $(u(r), v(r), r)$ of (13) with respect to the set

$$\omega = \{(u, v, r) \in \Omega : |u - \varphi| < \rho, \ |v - \varphi'| < -\rho', \ r > a\},$$

where $\varphi, \rho \in C^2(\mathbb{R}_+, \mathbb{R}_+)$ such that the conditions (9) and (10) hold true.

The boundary surfaces of ω are

$$\begin{aligned} U_i &= \left\{T \in \bar{\omega} : \Phi_i := (-1)^i [u - \varphi_i(r)] = 0\right\}, \\ V_i &= \left\{T \in \bar{\omega} : \Psi_i := (-1)^i [v - \psi_i(r)] = 0\right\}, \ i = 1, 2, \end{aligned}$$

where

$$\varphi_i(r) := \varphi(r) + (-1)^i \rho(r), \quad \psi_i(r) := \varphi'(r) - (-1)^i \rho'(r).$$

Let us denote the tangent vector field to an integral curve $(u(r), v(r), r)$ of (13) by X, i.e.

$$X = \left(v, \ -\left(\frac{n-1}{r} + \frac{r}{2}\right) v - u^p + \lambda u^q, \ 1\right).$$

The vectors $\nabla\Phi_i$ and $\nabla\Psi_i$, $i = 1, 2$, are the external normals on surfaces U_i and V_i, respectively:

$$\begin{aligned} \nabla\Phi_i &= \left((-1)^i, \ 0, \ (-1)^{i+1} \varphi_i'(r)\right), \\ \nabla\Psi_i &= \left(0, \ (-1)^i, \ (-1)^{i+1} \psi_i'(r)\right), \ i = 1, 2. \end{aligned}$$

By means of scalar products

$$P_i = (\nabla\Phi_i,\ X)\ \text{ on } \mathrm{U_i},\quad \mathrm{Q_i} = (\nabla\Psi_i,\ \mathrm{X})\ \text{ on } \mathrm{V_i},\ \mathrm{i} = 1,2,$$

we shall establish the behaviour of integral curves of the system (13) with respect to the set ω. For the proof of *Theorems 1-4* we shall use the same scalar products P_i, $i = 1,2$. We have

$$P_1 = -v + \varphi_1' > -\psi_2 + \varphi_1' \equiv 0 \ on\ U_1 \backslash L,\quad P_1 \equiv 0 \ on\ L = U_1 \cap V_2,$$

$$P_2 = v - \varphi_2' > \psi_1 - \varphi_2' \equiv 0 \ on\ U_2 \backslash M,\quad P_2 \equiv 0 \ on\ M = U_2 \cap V_1$$

and

$$\begin{aligned} u'(L) &= v(L) = \psi_2 = \varphi_1', \\ u''(L) &= v'(L) = -\left(\frac{n-1}{r} + \frac{r}{2}\right)\psi_2 - \varphi_1^p > \varphi_1'', \\ u'(M) &= v(M) = \psi_1 = \varphi_2', \\ u''(M) &= v'(M) = -\left(\frac{n-1}{r} + \frac{r}{2}\right)\psi_1 - \varphi_2^p < \varphi_2''. \end{aligned}$$

For scalar products Q_i, $i = 1,2$, first note that (in all theorems)

$$\begin{aligned} Q_1 = &\ \left(\tfrac{n-1}{r} + \tfrac{r}{2}\right) v + u^p - \lambda u^q + \psi_1' \\ \geq &\ \left(\tfrac{n-1}{r} + \tfrac{r}{2}\right)\psi_1 + \psi_1' + u^p - \lambda u^q, \end{aligned} \tag{14}$$

$$\begin{aligned} Q_2 = &\ -\left(\tfrac{n-1}{r} + \tfrac{r}{2}\right) v - u^p + \lambda u^q - \psi_2' \\ \geq &\ -\left(\tfrac{n-1}{r} + \tfrac{r}{2}\right)\psi_2 - \psi_2' - u^p + \lambda u^q. \end{aligned} \tag{15}$$

Moreover, in case of *Theorem 1* we have

$$\varphi(r) = \alpha r^{-n}\exp\left(-\frac{r^2}{4}\right),\quad \rho(r) = \eta r^{-n-s}\exp\left(-\frac{r^2}{4}\right) \tag{16}$$

and in view of (14) and (15) scalar products Q_i on V_i are

$$Q_1 \geq \left[2n\alpha + \frac{1}{2}\eta s r^{2-s} + \eta(n+s)(s+2)r^{-s}\right] r^{-n-2}\exp\left(-\frac{r^2}{4}\right) > 0 \quad \text{on } \mathrm{V}_1,$$

$$\begin{aligned} Q_2 \ \geq &\ \left[\frac{\eta s}{2} - 2n\alpha r^{s-2} + \eta(n+s)(s+2)r^{-2}\right] r^{-n-s}\exp\left(-\frac{r^2}{4}\right) \\ &\ - (2\alpha)^p r^{-np}\exp\left(-\frac{r^2}{4}p\right) \\ > &\ \left[\frac{\eta s}{2} - 2n\alpha r^{s-2} + \eta(n+s)(s+2)r^{-2} - (2\alpha)^p\right] r^{-n-s}\exp\left(-\frac{r^2}{4}\right) \\ > &\ \left[\frac{\eta s}{2} - 2n\alpha r^{s-2} - (2\alpha)^p\right] r^{-n-s}\exp\left(-\frac{r^2}{4}\right) \geq 0 \quad \text{on } \mathrm{V}_2. \end{aligned}$$

Consequently, $E = U_1 \cup U_2 \cup V_1 \cup V_2$ is a set of points of strict exit of integral curves of (13) with respect to sets Ω and ω. Hence, according to the retraction method (see [5]), the system (13) has at least one solution $(u(r), v(r))$ which satisfies the conditions

$$|u(r) - \varphi(r)| < \rho(r), \quad |v(r) - \varphi'(r)| < -\rho'(r) \quad \forall r \geq a.$$

That means that *Theorem 1* holds true.

In case of *Theorem 2* we have the functions φ and ρ which are defined by (16). Here, it is sufficient to notice that scalar products Q_i are

$$\begin{aligned} Q_1 &> \left[2n\alpha r^{-n-2} + \frac{1}{2}\eta s r^{-n-s}\right] \exp\left(-\frac{r^2}{4}\right) - \lambda (2\varphi)^q \\ &> \left[\frac{1}{2}\eta s - \lambda (2\alpha)^q\right] r^{-n-s} \exp\left(-\frac{r^2}{4}\right) > 0 \quad \text{on } \mathrm{V}_1, \\ Q_2 &> \left[-2n\alpha r^{-n-2} + \frac{1}{2}\eta s r^{-n-s}\right] \exp\left(-\frac{r^2}{4}\right) - (2\varphi)^p \\ &> \left[\frac{1}{2}\eta s - 2n\alpha a^{s-2} - (2\alpha)^p\right] r^{-n-s} \exp\left(-\frac{r^2}{4}\right) \geq 0 \quad \text{on } \mathrm{V}_2. \end{aligned}$$

In case of the *Theorem 3* we have

$$\begin{aligned} Q_1 &\geq \varphi'' + \left(\frac{n-1}{r} + \frac{r}{2}\right)\varphi' + \rho'' + \left(\frac{n-1}{r} + \frac{r}{2}\right)\rho' + (\varphi - \rho)^p \\ &= H(\varphi) + H(\rho) + (\varphi - \rho)^p > 0 \quad \text{on} \mathrm{V}_1, \\ Q_2 &\geq -\varphi'' - \left(\frac{n-1}{r} + \frac{r}{2}\right)\varphi' + \rho'' + \left(\frac{n-1}{r} + \frac{r}{2}\right)\rho' - (\varphi + \rho)^p \\ &= -H(\varphi) + H(\rho) - (\varphi + \rho)^p > 0 \quad \text{on} \mathrm{V}_2. \end{aligned}$$

For the proof of *Theorem 4* it is sufficient to notice that

$$\begin{aligned} Q_1 &\geq \varphi'' + \left(\frac{n-1}{r} + \frac{r}{2}\right)\varphi' + \rho'' + \left(\frac{n-1}{r} + \frac{r}{2}\right)\rho' + (\varphi - \rho)^p - \lambda(\varphi + \rho)^q \\ &= H(\varphi) + H(\rho) + (\varphi - \rho)^p - \lambda(\varphi + \rho)^q > 0 \quad \text{on} \mathrm{V}_1, \\ Q_2 &\geq -\varphi'' - \left(\frac{n-1}{r} + \frac{r}{2}\right)\varphi' + \rho'' + \left(\frac{n-1}{r} + \frac{r}{2}\right)\rho' - (\varphi + \rho)^p + \lambda(\varphi - \rho)^q \\ &= -H(\varphi) + H(\rho) - (\varphi + \rho)^p + \lambda(\varphi - \rho)^q > 0 \quad \text{on } \mathrm{V}_2. \end{aligned}$$

4. Some particular results

Using the obtained theorems (and their proofs), we can give the following particular results, with

$$\varphi(r) = \alpha r^{-n} \exp\left(-\frac{r^2}{4}\right).$$

Corollary 1. *If* $\lambda = 0$,

$$p \geq 1 + \frac{1}{n}, \quad a \geq \max\left\{4, \ \frac{n}{1-(2\alpha)^{p-1}}\right\}, \quad 0 < \alpha < \frac{1}{2}, \tag{17}$$

then the equation (2) $(\lambda = 0)$ *has at least one positive solution* $u(r)$ *satisfying the conditions*

$$\left| u(r) - \alpha r^{-n} \exp\left(-\frac{r^2}{4}\right)\right| < 4\alpha r^{-n-1} \exp\left(-\frac{r^2}{4}\right), \tag{18}$$

$$\left| u'(r) + \alpha\left(\frac{1}{2}+\frac{n}{r^2}\right) r^{-n+1} \exp\left(-\frac{r^2}{4}\right)\right| < 4\alpha\left(\frac{1}{2}+\frac{n+1}{r^2}\right) r^{-n} \exp\left(-\frac{r^2}{4}\right) \tag{19}$$

$\forall r \geq a$.

Here we have

$$\rho(r) = 4\alpha r^{-n-1} \exp\left(-\frac{r^2}{4}\right).$$

Corollary 2. *If* $\lambda = 0$,

$$p \geq 1 + \frac{2}{n}, \quad a^2 \geq 2n + 2(2\alpha)^{p-1}, \quad \alpha > 0, \tag{20}$$

then the equation (2) $(\lambda = 0)$ *has at least one positive solution* $u(r)$ *satisfying the conditions*

$$\left| u(r) - \alpha r^{-n} \exp\left(-\frac{r^2}{4}\right)\right| < \alpha a^2 r^{-n-2} \exp\left(-\frac{r^2}{4}\right), \tag{21}$$

$$\left| u'(r) + \alpha\left(\frac{1}{2}+\frac{n}{r^2}\right) r^{-n+1} \exp\left(-\frac{r^2}{4}\right)\right| < \alpha a^2\left(\frac{1}{2}+\frac{n+1}{r^2}\right) r^{-n-1} \exp\left(-\frac{r^2}{4}\right) \tag{22}$$

$\forall r \geq a$.

Here we have

$$\rho(r) = \alpha a^2 r^{-n-2} \exp\left(-\frac{r^2}{4}\right). \tag{23}$$

Corollary 3. *Let* (17) *hold true and*

$$q \geq 1 + \frac{1}{n}, \quad 0 < \lambda < (2\alpha)^{1-q}.$$

Then the equation (2) *has at least one positive solution* $u(r)$ *satisfying the conditions* (18) *and* (19) $\forall r \geq a$.

Corollary 4. *Let* (20) *hold true and*

$$q \geq 1 + \frac{2}{n}, \quad 0 < \lambda < (2n\alpha + \eta)(2\alpha)^{-q}.$$

Then the equation (2) *has at least one positive solution* $u(r)$ *satisfying the conditions* (21) *and* (22) $\forall r \geq a$.

Remark 1. *We can note that the obtained results also contain an answer to the question on approximation of solutions* $u(r)$ *whose existence is established. The* *errors of the approximations for solutions* $u(r)$ *and the first derivative* $u'(r)$ *are defined by the function* $\rho(r)$ *which can be sufficiently small* $\forall r \geq a$. *For example, in case of the Corollaries 2 and 4 the function* $\rho(r)$ *is defined by* (23). *This function tends to zero as* $r \to \infty$ *and can be sufficiently small* $\forall r \geq a$, *because parameter* $\alpha > 0$ *can be arbitrarily small.*

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