Two methods for replacing Dirichlet's boundary condition by Robin's boundary condition via penalization*

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Abstract. *In this paper we present two methods for replacing Dirichlet's problem by a sequence of Robin's problems. We study the linear parabolic equation as a model problem. We use the first method for the problem with irregular boundary data and the second for irregular domain.*

Key words: *penalization, Robin's boundary condition, parabolic equation, singular perturbation*

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1. Introduction

We study the Dirichlet's problem for a partial differential equation (we use a linear parabolic equation as a model problem). Our goal is to prove that the (geometric) Dirichlet's condition can be *replaced* by penalized (natural) Robin's condition. In fact, we construct a sequence of Robin's problems (easier to solve than the original Dirichlet's problem) having a sequence of solutions that converges to the solution of our Dirichlet's problem. We use two different approaches: *the method of growing friction* and *the method of artificial domain*, introduced in [6] and [4], respectively, for the Stokes system. The first method should be used for irregular boundary data while the second is convenient when the domain is not smooth or not simply connected.

Our model problem is the following:
Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain, $T > 0$, and let $A \in L^\infty(\Omega; Sym(\mathbb{R}))$ be a strictly positive matrix function (i.e. $\xi \cdot A(x)\xi \geq c_0|\xi|^2$, $x \in \Omega$, $\xi \in \mathbb{R}^n$). For $f \in L^2(]0,T[\times\Omega)$, $u_0 \in L^2(\Omega)$, $g \in L^2(]0,T[\times\partial\Omega)$ we pose the Cauchy-Dirichlet's problem

$$\frac{\partial u}{\partial t} - \operatorname{div}(A\nabla u) = f \text{ in } \Omega_T =]0,T[\times\Omega \tag{1}$$

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$$u = g \ \text{ on } \ \Gamma_T =]0,T[\times \partial\Omega \tag{2}$$

$$u(0,x) = u_0(x) \ , \ x \in \Omega \tag{3}$$

2. Method of growing friction

This method, originally introduced in [6] for the Stokes system, is based on the simple fact that the slip boundary condition for fluid becomes a no-slip condition as the friction coefficient tends to $+\infty$. In this section we assume in addition that the domain Ω is of class C^2 and that $A \in C^1(\overline{\Omega}; M_n(\mathbb{R}))$. Since the boundary condition $g \notin H^1(0,T;H^{1/2}(\partial\Omega)$ but only $g \in L^2(\Gamma_T)$, we can not apply the lift operator and prove the existence of a weak solution. In fact, we know that $u \notin L^2(0,T;H^1(\Omega))$. We are, therefore, forced to use the notion of the very weak solution.

Definition 1. *We say that $u \in L^2(\Gamma_T)$ is a* very weak *(transposed) solution of (1)-(3) if for any $\phi \in L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))$ such that $\frac{\partial\phi}{\partial t} \in L^2(\Omega_T)$ and $\phi(\,\cdot\,,T) = 0$, we have*

$$-\int_{\Omega_T} u\frac{\partial\phi}{\partial t} - \int_{\Omega_T} div(A\nabla\phi)u + \int_{\Gamma_T} \frac{\partial\phi}{\partial \mathbf{n}_A} g = \int_{\Omega_T} f\phi + \int_\Omega u_0\phi(\,\cdot\,,0) \ , \tag{4}$$

where

$$\frac{\partial}{\partial \mathbf{n}_A} = \sum_{i=1}^n \nu_i \, a_{ij} \, \frac{\partial}{\partial x_j} \ , \ \mathbf{n} = (\nu_1,\ldots,\nu_n) \text{ - exterior unit normale on } \Gamma_T, \ [A]_{ij} = a_{ij} \ .$$

(See Lions and Megenes [5] for a more general definition of the very weak solution for a parabolic equation.)

Remark 1.

(a) If $g \in H^1(0,T;H^{1/2}(\partial\Omega))$ and $u \in L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$ is a weak solution for (1)-(3), then u is also a very weak solution for (1)-(3).

(b) A very weak solution satisfies the equation (1) in the sense of distributions $\mathcal{D}'(\Omega_T)$.

(c) We can define the generalized trace operator

$$\tau : L^2(\Omega_T) \to H^{-1}(0,T;H^{-1/2}(\Omega))$$

by

$$\langle\tau(u)|\psi\rangle = \int_{\Omega_T} u\frac{\partial\phi}{\partial t} + \int_{\Omega_T} div(A\nabla\phi)u + \int_{\Omega_T} f\phi \ , \ \psi \in H^1(0,T;H^{1/2}(\Gamma))$$

where $\phi \in L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))$ is such that $\frac{\partial\phi}{\partial t} \in L^2(\Omega_T)$, $\phi(\,\cdot\,,0) = \phi(\,\cdot\,,T) = 0$ and $\frac{\partial\phi}{\partial \mathbf{n}_A} = \psi$ on Γ_T . Then, the very weak solution satisfies the boundary condition (2) in the sense

$$\tau(u) = g$$

(d) *In order to give the meaning to the initial condition we define the mapping* $\mathcal{J} : L^2(\Omega_T) \to H^{-2}(\Omega)$ *by*

$$\langle \mathcal{J}(u)|z\rangle = -\int_{\Omega_T} u\frac{\partial \phi}{\partial t} - \int_{\Omega_T} div(A\nabla\phi)u - \int_{\Omega_T} f\phi \ , \ z \in H_0^2(\Omega)$$

where $\phi \in L^2(0,T;H_0^2(\Omega)) \cap C([0,T];L^2(\Omega))$ *is such that* $\frac{\partial\phi}{\partial t} \in L^2(\Omega_T)$, $\phi(\,\cdot\,,T) = 0$, $\phi(\,\cdot\,,0) = z$ *. Then our very weak solution satisfies the initial condition (3) in the generalized sense*

$$\mathcal{J}(u) = u_0 \ .$$

Remark 2. *Another approach to the boundary value problems with* L^2 *boundary data, based on the theory of potentials, was given in [1], [2]. They give a meaning to the boundary condition using the notion of the* nontangential convergence. *However, it is not clear whether this method could be used for evolutional problems.* The idea of the method is to form a sequence of problems by keeping the same equation and the same initial condition

$$\frac{\partial u^m}{\partial t} - \mathrm{div}\,(A\nabla u^m) = f \ \text{ in } \ \Omega_T =]0,T[\, \times \Omega \tag{5}$$

$$u^m(0,x) = u_0(x) \ , \ x \in \Omega \tag{6}$$

and to replace our Dirichlet's condition (2) by Robin's condition

$$\frac{\partial u^m}{\partial \mathbf{n}_A} + m(u^m - g) = 0 \ \text{ on } \ \Gamma_T \ \ , \ m \in \mathbb{N} \ . \tag{7}$$

Since Robin's condition is a dynamic (or natural) condition, problem (5)-(7) can be written in the variational form as

$$\frac{d}{dt}(u^m(t),v)_{L^2(\Omega)} + (A\nabla u^m(t),v)_{L^2(\Omega)} + m(u^m(t),v)_{L^2(\partial\Omega)} =$$

$$= (f(t),v)_{L^2(\Omega)} + m(g(t),v)_{L^2(\partial\Omega)} \ \ , \ v \in H^1(\Omega) \tag{8}$$

$$u^m(0) = u_0 \tag{9}$$

Using the standard Galerkin's procedure (see e.g. Raviart and Thomas [7]) and the Poincaré's inequality

$$|v|_{L^2(\Omega)} \le C(|\nabla v|_{L^2(\Omega)} + |u|_{L^2(\partial\Omega)}) \ , \ v \in H^1(\Omega) \ ,$$

we can prove that

Theorem 1. *For any* $m \in \mathbb{N}$ *problem (8)-(9) has a unique solution*

$$u^m \in L^2(0,T,H^1(\Omega)) \cap C([0,T];L^2(\Omega)).$$

Such solution is usually called *a weak solution* for (5)-(7).

From the variational formulation (8) we directly get

$$(A\nabla u^m, \nabla u^m)_{L^2(\Omega_T)} + m|u^m|^2_{L^2(\Gamma_T)} + |u^m(T)|^2_{L^2(\Omega)} =$$

$$= |u_0|^2_{L^2(\Omega)} + (f,u^m)_{L^2(\Omega_T)} + m(g,u^m)_{L^2(\Gamma_T)}$$

implying:

Lemma 1.

$$|u^m|_{L^2(\Gamma_T)} \leq C \tag{10}$$
$$|\nabla u^m|_{L^2(\Omega_T)} \leq C\sqrt{m} \ . \tag{11}$$

In order to pass to the limit we need the following estimate:

Lemma 2.

$$|u^m|_{L^2(\Omega_T)} \leq C \ . \tag{12}$$

Proof. Let v be the solution of the homogeneous Cauchy-Dirichlet's problem

$$-\frac{\partial v}{\partial t} - \operatorname{div}(A\nabla v) = u^m \ \text{ in } \ \Omega_T \tag{13}$$
$$v = 0 \ \text{ on } \ \Gamma_T \ , \ u(T,x) = 0 \ , \ x \in \Omega \ . \tag{14}$$

Using the regularity of the right-hand side ($u^m \in L^2(0,T;H^1(\Omega))$) we have that $v \in L^2(0,T;H^2(\Omega))$, $\frac{\partial v}{\partial t} \in L^2(\Omega_T)$ and

$$|v|_{L^2(0,T;H^2(\Omega))} + |\frac{\partial v}{\partial t}|_{L^2(\Omega_T)} \leq C|u^m|_{L^2(0,T;H^1(\Omega))} \tag{15}$$

(see e.g. [5]). Using u^m as a test function in (13)-(14), we obtain

$$|u^m|^2_{L^2(\Omega_T)} = (f,v)_{L^2(\Omega_T)} - (\frac{\partial v}{\partial \mathbf{n}_A}, u^m)_{L^2(\Gamma_T)} \ .$$

Now (10) and (15) imply (12). □

As the consequence of *Lemma 1* we can prove:

Lemma 3.

$$u^m|_{\Gamma_T} \rightharpoonup g \ \textit{ weakly in } \ L^2(\Gamma_T) \ . \tag{16}$$

Proof. The estimate (10) implies that, up to a subsequence, $u^m|_{\Gamma_T}$ converges weakly in $L^2(\Gamma_T)$ to some limit. To identify that limit we take arbitrary $\phi \in H^1_0(0,T;H^1_0(\Omega))$ and we use (5)-(7) to get

$$(u^m - g,\phi)_{L^2(\Gamma_T)} = \frac{1}{m}\{-(A\nabla u^m,\nabla\phi)_{L^2(\Omega_T)} + (u^m, f + \frac{\partial \phi}{\partial t})_{L^2(\Omega_T)}\} \leq \frac{C}{\sqrt{m}} \to 0$$

implying the claim. □

We are now ready to prove the main result.

Theorem 2.

$$u^m \rightharpoonup u \ \textit{ weakly in } \ L^2(\Omega_T) \ , \tag{17}$$

where u is a unique very weak solution of (1)-(3).

Proof. Let $\phi \in H^1(0,T;H^2(\Omega)\cap H^1_0(\Omega))$, $\phi(\,\cdot\,,T) = 0$. Then

$$\begin{aligned}&-(u^m,\frac{\partial\phi}{\partial t})_{L^2(\Omega_T)} - (\operatorname{div}(A\nabla\phi),u^m)_{L^2(\Omega_T)} + (\frac{\partial\phi}{\partial \mathbf{n}_A},u^m)_{L^2(\Gamma_T)} = \\ &= (f,\phi)_{L^2(\Omega_T)} + (u_0,\phi(\,\cdot\,,0))_{L^2(\Omega)} \ .\end{aligned} \tag{18}$$

Using (12) we extract a subsequence such that

$$u^m \rightharpoonup \tilde{u} \text{ weakly in } L^2(\Omega_T)$$

for some $\tilde{u} \in L^2(\Omega_T)$. Passing to the limit in (18) we prove that $\tilde{u} = u$ is the unique solution of (4). Uniqueness is the consequence of linearity and it proves that the whole sequence converges. □

Remark 3.

(a) *The described method can be analogously used for linear elliptic and hyperbolic equations. The sequence u^m, unfortunatelly, converges only weakly in $L^2(\Omega_T)$ so that the method can not be generalized to nonlinear problems.*

(b) Lemma 2 *is the only place where we have used the C^2 regularity of Ω to prove $L^2(0,T;H^2(\Omega))$ regularity of v, the solution of problem (13)-(14). Using the regularity results for polygonal domains from Grisvard [3], our result can be extended to the case when Ω is a convex polygon.*

3. Method of artificial domain

In this section we take the homogeneous boundary condition $g = 0$, but we do not suppose any regularity on the domain Ω. The idea of this method (due to Lions [4]) is to take the simplest possible smooth domain (e.g. ball) $\mathcal{O}$ containing Ω and consider the equation

$$\frac{\partial u^m}{\partial t} - \operatorname{div}(A\nabla u^m) + m\, \mathbf{1}_{\mathcal{O}\backslash\Omega}\, u^m = \tilde{f} \text{ in } \mathcal{O}_T =]0,T[\times\mathcal{O} \tag{19}$$

where $\mathbf{1}_{\mathcal{O}\backslash\Omega}$ is the characteristic function of the set $\mathcal{O}\backslash\Omega$, $\tilde{f} \in L^2(\mathcal{O}_T)$ is an extension by zero of f and $m \in \mathbb{N}$. Extending u_0 to $\mathcal{O}$ by zero and denoting that extension by $\tilde{u}_0$ we pose the initial condition

$$u^m(0,x) = \tilde{u}_0(x)\ ,\ x \in \mathcal{O}\ . \tag{20}$$

On $\Sigma = \partial\mathcal{O}$ we can actually pose any *reasonable* boundary condition (Dirichlet's, Neumann's, periodicity (if $\mathcal{O}$ is chosen as a cube), Robin's...). We choose Robin's condition

$$\frac{\partial u^m}{\partial \mathbf{n}_A} + u^m = 0 \text{ on } \Sigma_T =]0,T[\times\Sigma\ . \tag{21}$$

As in the previous section we conclude that the variational form of (19)-(21)

$$\frac{d}{dt}(u^m(t),v)_{L^2(\mathcal{O})} + (A\nabla u^m(t),v)_{L^2(\mathcal{O})} + (u^m(t),v)_{L^2(\Sigma)} + \tag{22}$$

$$+m(u^m(t),v)_{L^2(\mathcal{O}\backslash\Omega)} = (\tilde{f}(t),v)_{L^2(\mathcal{O})}\ ,\ v \in H^1(\mathcal{O}) \tag{23}$$

$$u^m(0) = \tilde{u}_0 \tag{24}$$

has a unique solution $u^m \in L^2(0,T;H^1(\mathcal{O})) \cap C([0,T];L^2(\mathcal{O}))$. Moreover, we have:

Lemma 4.

$$|u^m|_{L^\infty(0,T;L^2(\mathcal{O}))} \leq C \tag{25}$$
$$|u^m|_{L^2(0,T;H^1(\mathcal{O}))} \leq C \tag{26}$$
$$|u^m|_{L^2(0,T;L^2(\mathcal{O}\setminus\Omega))} \leq \frac{C}{m} \quad . \tag{27}$$

We can now prove the convergence of the penalization procedure.

Theorem 3. *There exists a subsequence* $\{u^{m'}\}$ *of* $\{u^m\}$ *and*

$$w \in L^2(0,T;H^1(\mathcal{O})) \cap L^\infty(0,T;L^2(\mathcal{O}))$$

such that

$$u^{m'} \rightharpoonup w \ \ \textit{weakly in} \ \ L^2(0,T;H^1(\mathcal{O})) \ \ \textit{and weak* in} \ \ L^\infty(0,T;L^2(\mathcal{O})) \tag{28}$$

where the restriction $u = w|_\Omega \in L^2(0,T;H_0^1(\Omega))$ *is a unique weak solution of (1)-(3).*

Proof. Estimates (25)-(26) imply (28) for some w. Furthermore, (27) gives $w|_{\mathcal{O}\setminus\Omega} = 0$ and then $w|_\Omega \in L^2(0,T;H_0^1(\Omega))$. For $\phi \in H^1(0,T;H^1(\mathcal{O}))$, such that $\phi(T,x) = 0$ we have

$$-(u^m, \frac{\partial\phi}{\partial t})_{L^2(\mathcal{O}_T)} + (A\nabla u^m, \nabla\phi)_{L^2(\mathcal{O}_T)} + m(u^m,\phi)_{L^2(\mathcal{O}_T\setminus\Omega_T)} =$$
$$= (f,\phi)_{L^2(\Omega_T)} + (u_0, \phi(0,\cdot))_{L^2(\Omega)} \quad .$$

Now, if ϕ is chosen such that $supp\,\phi(t,\cdot) \subset \Omega$ for (a.e.) $t \in]0,T[$, then the term $(u^m,\phi)_{L^2(\mathcal{O}_T\setminus\Omega_T)} = 0$. Passing to the limit as $m \to \infty$ we obtain

$$-(w, \frac{\partial\phi}{\partial t})_{L^2(\mathcal{O}_T)} + (A\nabla w, \nabla\phi)_{L^2(\mathcal{O}_T)} = (f,\phi)_{L^2(\Omega_T)} + (u_0, \phi(0,\cdot))_{L^2(\Omega)} \quad .$$

□

Remark 4. *We can use this method for a very large number of problems including the nonlinear ones, because the sequence* u^m *converges strongly in* $L^2(0,T;H^1(\Omega))$ *(under certain technical assumptions). Furthermore, one can prove the estimate for* $|u^m - u|_{L^2(0,T;H^1(\Omega))}$ *in terms of* m *(see [4] for details).*

References

[1] B. E. J. Dahlberg, *On the Poisson integral for Lipschitz and* C^1 *domains*, Studia Math. **66**(1979), 13–24.

[2] E. B. Fabes, C. E.Kenig, G. C. Verchota, *The Dirichlet problem for the Stokes system on Lipschitz domains*, Duke Math.J. **57**(1988), 769–193.

[3] P. Grisvard, *Singularities in boundary value problems*, Collection Recherche en Mathématiques Appliquées, RMA 22, Masson, Paris, 1992.

[4] J. L. Lions, *Some Methods in the Mathematical Analysis of Systems and Their Control*, Gordon and Breach, New York, 1981.

[5] J. L. LIONS, E. MEGENES, *Problèmes aux limites non homogènes et applications, Vol. 2*, Dunod, Paris, 1968.

[6] E. MARUŠIĆ-PALOKA, *Solvability of the Navier-Stokes system with L^2 boundary data*, Appl.Math.Optim., to appear.

[7] P. A. RAVIART, J. M. THOMAS, *Introduction à l'analyse numérique des équations aux dérivées partielles,* 2^e *tirage*, Masson, Paris, 1988.