On construction of fourth order Chebyshev splines*

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Abstract. *It is an important fact that general families of Chebyshev and L-splines can be locally represented, i.e. there exists a basis of B-splines which spans the entire space. We develop a special technique to calculate with* 4^{th} *order Chebyshev splines of minimum deficiency on nonuniform meshes, which leads to a numerically stable algorithm, at least in case one special Hermite interpolant can be constructed by stable explicit formulæ. The algebraic derivation of the algorithm involved makes it possible to apply the construction to L-splines. The underlying idea is an Oslo type algorithm, combined with the known derivative formula for Chebyshev splines.*

We then show that weighted polynomial and tension spline spaces satisfy the conditions imposed, and show how to apply the above general techniques to obtain local representations.

Key words: *Chebyshev spline, B-spline, knot insertion, recurrence*

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1. Introduction

One of the great things in the univariate spline theory is the fact that the most general spline functions can be represented as linear combinations of splines having a compact support, the so-called *B-splines*. This leads to sparse matrix systems in diverse fields of numerical analysis, where entries are usually calculated as linear combinations of B-splines and their derivatives. Therefore, one has to evaluate such splines as accurately as possible; the complexity in calculation can also play a significant role. Such algorithms have been found only in cases which are very important, but do not cover all interesting situations. Fully investigated cases include polynomial [16], hyperbolic [17], trigonometric [9], and also some rarely used *modified splines*, obtained by substitutions in the recurrence relations for known B-splines. Most algorithms are based on three term recurrence relation due to de Boor and Cox. For the sake of reference, and also to introduce the notation to be used further, let us choose the knot sequence $\Delta = \{x_0, \ldots, x_{k+1}\}$, and, for a given integer

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n and multiplicity vector $\boldsymbol{m} = (n_1, \ldots, n_k)^T$ let us define an extended partition of the interval $[a, b]$ $\{t_1 \ldots t_{2n+k}\}$ in the usual way:

$$\begin{aligned} t_1 = \ldots = t_n &= a \\ t_{n+k+1} = \ldots = t_{2n+k} &= b \\ t_{n+1} \leq \ldots \leq t_{n+k} = \underbrace{x_1, \ldots, x_1}_{n_1}, &\ldots, \underbrace{x_k, \ldots, x_k}_{n_k} \end{aligned} \tag{1}$$

where n_i are integers satisfying $1 \leq n_i < n$.

In the polynomial case, the three-term recurrence

$$B_i^n(x) = \frac{x - t_i}{t_{i+n-1} - t_i} B_i^{n-1}(x) + \frac{t_{i+n} - x}{t_{i+n} - t_{i+1}} B_{i+1}^{n-1}(x) \tag{2}$$

enables stable calculating with B-splines B_i^n; they are locally supported ($\mathrm{supp}(B_i^n) = [t_i, t_{i+n}]$), and smoothness is governed by the multiplicities, meaning that $n - n_i - 1$ derivatives match at the knot of the multiplicity n_i. No generalization of (2) exists, at least with analytic functions in convex combinations of lower order splines; for precise descriptions see [19]. It is interesting that matching other linear functionals at the knots (like jumps of derivatives) can lead to recurrence relations like (2) [13]. The most general case of four term recurrence is treated in [5]; there also exists a three term recurrence [8] in which the "lower order" B-splines are substituted by some other quantities which do not appear to be B-splines in any other space. Both constructs can not be readily used in algorithms, and involve a considerable amount of work to be done analytically.

It is not necessary, however, to have a recurrence relation belonging to the above classes in order to calculate stably with B-splines. Continuous and Hermite splines, for instance, can be *locally* determined by interpolation, since we know enough information on each interval, and thus seek for a nontrivial function that has some function and derivative values equal to zero. A spline can be determined locally in the case of the multiplicity vector $\boldsymbol{m} = (n_i)^{\mathrm{T}}$, $n_i \geq n - 2$. This can be done for all Chebyshev systems, since the very definition of a Chebyshev system [17]) implies the possibility of Hermite interpolation. Since B-splines of higher smoothness, that is, associated with the knots possessing smaller multiplicities, can be written as linear combinations of not so smooth B-splines of the same order, this can in turn yield an efficient algorithm – provided that:

1. the coefficients in the linear combination can be found exactly, and

2. the coefficients are positive.

These sufficient conditions will enable us to calculate with B-splines by making scalar products of positive quantities – numerically almost as sound a thing to do as performing convex combinations of lower order splines like in (2). The main result shows that it is possible, in the case of Chebyshev splines of order 4, to obtain analytic formulæ for the coefficients under reasonable assumptions. The technique used is that of some special (simultaneous) knot insertion, known in CAGD as *Oslo* algorithm (see [10] and references therein). Though it can not be extended to arbitrary order in a straightforward way, one can note that in practice, higher order splines are seldom used.

2. Notation and preliminaries

To make the complex notation involved in Chebyshev spline theory simpler, we begin by introducing the linear operators of duality and reduction:

$$\textit{duality:}\qquad D_i = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}, \qquad D_i : \mathcal{R}^i \to \mathcal{R}^i\ ,\ i = 1, \cdots n-1,$$

$$\textit{reduction:}\qquad R_i = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \qquad R_i : \mathcal{R}^i \to \mathcal{R}^{i-1},\ i = 2, \cdots n-1.$$

Now, let us consider an interval $\delta \subseteq [a,b]$, measurable with respect to the Stieltjes measures $d\sigma_2, \ldots d\sigma_n$. If we define $d\boldsymbol{\sigma} := (d\sigma_2(\delta), \ldots d\sigma_n(\delta))^T \in \mathcal{R}^{n-1}$ to be the *density vector*, then a canonical Chebyshev system (or CCT-system) $\mathcal{S}(n, d\boldsymbol{\sigma})$ of order n is a family of functions $\{u_1, \ldots u_n\}$ that can be represented in the form:

$$\begin{aligned} u_2(x) &= u_1(x) \int_a^x d\sigma_2(t_2) \\ &\vdots \\ u_n(x) &= u_1(x) \int_a^x d\sigma_2(t_2) \ldots \int_a^{t_{n-1}} d\sigma_n(t_n). \end{aligned}$$

If all of the measures $d\sigma_i$ are dominated by the Lebesgue measure, then they possess densities $\frac{1}{p_i}$, $i = 2, \ldots n$; if p_i are smooth, i.e. $\dfrac{1}{p_i} := \dfrac{d\sigma_i}{dt} \in C^{n-i+1}$, the functions form an *Extended Complete Chebyshev System (ECT-system)* [18].

The operators of duality and reduction can now be used to define reduced, dual, and reduced dual Chebyshev systems, as Chebyshev systems defined, respectively, by appropriate measure vectors:

j-reduced system:	$\mathcal{S}(n-j, R_{n-j} \cdots R_{n-1} d\boldsymbol{\sigma})$
dual system:	$\mathcal{S}(n, D_{n-1} d\boldsymbol{\sigma})$
j-reduced dual system:	$\mathcal{S}(n-j, R_{n-j} \cdots R_{n-1} D_{n-1} d\boldsymbol{\sigma})$

$(j = 1, \ldots n-1)$.

If we define operators D_j as certain measure derivatives via formulæ

$$D_j f(x) \ := \lim_{\delta \to 0+} \frac{f(x+\delta) - f(x)}{\sigma_{j+1}(x+\delta) - \sigma_{j+1}(x)} \qquad j = 1, \ldots, n-1,$$

then the *generalized derivatives* are defined to be the linear operators $L_{j,d\boldsymbol{\sigma}} : \mathcal{S}(n, d\boldsymbol{\sigma}) \to \mathcal{S}(n-j, R_{n-j} \cdots R_{n-1} d\boldsymbol{\sigma})$, $L_{j,d\boldsymbol{\sigma}} := D_j \cdots D_1 D_0$, where $D_0 f := f/u_1$. The generalized derivatives exhibit the same behaviour as the ordinary ones applied to powers of x:

$$L_{j,d\boldsymbol{\sigma}} u_i = \begin{cases} 0 & i = 1, 2, \ldots j \\ u_{j,i-j} & i = j+1, \ldots, n. \end{cases}$$

The version of the fundamental theorem of integral calculus also holds:

Lemma 1. *Let* $\{u_i\}_{i=1}^n$ *be the CCT-system on* $[a,b]$, $f \in \mathrm{L}\{u_1,\dots,u_n\}$ *and* $a \le c \le d \le b$. *Then*

$$\frac{f(d)}{u_1(d)} - \frac{f(c)}{u_1(c)} = \int_c^d L_{1,d\boldsymbol{\sigma}} f(t) d\sigma_2(t). \tag{3}$$

Proof. Because of linearity, we can prove (3) for $f = u_j$; let also $j = 3$ for the sake of brevity. Then

$$\begin{aligned}
\int_c^d D_1 D_0 u_3(t) d\sigma_2(t) = & \int_c^d D_1 \tfrac{u_3}{u_1}(t) d\sigma_2(t) = \int_c^d D_1 \int_a^t \int_a^{t_2} d\sigma_3(t_3) d\sigma_2(t_2) d\sigma_2(t) \\
= & \int_c^d \lim_{\delta\to 0+} \tfrac{\int_a^{t+\delta}\int_a^{t_2} d\sigma_3 d\sigma_2(t_2) - \int_a^t \int_a^{t_2} d\sigma_3(t_3) d\sigma_2(t_2)}{\sigma_2(t+\delta)-\sigma_2(t)} d\sigma_2(t) \\
= & \int_c^d \lim_{\delta\to 0+} \tfrac{\int_t^{t+\delta} d\sigma_2(t_2) \int_a^{t_2} d\sigma_3(t_3)}{\sigma_2(t+\delta)-\sigma_2(t)} d\sigma_2(t) \\
= & \int_c^d \lim_{\delta\to 0+} (-\sigma_3(a) + \tfrac{1}{\sigma_2(t+\delta)-\sigma_2(t)} \int_t^{t+\delta} \sigma_3(t_2) d\sigma_2(t_2) d\sigma_2(t) \\
= & \int_c^d \{-\sigma_3(a) + \lim_{\delta\to 0+} \tfrac{1}{\sigma_2(t+\delta)-\sigma_2(t)} \xi(\delta) \int_t^{t+\delta} d\sigma_2(t_2)\} d\sigma_2(t)
\end{aligned}$$

where $\sigma_3(t) \le \xi(\delta) \le \sigma_3(t+\delta)$.[1] The above expression then simplifies to

$$\begin{aligned}
\int_c^d (-\sigma_3(a) + \sigma_3(t) d\sigma_2(t) = & \int_c^d \int_a^{t_2} d\sigma_3(t_3) d\sigma_2(t_2) \\
= & \int_a^d \int_a^{t_2} d\sigma_3(t_3) d\sigma_2(t_2) - \int_a^c \int_a^t d\sigma_3(t_3) d\sigma_2(t_2) \\
= & \tfrac{u_3(d)}{u_1(d)} - \tfrac{u_3(c)}{u_1(c)}. \qquad \square
\end{aligned}$$

We conclude next that the CCT-system $\mathcal{S}(n, d\boldsymbol{\sigma}) = \{u_1, \dots y_n\}$ consists of functions with positive Wronskian, that is, if $l_i = \max\{j :\ t_i = \cdots = t_{i-j}\}, \quad i = 1,\dots,n,$ then

$$\det[\mathrm{L}_{l_i} u_j(t_i)] > 0, \qquad i = 1,\dots,n;\ j = 1,\dots,n.$$

This follows from the fact that the proof given in [18] relies only on *Lemma 3*.

We define Chebyshev spline spaces in the usual way: for a *partition* (1) $\Delta = \{x_i\}_{i=0}^{k+1}$ of an interval $[a,b]$ and a given *multiplicity vector* $\boldsymbol{m} = (n_1,\dots,n_k)^T$, we define Chebyshev spline space $\mathcal{S}(n, \boldsymbol{m}, d\boldsymbol{\sigma}, \Delta)$ as a space of functions satisfying

(i) for all $s \in \mathcal{S}(n, \boldsymbol{m}, d\boldsymbol{\sigma}, \Delta)$, $i = 0,\dots,k$, there exists $s_i \in \mathcal{S}(n, d\boldsymbol{\sigma})$ such that $s|_{\Delta_i} = s_i|_{\Delta_i}$, ($\Delta_i := (x_i, x_{i+1})$ and

(ii) $L_{j,d\boldsymbol{\sigma}} s_{i-1}(x_i) = L_{j,d\boldsymbol{\sigma}} s_i(x_i)$ for $j = 0,\dots n-1-n_i$, $i = 1,\dots,k$.

[1]Should σ_3 be continuous, by the classical mean value theorem one obtains:

$$\int_t^{t+\delta} \sigma_3(t_2) d\sigma_2(t_2) = \sigma_3(\xi) \int_t^{t+\delta} d\sigma_2(t_2)$$

for some ξ, $t \le \xi \le t+\delta$

Since the construction of locally supported splines in $\mathcal{S}(n, \boldsymbol{m}, d\boldsymbol{\sigma}, \Delta)$ is purely algebraic, by using only *Lemma 1* and determinant identities, we infer that such splines exist, even for noncontinuous σ_i [16, 19]. In what follows we shall assume that $T^n_{i,d\boldsymbol{\sigma}} \in \mathcal{S}(n, \boldsymbol{m}, d\boldsymbol{\sigma}, \Delta)$ are unique such splines, called *Chebyshev B-splines*, possessing compact support $[t_i, t_{i+n}]$, over which they are positive, and satisfy the *partition of unity*:

$$\sum_{i=1}^{n+K} T^n_{i,d\boldsymbol{\sigma}}(x) = 1, \qquad K := \sum_i n_i. \tag{4}$$

By a more recent result $T^n_{i,d\boldsymbol{\sigma}}$ can also be defined recursively by B-splines in reduced systems. The well known derivative formula for polynomial B-splines can be generalized to Chebyshev splines:

Theorem 1. *Let $L_{1,d\boldsymbol{\sigma}}$ be the first generalized derivative with respect to the CCT-system $\mathcal{S}(n, d\boldsymbol{\sigma})$, and let the multiplicity vector $\boldsymbol{m}$ satisfy $n_i < n-1$ for $i = 1, \dots k$. Then for $x \in [a,b]$ and $i = 1, \dots, n + \sum_{i=1}^{k} n_i$ the following derivative formula holds:*

$$L_{1,d\boldsymbol{\sigma}} T^n_{i,d\boldsymbol{\sigma}}(x) = \frac{T^{n-1}_{i,R_{n-1}d\boldsymbol{\sigma}}(x)}{C_{n-1}(i)} - \frac{T^{n-1}_{i+1,R_{n-1}d\boldsymbol{\sigma}}(x)}{C_{n-1}(i+1)} \tag{5}$$

where

$$C_{n-1}(i) := \int_{t_i}^{t_{i+n-1}} T^{n-1}_{i,R_{n-1}d\boldsymbol{\sigma}} d\sigma_2. \tag{6}$$

Proof. The “smooth” version of (5) was used in [2] to define Chebyshev B-splines, and is also used in [14]. The purely algebraic proof is somewhat longer and can be found in [15]. □

The recursive definition of Chebyshev splines implied by integration of the derivative formula (5) cannot be used efficiently for numerical calculation of local basis, even for polynomial splines. We will show that for $n = 4$, under some mild additional hypotheses, one can bypass the inherent instability involved in (5), and obtain a stable numerical algorithm.

3. Local basis

Let $n = 4$, and $d\lambda$ denote the *Lebesgue measure*. Consider the special CCT-system $\{1, u_2, u_3, u_4\}$:

$$\begin{aligned}
u_2(x) &= \int_0^x d\sigma_2(t_2) \\
u_3(x) &= \int_0^x d\sigma_2(t_2) \int_0^{t_2} d\sigma_3(t_3) \\
u_4(x) &= \int_0^x d\sigma_2(t_2) \int_0^{t_2} d\sigma_3(t_3) \int_0^{t_3} d\lambda(t_4).
\end{aligned}$$

We wish to construct a local basis for the space spanned by these functions, that is B-splines in $\mathcal{S}(4,\boldsymbol{m},d\boldsymbol{\sigma},\Delta)$, $d\boldsymbol{\sigma}=(d\sigma_2,d\sigma_3,d\lambda)^T$; the Chebyshev analog of cubic splines. To this end, we shall make the following hypothesis:

Hypothesis Let $\mathcal{S}(4,\boldsymbol{m},d\boldsymbol{\sigma},\Delta)$ be such that third order Chebyshev B-splines $T^3_{i,R_3d\boldsymbol{\sigma}}$ in the reduced system $\mathcal{S}(4,\boldsymbol{m},R_3d\boldsymbol{\sigma},\Delta)$ can be evaluated at the knots.

By hypothesis, we can evaluate B-splines regardless of the multiplicity of the knots; therefore, we can reinsert each knot, that is define the multiplicity vector $\tilde{\boldsymbol{m}}=(2,\ldots 2)^T$ on the same knot sequence. Since $\mathcal{S}(3,\boldsymbol{m},R_3d\boldsymbol{\sigma},\Delta)\subset\mathcal{S}(3,\tilde{\boldsymbol{m}},R_3d\boldsymbol{\sigma},\Delta)$, we conclude that $\delta^3_j\in\mathcal{R}$ exist such that $T^3_j(x)=\sum_i\delta^3_j(i)\tilde{T}^3_j(x)$. It is not difficult to prove that $\delta^3_j(i)=0$ for $i\notin\{r,r+1,r+2\}$, where r is an index such that $t_j=\tilde{t}_r<\tilde{t}_{r+1}$. It also follows that othherwise $\delta^3_j(i)>0$. In fact, it is not difficult to calculate the $\delta^3_j(i)$ coefficients:

Lemma 2. *Let $T^3_{j,R_3d\boldsymbol{\sigma}}\in\mathcal{S}(3,\boldsymbol{m},R_3d\boldsymbol{\sigma},\Delta)$ be a Chebyshev 3^{rd} order spline associated with the multiplicity vector $\boldsymbol{m}=(1,\ldots 1)^T$, and let us assume that $\tilde{T}^3_{j,R_3d\boldsymbol{\sigma}}\in\mathcal{S}(3,\tilde{\boldsymbol{m}},R_3d\boldsymbol{\sigma}),\Delta)$ are B-splines associated with the multiplicity vector $\tilde{\boldsymbol{m}}=(2,\ldots 2)^T$ on the same knot sequence. If $\{t_1,\ldots t_{k+6}\}$ and $\{\tilde{t}_1,\ldots\tilde{t}_{2k+6}\}$ are the associated extended partitions, and r an index such that $t_j=\tilde{t}_r<\tilde{t}_{r+1}$, then for $j=1,\ldots k+3$:*

$$T^3_{j,R_3d\boldsymbol{\sigma}}=T^3_{j,R_3d\boldsymbol{\sigma}}(t_{j+1})\tilde{T}^3_{r,R_3d\boldsymbol{\sigma}}+\tilde{T}^3_{r+1,R_3d\boldsymbol{\sigma}}+T^3_{j,R_3d\boldsymbol{\sigma}}(t_{j+2})\tilde{T}^3_{r+2,R_3d\boldsymbol{\sigma}}.$$

Proof. Let us temporarily drop the second index in $T^3_{j,R_3d\boldsymbol{\sigma}}$, since $d\sigma_3$ does not play any role in the proof. By utilizing the fact that $T^3_j(t_j)=T^3_j(t_{j+3})=0$, the same being true for the first generalized derivatives, we conclude that two out of three coefficients representing T^3_j on each interval of its support are zero. For $x\in(t_j,t_{j+1})$ we have $T^3_j(t_{j+1})=\delta^3_j(r)\tilde{T}^3_r(t_{j+1})$. Since (4) applied to $\mathcal{S}(3,\tilde{\boldsymbol{m}},R_3d\boldsymbol{\sigma},\Delta)$ implies that $\tilde{T}^3_r(t_{j+1}=1$, it remains to show that the middle coefficient is equal to 1. For $x\in(t_j,t_{j+1})$ partition of unity (4) gives

$$T^3_{j-1}(x)+T^3_j(x)+T^3_{j+1}(x)\quad=\quad 1.$$

We expand $T^3_i,\ i\in\{j-1,j,j+1\}$ in $\mathcal{S}(3,\tilde{\boldsymbol{m}},R_3d\boldsymbol{\sigma},\Delta)$, and rearrange the terms to obtain

$$\begin{aligned}1\quad=\quad&\tilde{T}^3_r(x)[T^3_{j-1}(t_{j+1})+T^3_j(t_{j+1})]+\delta^3_j(r+1)\tilde{T}^3_{r+1}(x)+\\&\tilde{T}^3_{r+2}(x)[T^3_j(t_{j+2})+T^3_{j+2}(t_{j+2})].\end{aligned}$$

Expressions in [ ] are equal to 1 by (4) applied at the knots t_{j+1},t_{j+2}. But (4) must also hold for B-splines in $\mathcal{S}(3,\tilde{\boldsymbol{m}},R_3d\boldsymbol{\sigma},\Delta)$, and, since the expansion of unity in this space is unique, we must have $\delta^3_j(r+1)=1$. □

Let us proceed towards a more interesting "cubic" case. First we note that, at least in theory, one can construct two-interval supported $T^4_{j,d\boldsymbol{\sigma}}\in\mathcal{S}(4,\tilde{\boldsymbol{m}},d\boldsymbol{\sigma},\Delta)$ by Hermite interpolation ([3]), since the function and its $L_{1,d\boldsymbol{\sigma}}$ derivatives are known at the knots; this construction is purely local, and uniqueness is guaranteed by (4). Assume that such an interpolant can be constructed and evaluated numerically. We can then try to expand the general B-spline in terms of the less smooth ones which we know, and thus obtain the evaluation formulæ by this special simultaneous knot insertion algorithm, belonging to the class of Oslo algorithms [4].

Theorem 2. *Let* $T^4_{j,d\boldsymbol{\sigma}} \in \mathcal{S}(4, \boldsymbol{m}, d\boldsymbol{\sigma}, \Delta)$, $\tilde{T}^4_{j,d\boldsymbol{\sigma}} \in \mathcal{S}(4, \tilde{\boldsymbol{m}}, d\boldsymbol{\sigma}, \Delta)$, *multiplicity vectors* $\boldsymbol{m}$, $\tilde{\boldsymbol{m}}$ *being as in* Lemma 2 *Then positive* $\delta^4_j(i)$, *dependent on* $d\sigma_3$ *exist, such that*

$$T^4_{j,d\boldsymbol{\sigma}} = \sum_{i=r}^{r+3} \delta^4_j(i)\tilde{T}^4_{i,d\boldsymbol{\sigma}},$$

where $r = r_j$ *satisfies*

$$t_j = \tilde{t}_{r_j} < \tilde{t}_{r_j+1}.$$

Let the extended partitions be $\{t_1, \dots t_{k+8}\}$ *and* $\{\tilde{t}_1, \dots \tilde{t}_{2k+8}\}$. *Then* $\delta^4_j(i)$, $i = r, \dots r+3$ *are determined by the formulæ:*

$$\begin{aligned}
\delta^4_j(r) &= \frac{T^3_{j,R_3d\boldsymbol{\sigma}}(t_{j+1})\tilde{C}(r)}{T^3_{j,R_3d\boldsymbol{\sigma}}(t_{j+1})\tilde{C}(r)+\tilde{C}(r+1)+T^3_{j,R_3d\boldsymbol{\sigma}}(t_{j+2})\tilde{C}(r+2)}\\
\delta^4_j(r+1) &= \frac{T^3_{j,R_3d\boldsymbol{\sigma}}(t_{j+1})\tilde{C}(r)+\tilde{C}(r+1)}{T^3_{j,R_3d\boldsymbol{\sigma}}(t_{j+1})\tilde{C}(r)+\tilde{C}(r+1)+T^3_{j,R_3d\boldsymbol{\sigma}}(t_{j+2})\tilde{C}(r+2)}\\
\delta^4_j(r+2) &= \frac{T^3_{j+1,R_3d\boldsymbol{\sigma}}(t_{j+3})\tilde{C}(r+4)+\tilde{C}(r+3)}{T^3_{j+1,R_3d\boldsymbol{\sigma}}(t_{j+2})\tilde{C}(r+2)+\tilde{C}(r+3)+T^3_{j+1,R_3d\boldsymbol{\sigma}}(t_{j+3})\tilde{C}(r+4)}\\
\delta^4_j(r+3) &= \frac{T^3_{j+1,R_3d\boldsymbol{\sigma}}(t_{j+3})\tilde{C}(r+4)}{T^3_{j+1,R_3d\boldsymbol{\sigma}}(t_{j+2})\tilde{C}(r+2)+\tilde{C}(r+3)+T^3_{j+1,R_3d\boldsymbol{\sigma}}(t_{j+3})\tilde{C}(r+4)}
\end{aligned}$$

where, as in (6)

$$\tilde{C}(i) = \int_{\text{support}} \tilde{T}^3_{i,R_3d\boldsymbol{\sigma}} d\sigma_2.$$

Proof. By an argument similar to that in *Lemma 2*, we obtain $T^4_{j,d\boldsymbol{\sigma}}(x) = \sum_{i=r-2}^{r+6} \delta^4_j(i)\tilde{T}^4_{i,d\boldsymbol{\sigma}}(x)$. After applying the first generalized derivative $L_{1,d\boldsymbol{\sigma}}$ to both sides, according to (5), and rearranging, we have

$$\frac{T^3_{j,R_3d\boldsymbol{\sigma}}(x)}{C_3(j)} - \frac{T^3_{j+1,R_3d\boldsymbol{\sigma}}(x)}{C_3(j+1)} = \sum_i \frac{\delta^4_j(i) - \delta^4_j(i-1)}{\tilde{C}_3(i)} \tilde{T}^3_{i,R_3d\ sig}(x), \tag{7}$$

where $C_3(i) = \int_{t_i}^{t_{i+3}} T^3_{i,R_3d\boldsymbol{\sigma}} d\sigma_2$ and $\tilde{C}_3(i) = \int_{\tilde{t}_i}^{\tilde{t}_{i+3}} \tilde{T}^3_{i,R_3d\ sig} d\sigma_2$. We may integrate the expression for $T_{j,R_3d\boldsymbol{\sigma}}$ in *Lemma 2* with respect to the measure $d\boldsymbol{\sigma}_2$ to obtain

$$C_3(i) = \sum_l \delta^3_i(l)\tilde{C}_3(l),$$

$\tilde{C}_3(l)$ being defined in *Lemma 2* Then (7) leads to the linear system for $\delta_j^4(i)$, $i = r, \ldots r+4$; other coefficients are zero by compact the support argument. The matrix of the system and its inverse are:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

respectively, and the proof follows when we apply the inverse to the right-hand side of the system. □

4. Applications

There is a number of interesting special cases. Let us assume that a partition and a refined partition are as shown in *Figure 1*:

t_1	t_2	t_3	t_4	t_5	t_6	t_7
$\tilde{t}_1$	$\tilde{t}_3$	$\tilde{t}_5$	$\tilde{t}_7$	$\tilde{t}_9$	$\tilde{t}_{11}$	$\tilde{t}_{13}$
$\tilde{t}_2$	$\tilde{t}_4$	$\tilde{t}_6$	$\tilde{t}_8$	$\tilde{t}_{10}$	$\tilde{t}_{12}$	$\tilde{t}_{14}$

Figure 1. *Partition and a refined partition*

If $d\boldsymbol{\sigma} = (d\lambda, d\lambda, d\lambda)^T$, we have the well known formula for polynomial B-splines [4]:

$$B_2^4 \quad = \quad \frac{t_3 - t_2}{t_5 - t_2}\tilde{B}_4^4 + \frac{t_4 - t_2}{t_5 - t_2}\tilde{B}_5^4 + \frac{t_6 - t_4}{t_6 - t_3}\tilde{B}_6^4 + \frac{t_6 - t_4}{t_6 - t_3}\tilde{B}_7^4. \tag{8}$$

If $d\sigma$ is a measure with positive piecewise constant density, and $d\boldsymbol{\sigma} = (d\lambda, d\sigma, d\lambda)^T$, we obtain polynomial B-splines with prescribed jumps in second derivatives, appearing in some convexity preserving approximations. This explicitly solves the problem of finding the local basis for Foley's *weighted splines* [14, 6]. The formula becomes more involved:

$$T_2^4 \quad = \quad \frac{\gamma_2^3(4)}{\|\gamma_2^3\|}\tilde{B}_4^4 + \frac{\gamma_2^3(4) + \gamma_2^3(5)}{\|\gamma_2^3\|}\tilde{B}_5^4 + \frac{\gamma_3^3(7) + \gamma_3^3(8)}{\|\gamma_3^3\|}\tilde{B}_6^4 + \frac{\gamma_3^3(8)}{\|\gamma_3^3\|}\tilde{B}_7^4$$

where

$$\begin{aligned} \gamma_2^3(4) &= \frac{d\sigma(t_2, t_3)}{d\sigma(t_2, t_4)}(t_4 - t_2) \\ \gamma_2^3(5) &= t_4 - t_3 \\ \gamma_2^3(7) &= t_5 - t_4 \\ \gamma_2^3(8) &= \frac{d\sigma(t_5, t_6)}{d\sigma(t_4, t_6)}(t_6 - t_4)\,. \end{aligned}$$

In the weighted spline case the $\tilde{B}^4$ stands for polynomial splines as in the first example. This is due to the fact that splines on the refined knot sequence do not have their second derivatives constrained in any way, and must therefore be equal to the polynomial B-splines by uniqueness argument: $\tilde{T}^4_{d\boldsymbol{\sigma}} \equiv \tilde{B}^4$.

Finally, let us consider splines piecewisely in $\mathrm{L}\{1, x, \exp(px), \exp(-px)\}$, where $p > 0$ [1]. This is a space of *tension splines* [11, 12], with uniform tension parameter p. The functions span an ECT-space, but it is less obvious that we may take $d\boldsymbol{\sigma} = (dt \cosh(pt), dt/\cosh(pt), dt \cosh(pt))^T$. The first generalized derivative $L_1 f(t) = f'(t)/\cosh(pt)$ takes the tension spline space to the very simple space of *hyperbolic splines* [17]. The hypothesis is verified directly, since a de Boor-Cox type formula exists for B-splines in $\mathcal{S}(4, \boldsymbol{m}, R_3 d\boldsymbol{\sigma}, \Delta)$, namely:

$$T^3_{j,R_3 d\boldsymbol{\sigma}} = \text{const.} \begin{cases} \frac{\sinh^2(p/2)(x-t_j)}{\sinh(p/2)(t_{j+2}-t_j)\sinh(p/2)(t_{j+1}-t_j)}, & x \in (t_j, t_{j+1}) \\ \frac{\sinh^2(p/2)(x-t_j)\sinh^2(p/2)(t_{j+2}-x)}{\sinh(p/2)(t_{j+2}-t_j)\sinh(p/2)(t_{j+2}-t_{j+1})} + & \\ \frac{\sinh^2(p/2)(t_{j+3}-x)\sinh^2(p/2)(x-t_{j+1}x)}{\sinh(p/2)(t_{j+3}-t_{j+1})\sinh(p/2)(t_{j+2}-t_{j+1})}, & x \in (t_{j+1}, t_{j+2}) \\ \frac{\sinh^2(p/2)(t_{j+3}-x)}{\sinh(p/2)(t_{j+3}-t_{j+1})\sinh(p/2)(t_{j+3}-t_{j+2})}, & x \in (t_{j+2}, t_{j+3}) \end{cases},$$

where const. $= \cosh\frac{p}{2}(t_{j+2} - t_{j+1})$. To apply *Theorem 2*, we need to calculate the integrals in (7), what can be done by positive weights integration formula, or analytically.

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