On collocation methods for Volterra integral equations with delay arguments*†

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Abstract. *In this paper we construct and give an analysis of the global convergence and local superconvergence properties of polynomial collocation solution* $u \in S^{(d)}_{m+d}(Z_N)$ *of Volterra integral equations with constant delay, thus extending the existing theory for* $d = -1$ *to the general case.*

Key words: *Volterra integral equations, delay, collocation methods*

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1. Introduction

We consider the numerical discretization of Volterra integral equations with (constant) delay $\tau > 0$,

$$y(t) = g(t) + \int_0^t k_1(t, s, y(s))ds + \int_0^{t-\tau} k_2(t, s, y(s))ds, \quad t \in I := [0, T], \tag{1}$$

with

$$y(t) = \phi(t), \qquad t \in [-\tau, 0), \tag{2}$$

by collocation methods in certain polynomial spline spaces. The aim of this paper is to introduce a new polynomial collocation method to approximate the solution of (1). It will be assumed that the given functions, $\phi : [-\tau, 0] \to \mathbb{R}$, $g : I \to \mathbb{R}$, $k_1 : S \times \mathbb{R} \to \mathbb{R}$ ($S := \{(t, s) : 0 \le s \le t \le T\}$), and $k_2 : S_\tau \times \mathbb{R} \to \mathbb{R}$ ($S_\tau := I \times [-\tau, T - \tau]$) are at least continuous on their domains. We will not discuss the "classical" Volterra integral equations. So, assume that $k_2(t, s, y)$ does not vanish identically on its domain. Existence and uniqueness results for (1) can be found in, for example, [2] and [6].

The exact solution of (1), in the case $k_2 = 0$, has been approximated by the collocation method in polynomial spline spaces $S^{(-1)}_{m-1}(Z_N)$ (see [3], [4]) and $S^{(d)}_{m+d}(Z_N)$ for $d > -1$ (see [7], [8]). Details of the notation are given in §2. Recently, the

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collocation solution of (1) has been constructed in the piecewise polynomial space $S^{(-1)}_{m-1}(Z_N)$ (see [5]). The aim of this paper is to construct an approximate solution of (1) in the polynomial space $S^{(d)}_{m+d}(Z_N)$. The approximation $u \in S^{(d)}_{m+d}(Z_N)$ will be determined by collocation. The attainable order of global and local convergence of these methods, both in exact and discretized case, is analysed in detail.

2. Collocation

Let $t_n = nh$ $(n = 0, ..., N-1,\ t_N = T)$ define a uniform partition for $I = [0, T]$, and let $\Pi_N := \{t_0, ..., t_N\}$, $\sigma_0 := [t_0, t_1], \sigma_n := (t_n, t_{n+1}]$ $(1 \le n \le N-1)$. The mesh Π_N is constrained in the following sense:

$$h = \frac{\tau}{r} \ \text{ for some } \ r \in \mathbb{N}. \tag{3}$$

With a given mesh Π_N we associate the set of its interior points, $Z_N := \{t_n : n = 1, ..., N-1\}$. For a fixed $N \ge 1$ and, for given integers $d \ge -1$ and $m \ge 1$, the piecewise polynomial space $S^{(d)}_{m+d}(Z_N)$ is defined by

$$S^{(d)}_{m+d}(Z_N) := \{u : I \to \mathbb{R}; u|_{\sigma_n} =: u_n \in \pi_{m+d}, u^{(\nu)}_{n-1}(t_n) = u^{(\nu)}_n(t_n), \nu = 0, ..., d\}, \tag{4}$$

where π_{m+d} denotes the set of (real) polynomials of a degree not exceeding $m+d$. The dimension of this space is given by $\dim S^{(d)}_{m+d}(Z_N) = Nm+d+1$. Let $u_n = u\mid_{\sigma_n}$, $u \in S^{(d)}_{m+d}(Z_N)$, for all $t \in \sigma_n$ we have

$$u_n(t) = \sum_{l=0}^{d} \frac{u^{(l)}_{n-1}(t_n)}{l!}(t-t_n)^l + \sum_{l=1}^{m} a_{n,l}(t-t_n)^{d+l}, \quad n = 0, ..., N-1, \tag{5}$$

where $u^{(l)}_{-1}(0) = y^{(l)}(0), \quad l = 0, ..., d.$

From (5) we see that an element $u \in S^{(d)}_{m+d}(Z_N)$ is well defined when we know the coefficients $\{a_{n,l}\}$ for all $l = 0, ..., N-1$. In order to compute these coefficients, we consider the set of collocation parameters $\{c_j\}$, where $0 \le c_1 < ... < c_m \le 1$, and define the set $X_N := \{t_{n,j}\}^{m,N-1}_{j=1,n=0}$ of collocation points by

$$t_{n,j} := t_n + c_j h \quad j = 1, ..., m;\ n = 0, ..., N-1. \tag{6}$$

The collocation solution $u \in S^{(d)}_{m+d}(Z_N)$ will be determined by imposing the condition that u satisfies the integral equation (1) on the finite set X_N

$$u(t) \ = \ g(t) + \int_0^t k_1(t, s, u(s))ds + \int_0^{t-\tau} k_2(t, s, u(s))ds, \quad \text{with} \tag{7}$$

$$u(t) \ = \ \phi(t) \ \text{ on } \ [-\tau, 0). \tag{8}$$

If $t = t_{n,j}$ is such that $t_{n,j} - \tau$ $(= t_{n-r,j}) < 0$ (recall that, by (3), $\tau = rh = t_r$), then (1) becomes

$$u(t) = g(t) + \int_0^t k_1(t, s, u(s))ds - \Phi(t),\ t = t_{n,j},\ (j = 1, ..., m;\ n = 0, ..., r-1), \tag{9}$$

where

$$\Phi(t) := \int_{t-\tau}^{0} k_2(t,s,\phi(s))ds. \tag{10}$$

In contrast to classical Volterra integral equations corresponding to $k_2 = 0$, the occurrence of the term $\Phi(t)$ in the collocation equation (9) reveals that, for $t = t_{n,j} < \tau$, we have to evaluate (or approximate) a *functional* containing the given initial function $\phi(t)$.

In order to put (7) into a form amenable to numerical computation, let $t \in \sigma_n$, and define

$$F_n(t) := \int_0^{t_n} k_1(t,s,u(s))ds = h\sum_{i=0}^{n-1}\int_0^1 k_1(t,t_i+vh,u(t_i+vh))dv. \tag{11}$$

Moreover, let

$$\begin{aligned} D(t) &:= \int_0^{t-\tau} k_2(t,s,u(s))ds \quad \text{with} \\ D(t) &:= -\Phi(t) \text{ if } t<\tau. \end{aligned} \tag{12}$$

By using (6), equation (7) can be written as

$$u(t_{n,j}) = g(t_{n,j}) + F_n(t_{n,j}) + D(t_{n,j}) + h\int_0^{c_j} k_1(t_{n,j}, t_n+vh, u(t_n+vh))dv, \tag{13}$$

for $j = 1, ..., m$.

Consider now (13). Generally, the integrals on the right-hand side including those in $F_n(t_{n,j})$ and $D(t_{n,j})$ cannot be evaluated analytically, but have to be approximated by suitable quadrature formulae.

Let μ_0 and μ_1 be given positive integers. Suppose that the quadrature parameters $\{d_l\}$ and $\{d_{j,l}\}$ satisfy $0 \le d_1 < ... < d_{\mu_1} \le 1$ and $0 \le d_{j,1} < ... < d_{j,\mu_0} \le c_j$ $(j = 1, ..., m)$, respectively. The quadrature weights are then given by

$$\begin{aligned} w_l &:= \int_0^1 \prod_{r=1,r\neq l}^{\mu_1} \frac{v-d_r}{d_l-d_r}dv, \ l=1,...,\mu_1 \quad \text{and} \\ w_{j,l} &:= \int_0^{c_j} \prod_{r=1,r\neq l}^{\mu_0} \frac{v-d_{j,r}}{d_{j,l}-d_{j,r}}dv, \ l=1,...,\mu_0; \ j=1,...,m. \end{aligned}$$

The fully discretized collocation equation corresponding to (13) is thus given by

$$\hat{u}(t_{n,j}) = g(t_{n,j}) + \hat{F}_n(t_{n,j}) + \hat{D}(t_{n,j}) + h\sum_{l=1}^{\mu_0} w_{j,l}k_1(t_{n,j}, t_n+d_{j,l}h, \hat{u}(t_n+d_{j,l}h)), \tag{14}$$

$j = 1, ..., m$, with

$$\hat{F}_n(t_{n,j}) := h\sum_{i=0}^{n-1}\sum_{l=1}^{\mu_1} w_l k_1(t_{n,j}, t_i+d_lh, \hat{u}(t_i+d_lh)), \tag{15}$$

and, if $n - r \geq 0$, then

$$\begin{aligned} \hat{D}_n(t_{n,j}) &:= h \sum_{i=0}^{n-r-1} \sum_{l=1}^{\mu_1} w_l k_2(t_{n,j}, t_i + d_l h, \hat{u}(t_i + d_l h)) \\ &\quad + h \sum_{l=1}^{\mu_0} w_{j,l} k_2(t_{n,j}, t_{n-r} + d_{j,l} h, \hat{u}(t_{n-r} + d_{j,l} h)). \end{aligned} \tag{16}$$

If $n - r < 0$, $\hat{D}_n(t_{n,j})$ is given either by the exact value of $-\Phi(t_{n,j})$ (recall (10)),

$$\begin{aligned} \hat{D}_n(t_{n,j}) = D_n(t_{n,j}) &= -h \int_{c_j}^{1} k_2(t_{n,j}, t_{n-r} + vh, \phi(t_{n-r} + vh))dv \\ &\quad -h \sum_{i=n-r+1}^{-1} k_2(t_{n,j}, t_i + vh, \phi(t_i + vh))dv, \end{aligned} \tag{17}$$

or by a suitable quadrature approximation to $-\Phi(t_{n,j})$,

$$\begin{aligned} \hat{D}_n(t_{n,j}) &= -h \sum_{l=1}^{\mu_1} \tilde{w}_{j,l} k_2(t_{n,j}, t_{n-r} + \xi_{j,l} h, \phi(t_{n-r} + \xi_{j,l} h)) \\ &\quad -h \sum_{i=n-r+1}^{-1} \sum_{l=1}^{\mu_1} w_l k_2(t_{n,j}, t_i + d_l h, \phi(t_i + d_l h)) \end{aligned} \tag{18}$$

where $\xi_{j,l} := c_j + (1 - c_j)d_l$, $\tilde{w}_{j,l} := (1 - c_j)w_l \qquad (j, l = 1, ..., m)$.
Since the quadrature error terms will be disregarded, we generate an approximation $\hat{u} \in S_{m+d}^{(d)}(Z_N)$ of the following form:

$$\begin{aligned} \hat{u}(t) = \hat{u}_n(t) &= \sum_{l=0}^{d} \frac{\hat{u}_{n-1}^{(l)}(t_n)}{l!}(t - t_n)^l + \sum_{l=1}^{m} \hat{a}_{n,l}(t - t_n)^{d+l} \\ \hat{u}_{-1}^{(l)}(0) &= y^{(l)}(0), \quad l = 0, ..., d \qquad \text{for all } t \in \sigma_n, \quad n = 0, ..., N-1. \end{aligned} \tag{19}$$

Equations (13) and (14) represent for each $n = 0, ..., N-1$, a recursive system of m nonlinear algebraic equations with the unknowns $\{a_{n,r}\}$ and $\{\hat{a}_{n,r}\}$, respectively. Since the solutions of the systems have been found, the values u and $\hat{u}$ and their derivatives on σ_n are determined by (5) and by (19), respectively.

3. Global convergence

Let $u \in S_{m+d}^{(d)}(Z_N)$ denote the (exact) collocation solution to (1) defined by (7)-(9). For simplicity of exposition we will focus on the linear version of (1),

$$y(t) = g(t) + \int_0^t K_1(t, s)y(s)ds + \int_0^{t-\tau} K_2(t, s)y(s)ds, \quad t \in I, \tag{20}$$

where $K_1 \in C(S)$, $K_2 \in C(S_\tau)$. A comment on the extension of the convergence results to the nonlinear equation (1) can be found at the end of this section.

Theorem 1. *Assume that the given functions in (20) and (2) satisfy* $g \in C^{m+d+1}(I)$, $K_1 \in C^{m+d+1}(S)$, $K_2 \in C^{m+d+1}(S_\tau)$, $\phi \in C^{m+d+1}([-\tau, 0])$, *and that for* $t \in [0, \tau]$ *the integral (10)*

$$\Phi(t) := \int_{t-\tau}^{0} K_2(t,s)\phi(s)ds \tag{21}$$

is known exactly. Then for all sufficiently small $h = \tau/t$ ($r \in \mathbb{N}$) *the constrained mesh collocation solution* $u \in S_{m+d}^{(d)}(Z_N)$ *to (20) satisfies*

$$\|y^{(k)} - u^{(k)}\|_\infty \leq C_k h^{m+d+1-k}, \text{ for all } k = 0, ..., m+d, \tag{22}$$

where C_k *are positive constants not depending on* h. *This estimate holds for all collocation parameters* $\{c_j\}$ *with* $0 \leq c_1 < ... < c_m \leq 1$.

Proof. Assume for simplicity, and without any loss of generality, that $T = M\tau$ for some $M \in \mathbb{N}$. In each interval $J_\mu := (\mu\tau, (\mu+1)\tau)$, the exact solution y of (20) is $m+d+1$ times continuously differentiable. This follows from the smoothness hypotheses we have imposed on ϕ, g, K_1 and K_2, and from the expressions for $y^{(\nu)}(t)$ obtained by successively differentiating (20) with respect to t. From this it is obvious that both the left and right limits of $y^{(\nu)}(t)$ ($\nu = 0, ..., m+d+1$), as t tends to $\mu\tau$, exist and are finite.

We will prove the estimate (22) by induction. For $n = 0, 1, ..., N-1$, and all $t = t_n + vh \in \sigma_n$ ($v \in (0,1]$), the exact solution of y can be expanded in Taylor series:

$$\begin{aligned} y(t_n + vh) &= \sum_{l=0}^{m+d} \frac{y^{(l)}(t_n)}{l!} v^l h^l + R_n(v) h^{m+d+1}, \quad \text{where} \\ R_n(v) &= \frac{1}{(m+d)!} \int_0^v y^{(m+d+1)}(t_n + sh)(v-s)^{m+d} ds. \end{aligned} \tag{23}$$

So, by (5) and (23) we have

$$e(t_n + vh) = \sum_{l=0}^{d} \frac{e_{n-1}^{(l)}(t_n)}{l!} h^l v^l + h^{m+d+1} \left(\sum_{l=1}^{m} \beta_{n,l} v^{d+l} + R_n(v) \right), \tag{24}$$

and

$$h^{m+1}\beta_{n,l} = \left(\frac{y^{(d+l)}(t_n) - a_{n,l}}{(d+l)!} \right) h^l \qquad (l = 1, ..., m). \tag{25}$$

As y is the solution of (20), and $u \in S_{m+d}^{(d)}(Z_N)$ satisfies the exact collocation equation (7)-(9), the collocation error $e := y - u$ satisfies

$$e(t_{n,j}) = \int_0^{t_{n,j}} K_1(t_{n,j}, s)e(s)ds + \int_0^{t_{n-r,j}} K_2(t_{n,j}, s)e(s)ds, \tag{26}$$

$$j = 1, ..., m \quad (n = 0, ..., N-1).$$

If $t_n < \tau \ (= t_r)$, then $t_{n-r,j} = t_n + c_j h - \tau \leq 0$. Since $u(t) = \phi(t)$ on $[-\tau, 0)$, equation (26) is reduced to

$$e(t_{n,j}) = \int_0^{t_{n,j}} K_1(t_{n,j}, s)e(s)ds \tag{27}$$

and can be written as

$$e(t_{n,j}) = h\int_0^{c_j} K_1(t_{n,j}, t_n + vh)e(t_n + vh)dv + h\sum_{i=0}^{n-1} K_1(t_{n,j}, t_i + vh)e(t_i + vh)dv. \tag{28}$$

If $t_n \geq t_r$, the equation (26) can be written as

$$\begin{aligned} e(t_{n,j}) \ = \ & h\int_0^{c_j} K_1(t_{n,j}, t_n + vh)e(t_n + vh)dv + h\sum_{i=0}^{n-1}\int_0^1 K_1(t_{n,j}, t_i + vh)e(t_i + vh)dv \\ & +h\int_0^{c_j} K_2(t_{n,j}, t_{n-r} + vh)e(t_{n-r} + vh)dv \\ & +h\sum_{i=0}^{n-r-1}\int_0^1 K_2(t_{n,j}, t_i + vh)e(t_i + vh)dv. \end{aligned} \tag{29}$$

In order to make the following analysis more obvious, let

$$\begin{aligned} \beta_n &:= (\beta_{n,1}, ..., \beta_{n,m})^\tau, \ n = 0, ..., N-1, \\ q_n^{(1)} &:= (q_{n,1}^{(1)}, ..., q_{n,m}^{(1)})^\tau, \ n = 0, ..., r-1, \\ q_n^{(2)} &:= (q_{n,1}^{(2)}, ..., q_{n,m}^{(2)})^\tau, \ n = r, ..., N-1, \end{aligned}$$

with

$$\begin{aligned} q_{n,j}^{(1)} := \quad & -R_n(c_j) + h\int_0^{c_j} K_1(t_{n,j}, t_n + vh)R_n(v)dv \\ & +h\sum_{i=0}^{n-1}\int_0^1 K_1(t_{n,j}, t_i + vh)R_i(v)dv \end{aligned}$$

and

$$\begin{aligned} q_{n,j}^{(2)} := \quad & q_{n,j}^{(1)} + h\int_0^{c_j} K_2(t_{n,j}, t_{n-r} + vh)R_{n-r}(v)dv \\ & +h\sum_{i=0}^{n-r-1}\int_0^1 K_2(t_{n,j}, t_i + vh)R_i(v)dv. \end{aligned}$$

By using the expression (25) for e, we obtain the following recurrence relation from (28) and (30) for the vectors β_n of the form

$$(V - hQ_{n,n}^{(1)})\beta_n = \begin{cases} h\sum_{i=0}^{n-1} Q_{n,i}^{(1)}\beta_i + r_n^{(1)} + q_n^{(1)}, & n = 0, ..., r-1 \\ h\sum\limits_{i=0}^{n-1} Q_{n,i}^{(1)}\beta_i + h\sum\limits_{i=0}^{n-r} Q_{n,i}^{(2)}\beta_i + r_n^{(2)} + q_n^{(2)}, & n = r, ..., N-1 \end{cases} \tag{30}$$

where the square matrices V, W, $Q^{(1)}_{n,i}$, $F^{(1)}_{n,i}$, $Q^{(2)}_{n,i}$, $F^{(2)}_{n,i}$ and the vectors $r^{(1)}_n$, $r^{(2)}_n$ are given by

$$
\begin{aligned}
V &:= (c_j^{d+l}), \quad W := (c_j^l) \qquad (j,l=1,...,m),\\
Q^{(1)}_{n,i} &:= \begin{cases} (\int_0^1 K_1(t_{n,j},t_i+vh)v^{d+l}dv) & \text{if} \quad 0\le i\le n-1,\\ (\int_0^{c_j} K_1(t_{n,j},t_i+vh)v^{d+l}dv) & \text{if} \quad i=n\ (j,l=1,...,m),\end{cases}\\
F^{(1)}_{n,i} &:= \begin{cases} (\int_0^1 K_1(t_{n,j},t_i+vh)v^{l}dv) & \text{if} \quad 1\le i\le n-1,\\ (\int_0^{c_j} K_1(t_{n,j},t_i+vh)v^{l}dv) & \text{if} \quad i=n\ (j,l=1,...,m),\end{cases}\\
Q^{(2)}_{n,i} &:= \begin{cases} (\int_0^1 K_2(t_{n,j},t_i+vh)v^{d+l}dv) & \text{if} \quad 0\le i\le n-r-1,\\ (\int_0^{c_j} K_2(t_{n,j},t_i+vh)v^{d+l}dv) & \text{if} \quad i=n-r\ (j,l=1,...,m),\end{cases}\\
F^{(2)}_{n,i} &:= \begin{cases} (\int_0^1 K_2(t_{n,j},t_i+vh)v^{l}dv) & \text{if} \quad 1\le i\le n-r-1,\\ (\int_0^{c_j} K_2(t_{n,j},t_i+vh)v^{l}dv) & \text{if} \quad i=n-r\ (j,l=1,...,m),\end{cases}\\
h^p r^{(1)}_n &:= h\sum_{i=1}^{n-1} F^{(1)}_{n,i}\gamma_i + (hF^{(1)}_{n,n}-W)\gamma_n,\\
h^p r^{(2)}_n &:= h^{m+d+1} r^{(1)}_n + h\sum_{i=1}^{n-r} F^{(2)}_{n,i}\gamma_i,
\end{aligned}
$$

with $p=m+d+1$ and $\gamma_i := (\gamma_{i,1},...,\gamma_{i,m})^\tau$, $i=1,...,n$ where $\gamma_{i,l} := h^l \frac{e^{(l)}_{i-1}(t_i)}{l!}$.

For all collocation parameters $\{c_j\}$ with $0\le c_1<...<c_m\le 1$, V and W are the Vandermonde matrices and thus nonsingular. Let $\bar{K}_1 := \max\{|K_1(t,s)| : (t,s)\in S\}$ and $\bar{K}_2 := \max\{|K_2(t,s)| : (t,s)\in S_\tau\}$, since by the assumptions, $K_1\in C(S)$ and $K_2\in C(S_\tau)$, thus $\bar{K}_1$ and $\bar{K}_2$ are finite.

For $n=0$, by (5) and (27), we obtain:

$$(V-hQ^{(1)}_{0,0})\beta_0 = q^{(1)}_0. \tag{31}$$

Since V is nonsingular, and by the assumptions of *Theorem 1*, it follows that there exists $\bar{h}>0$, such that the matrix $V-hQ^{(1)}_{0,0}$ possesses uniformly bounded inverse for all $h\in(0,\bar{h})$. Since $|R_0(v)|\le M_0$ for all $v\in[0,1]$ ($M_0>0$ is a finite constant), we obtain $\|q^{(1)}_0\|_1 \le m(M_0+hM_0\bar{K}_1) =: q$. By using these estimations in (31), we find:

$$\|\beta_0\|_1 \le \|(V-hQ^{(1)}_{0,0})^{-1}\|_1\|q^{(1)}\|_1 \le D_0 q =: B. \tag{32}$$

Thus, (32) together with (24) proves that

$$|e(t_0+vh)| \le C_0 h^{m+d+1}, \text{ for all } v\in[0,1]. \tag{33}$$

We take the derivative in relation (25) k times ($k=1,...,m+d$), and use (32) to obtain

$$|e^{(k)}(t_0+vh)| \le C_{0,k} h^{m+d+1-k}, \text{ for all } v\in[0,1]. \tag{34}$$

Suppose now that for $j=0,...,n-1$

$$|e^{(k)}(t_j+vh)| \le C_{j,k} h^{m+d+1-k}, \text{ for all } v\in(0,1], \quad k=0,\dots,m+d. \tag{35}$$

We shall prove that (35) holds for $j = n\,(< r)$.

Since by the assumption, $K_1 \in C(S)$, it follows that there exists $\bar{h} > 0$, such that $\|\bar{h}Q^{(1)}_{n,n}\|_1 < 1$, by a standard Neumann series argument, it follows that the matrix $V - hQ^{(1)}_{n,n}$ possesses a uniformly bounded inverse for all $h \in (0, \bar{h})$. We also note that the matrices $F^{(1)}_{n,i}$, $i = 1, ..., n$ are bounded, so there exists a positive constant C_F such that $\|hF^{(1)}_{n,n} - W\| \le C_F$. By using (35), we get $\|\frac{1}{h^{m+d+1}}\gamma_i\| \le g_i$, $i = 1, \ldots, n$. Since $|R_n(v)| \le M_n$, for all $v \in (0, 1]$ ($M_n > 0$ is a finite constant), we obtain

$$\|q\|_1 \le m(M_n + nh\bar{K}_1 M_n) =: q. \tag{36}$$

By using these estimates, it follows from (33) that

$$\|\beta_n\|_1 \le hC_0 \sum_{i=0}^{n-1} \|\beta_i\|_1 + C_1. \tag{37}$$

A well known result on discrete Gronwall inequalities (see [3, p.41]) leads to

$$\|\beta_n\|_1 \le C_1 \exp(C_0\tau) =: B, \tag{38}$$

($0 \le n \le r - 1$) uniformly as $h \to 0$ (where $rh = \tau$). This by (24) implies that

$$|e(t_n + vh)| \le Ch^{m+d+1} \text{ for all } v \in (0, 1]. \tag{39}$$

We take the derivative in relation (24) k times ($k = 1, ..., m + d$) and by using (38), we obtain

$$|e^{(k)}(t_n + vh)| \le C_{n,k} h^{m+d+1-k} \text{ for all } v \in (0, 1]. \tag{40}$$

Now let $t_n \ge t_r (= \tau)$, then for $n = r$, by (32) we obtain

$$(V - hQ^{(1)}_{r,r})\beta_r = h \sum_{i=0}^{r-1} Q^{(1)}_{r,i}\beta_i + r^{(1)}_r + q^{(2)}_r. \tag{41}$$

Similarly as in the case $t_n < t_r$, by the assumptions of *Theorem 1*, for all sufficiently small $h > 0$, the matrix $V - hQ^{(1)}_{r,r}$ is nonsingular and has a uniformly bounded inverse. Other matrices involved in (41) have bounded norms, so by using these estimates and (39), the equation (41) yields a discrete Gronwall inequality like (37) with $n = r$. Thus, the results on discrete Gronwall inequalities ([3, p.41]) lead to

$$\|\beta_r\|_1 \le B. \tag{42}$$

Thus, (42) with (24) proves that (39) holds for $n = r$. If we take the derivative in relation (24) k times ($k = 1, .., m + d$), and by using (42), we obtain that (40) holds for $n = r$. Suppose now that (40) holds for $j = 0, ..., n - 1$ ($n > r$). In complete analogy with the above results, by the assumptions of *Theorem 1* and the induction argument for all sufficiently small $h > 0$, the equation (30) yields a discrete Gronwall inequality

$$\|\beta_n\|_1 \le hC_0 \sum_{i=0}^{n-1} \|\beta_i\|_1 + hD_0 \sum_{i=0}^{n-r} \|\beta_i\|_1 + C_1, \tag{43}$$

where now $r \le n \le N-1$. The estimates (38) and the results [3, p.41], lead to

$$\|\beta_n\|_1 \le B \quad (r \le n \le N-1) \tag{44}$$

uniformly as $h \to 0$ (with $Nh = T$). So, by (24) the estimate (39) holds for all $n = r, ..., N-1$. By using the same argument as in (44), the estimate (40) holds for all $n = r, ..., N-1$. Hence, *Theorem 1* holds. □

If $\Phi(t)$ in (10) cannot be found analytically, it has to be approximated by a suitable numerical quadrature.

Theorem 2. *Let the assumptions of* Theorem 1 *hold, except that the integrals*

$$\Phi(t) = \int_{t-\tau}^{0} K_2(t,s)\phi(s)ds, \qquad t = t_{n,j} \qquad (n = 0, ..., r-1),$$

are now approximated by quadrature formulas $\hat{\Phi}(t)$, with corresponding quadrature errors $E_0(t) := \Phi(t) - \hat{\Phi}(t)$, such that

$$|E_0(t)| \le h^q, \qquad t = t_{n,j} \qquad (0 \le n < r), \tag{45}$$

for some $q > 0$. Then the collocation solution $u \in S_{m+d}^{(d)}(Z_N)$ satisfies, for all sufficiently small $h > 0$,

$$\|e^{(k)}\|_\infty \le C_k h^{p-k}, \text{ for all } k = 0, \ldots, p-1, \tag{46}$$

with $p := \min\{m+d+1, q\}$, where C_k are finite constants not depending on h.

Proof. Because of (13), we have to calculate the integrals $\Phi(t)$ only if $t_n < t_r$ $(= \tau)$. Thus, if $t_n \ge t_r$, the estimate (22) holds (with $k = 0$). So, we assume that $0 \le n < r$. By subtracting (13) from (20) (with $t = t_{n,j}$), we obtain

$$e(t_{n,j}) = \int_0^{t_{n,j}} K_1(t_{n,j}, s)e(s)ds - (\Phi(t_{n,j}) - \hat{\Phi}(t_{n,j})). \tag{47}$$

Instead of (24), let

$$e(t_n + vh) = \sum_{l=0}^{d} \frac{e_{n-1}^{(l)}(t_n)}{l!} h^l v^l + h^{d+p} \sum_{l=1}^{m} \beta_{n,l} v^{d+l} + h^{m+d+1} R_n(v), \text{ with} \tag{48}$$

$$h^p \beta_{n,l} = \left(\frac{y^{(d+l)}(t_n) - a_{n,l}}{(d+l)!} \right) h^l \quad (l = 1, ..., m), \tag{49}$$

and with suitable $p > 0$ to be determined. Substitution of this expression for the collocation error in the above error equation yields, after division by h^p, the recurrence relation for the vectors β_n of the form

$$(V - hQ_{n,n}^{(1)})\beta_n = h \sum_{i=0}^{n-1} Q_{n,i}^{(1)} \beta_i + r_n^{(1)} + h^{m+d+1-p} q_n^{(1)} - h^{-p} E_0^n, \;\; n = 0, ..., r-1,$$

where $E_0^n := (E_0(t_{n,1}), ..., E_0(t_{n,m}))^\tau$. Proceding as in the proof of *Theorem 1*, we are in a situation in which we may apply the discrete Gronwall inequality, in analogy to (43),

$$\|\beta_n\|_1 \leq hC_0 \sum_{i=0}^{n-1} \|\beta_i\|_1 + \hat{C}_1, \ 0 \leq n < r,$$

where $\hat{C}_1 := h^{m+d+1-p}C_1 + h^{q-p}Q_1$. This implies that $\|\beta_n\|_1 \leq \hat{C}_1 \exp(C_0\tau)$, $0 \leq n < r$. Hence, $\|\beta_n\|_1$ will remain uniformly bounded as $h \to 0$ $(rh = \tau)$ iff $p \leq \min\{m + d + 1, q\}$. Now, we take the derivative in relation (48) k times ($k = 1, ..., \min\{m+d, q-1\}$), and by using the fact that $\|\beta_n\|_1 \leq B$ for all $n = 0, ..., N-1$, we obtain for $p = \min\{m + d + 1, q\}$

$$|e_n^{(k)}(t)| \leq C_{n,k}h^{p-k}, \text{ for all } t \in \sigma_n. \tag{50}$$

□

The global convergence of the fully discretized collocation method is described in the following theorem.

Theorem 3. *Let the assumptions of* Theorem 2 *hold, and assume that the approximations* $\hat{\Phi}(t)$ *at* $t = t_{n,j}$ $(0 \leq n < r)$ *are given by the interpolatory quadrature formulas (18). Suppose that the quadrature formulas used in (14)-(16) satisfy:*

$$\int_0^1 \rho(t_i + vh)dv - \sum_{l=1}^{\mu_1} w_l \rho(t_i + d_l h) = O(h^{r_1}), \text{ and, for } j = 1, ..., m: \tag{51}$$

$$\int_0^{c_j} \rho(t_n + vh)dv - \sum_{l=1}^{\mu_0} w_{j,l} \rho(t_n + d_{j,l}h) = O(h^{r_0}). \tag{52}$$

Then the error $\hat{e} := y - \hat{u}$ *associated with the collocation solution* $\hat{u} \in S_{m+d}^{(d)}(Z_N)$ *(defined by (14)-(16), (18) and (19)) satisfies, for all sufficiently small* $h = \tau/r$,

$$\|\hat{e}^{(k)}\|_\infty \leq C_k h^{p-k}, \text{ for all } k = 0, ..., p-1, \tag{53}$$

with $p := \min\{m + d + 1, r_0 + 1, r_1\}$, *where* C_k *are finite constants not depending on* h.

Proof. Since $\|\hat{e}^{(k)}\|_\infty \leq \|y^{(k)} - u^{(k)}\|_\infty + \|u^{(k)} - \hat{u}^{(k)}\|_\infty$, and by the statement (22) of *Theorem 1* $\|y^{(k)} - u^{(k)}\|_\infty = O(h^{m+d+1})$, we have to estimate only $\|u^{(k)} - \hat{u}^{(k)}\|_\infty$. Now, by subtracting (19) from (5), it follows that the error function $\varepsilon := u - \hat{u}$ can be written for all $n = 0, ..., N-1$ in the form

$$\varepsilon(t_n + vh) = \sum_{s=0}^{d} \frac{\varepsilon_{n-1}^{(s)}(t_n)}{s!} v^s h^s + h^p \sum_{s=1}^{m} \eta_{n,s} v^{d+s}, \text{ where} \tag{54}$$

$$h^p \eta_{n,s} := (a_{n,s} - \hat{a}_{n,s})h^{d+s}, \tag{55}$$

with p being a suitable positive integer to be determined. Consider the integrals in the (exact) collocation equation (13) (for the linear case (20)). By using the interpolatory quadrature formulas, we have the following error terms:

$$E_i^{n,j}=\int_0^1 K_1(t_{n,j},t_i+vh)u(t_i+vh)dv-\sum_{s=1}^{\mu_1} w_s K_1(t_{n,j},t_i+d_sh)u(t_i+d_sh),\quad 0\le i<n,$$

$$E_n^{n,j}=\int_0^{c_j} K_1(t_{n,j},t_n+vh)u(t_n+vh)dv-\sum_{s=1}^{\mu_0} w_{j,s}K_1(t_{n,j},t_n+d_{j,s}h)u(t_n+d_{j,s}h),$$

$$\bar{E}_i^{n,j}=\int_0^1 K_2(t_{n,j},t_i+vh)u(t_i+vh)dv-\sum_{s=1}^{\mu_1} w_s K_2(t_{n,j},t_i+d_sh)u(t_i+d_sh),\quad 0\le i<n-r,$$

$$\bar{E}_{n-r}^{n,j}=\int_0^{c_j} K_2(t_{n,j},t_{n-r}+vh)u(t_{n-r}+vh)dv-\sum_{s=1}^{\mu_0} w_{j,s}K_2(t_{n,j},t_{n-r}+d_{j,s}h)u(t_{n-r}+d_{j,s}h).$$

If $n<r$, we approximate the integrals $\Phi(t_{n,j})$ (recall (10)) by quadrature formulas (18), and have the error term $E_0=\sum_{i=n-r+1}^{0}\tilde{E}_i^{n,j}$. By subtracting (19) from (13), and by using the above quadrature error terms, we find that

$$\begin{aligned}\varepsilon(t_{n,j}) &= h\sum_{s=1}^{\mu_0} w_{j,s}K_1(t_{n,j},t_n+d_{j,s}h)\varepsilon(t_n+d_{j,s}h) \\ &\quad +h\sum_{i=0}^{n-1}\sum_{s=1}^{\mu_1} w_s K_1(t_{n,j},t_i+d_sh)\varepsilon(t_i+d_sh) \\ &\quad +h\sum_{i=0}^{n} E_i^{n,j}+\Delta(t_{n,j}),\quad j=1,...,m,\end{aligned}\tag{56}$$

where, if $n<r$, then $\Delta(t_{n,j})=E_0$. If $n\ge r$, then

$$\begin{aligned}\Delta(t_{n,j}) &= h\sum_{s=1}^{\mu_0} w_{j,s}K_2(t_{n,j},t_{n-r}+d_{j,s}h)\varepsilon(t_{n-r}+d_{j,s}h) \\ &\quad +h\sum_{i=0}^{n-r-1}\sum_{s=1}^{\mu_1} w_s K_2(t_{n,j},t_i+d_sh)\varepsilon(t_i+d_sh)+h\sum_{i=0}^{n-r}\bar{E}_i^{n,j}.\end{aligned}\tag{57}$$

Let $\eta_n:=(\eta_{n,1},...,\eta_{n,m})^\tau$, where $\eta_{n,l}$ is given by (55). Now, by replacing ε with (54) in (56) and (57), we obtain analogue recurrence relations for the vectors η_n, as for β_n in the proof of *Theorem 1.* We obtain

$$(V-h\hat{Q}_{n,n}^{(1)})\eta_n=\begin{cases} h\sum_{i=0}^{n-1}\hat{Q}_{n,i}^{(1)}\eta_i+\hat{r}_n^{(1)}+\hat{q}_n^{(1)}, & n=0,...,r-1 \\ h\sum_{i=0}^{n-1}\hat{Q}_{n,i}^{(1)}\eta_i+h\sum_{i=0}^{n-r}\hat{Q}_{n,i}^{(2)}\eta_i+\hat{r}_n^{(2)}+\hat{q}_n^{(2)}, & n=r,...,N-1\end{cases}$$

where the square matrices $\hat{Q}_{n,i}^{(s)}$, $\hat{F}_{n,i}^{(s)}$, $(s=1,2)$, are discrete version of matrices $Q_{n,i}^{(s)}$, $F_{n,i}^{(s)}$, $(s=1,2)$ obtained by replacing integrals by quadrature formulas (see the proof of *Theorem 1*). The associated vectors are given by

$$h^p\hat{r}_n^{(1)} := h\sum_{i=1}^{n-1}\hat{F}_{n,i}^{(1)}\hat{\gamma}_i + (h\hat{F}_{n,n}^{(1)} - W)\hat{\gamma}_n, \qquad h^p\hat{r}_n^{(2)} := h^{m+d+1}\hat{r}_n^{(1)} + h\sum_{i=1}^{n-r}\hat{F}_{n,i}^{(2)}\hat{\gamma}_i,$$

$$h^p\hat{q}_n^{(1)} := h\sum_{i=0}^{n}E_i^{n,j} + hE_0, \qquad h^p\hat{q}_n^{(2)} := h\sum_{i=0}^{n}E_i^{n,j} + h\sum_{i=0}^{n-r}\bar{E}_i^{n,j},$$

and $\hat{\gamma}_i := (\hat{\gamma}_{i,1}, ..., \hat{\gamma}_{i,m})^\tau$, $i = 1, .., n$, where $\hat{\gamma}_{i,l} := h^l\frac{\varepsilon_{i-1}^{(l)}}{l!}$.

Now, by using the same technique (an induction for n) as in the proof of *Theorem 1* and taking into account the assumptions (51) and (52), it is easy to show that the norms $\|\eta_n\|_1$ satisfy a discrete Gronwall inequality,

$$\begin{aligned}
\|\eta_n\|_1 &\leq hC_0\sum_{i=0}^{n-1}\|\eta_i\|_1 + h^pC_1 \quad \text{for } 0 \leq n < r,\\
\|\eta_n\|_1 &\leq hD_0\sum_{i=0}^{n-1}\|\eta_i\|_1 + hD_1\sum_{i=0}^{n-r}\|\eta_i\|_1 + h^pD_2 \quad \text{for } n \geq r.
\end{aligned}$$

where $p = \min\{r_0 + 1, r_1\}$. Hence,

$$|\eta_n\|_1 \leq Ch^p, \text{ for all } n = 0, ..., N-1. \tag{58}$$

Together with (54), the above estimate implies that

$$|\varepsilon(t_n + vh)| \leq \hat{C}h^p, \; p = \min\{\mu_0 + 1, \mu_1\} \text{ for all } v \in (0, 1].$$

Now, we take the derivative in relation (54) k times ($k = 1, ..., p-1$) and by using (58), we obtain

$$|\varepsilon^{(k)}(t_n + vh)| \leq \hat{C}_kh^{p-k}, \text{ for all } v \in (0, 1] \text{ and } k = 0, ..., p-1.$$

Finally, by using the results of *Theorem 1*, we obtain

$$\|\hat{e}(k)\|_\infty \leq C_kh^{p'-k}, \text{ for all } k = 0, ..., p'-1 \text{ with } p' = \min\{m + d + 1, p\}.$$

□

We conclude this section with a comment regarding the extension of the results of *Theorems 1 - 3* to the nonlinear delay equation (1). Under the assumption of the existence of a (unique) solution $y(t)$ on I, the nonlinear analogue of the error equation (26) is

$$\begin{aligned}
e(t_{n,j}) &= \int_0^{t_{n,j}}\{k_1(t_{n,j}, s, y(s)) - k_1(t_{n,j}, s, u(s))\}ds \qquad (59)\\
&+ \int_0^{t_{n-r,j}}\{k_2(t_{n,j}, s, y(s)) - k_2(t_{n,j}, s, u(s))\}ds \qquad (j = 1, ..., m).
\end{aligned}$$

If the partial derivatives $\partial k_i(t, s, y)/\partial y$ ($i = 1, 2$) are continuous and bounded on $S \times D$ and $S_\tau \times D_\tau$, respectively with $D := \{y \in \mathbb{R} : |y - y(s)| < Y, \; s \in I\}$ and

$D_\tau := \{y \in \mathbb{R} : |y - y(s)| < Y,\ s \in [-\tau, T-\tau]\}$, for some $Y < \infty$, and if $h > 0$ is sufficiently small (assuring the existence of a unique collocation solution u), then (59) may again be written in the form (26). The roles of the K_i are now assumed by

$$k_i^{(1)}(t,s) := \frac{\partial k_i}{\partial y}(t,s,z_i(s)) \ \ (i = 1,2),$$

where $z_i(s) := \theta_i y(s) + (1-\theta_i)u(s)$, $0 \le \theta_i = \theta_i(s) \le 1$. Hence, the above proofs are easily adapted to deal with the nonlinear case (1), and so the convergence results of *Theorems 1 - 3* remain valid for nonlinear delay integral equations.

4. Local superconvergence on $\bar{Z}_N$

The notion of local superconvergence is used when on a set of interior points Z_N (or $\bar{Z}_N$) the approximate solution has a convergence order greater than the global one. From *Theorem 1* we notice that the only conditions imposed on the collocation parameters $\{c_j\}$ are that they must be distinct and must belong to $(0,1]$. The local superconvergence on $\bar{Z}_N$ is closely connected with the choice of the collocation parameters and with the relation between their number and the number of the coefficients of the approximate solution determined from the smooth conditions (see [5] for delay and [3] for "classical" Volterra integral equations). Without loss of generality, we assume that $T = t_N = M\tau$ for some $M \in \mathbb{N}$.

Theorem 4. *Assume that the functions given in (20) and (2) satisfy* $g \in C^{m+p}(I)$, $K_1 \in C^{m+p}(S)$, $K_2 \in C^{m+p}(S_\tau)$, $\phi \in C^{m+p}([-\tau, 0])$ *for some (given) integer* p *with* $d+1 < p \le m$. *Suppose that the delay integral* $\Phi(t)$ *(10) can be evaluated analytically.*

If $m \ge d+2$, *and if the collocation parameters* $\{c_j\}$, *with* $0 < c_1 < ... < c_m = 1$, *have the ortogonality property*

$$\begin{aligned} J_k &:= \int_0^1 s^k \prod_{j=1}^{m} (s-c_j)ds = 0, \ \textit{for } k = 0,...,p-1; \qquad (60) \\ J_p &\neq 0. \end{aligned}$$

If $h = \tau/r$ *is sufficiently small, then the collocation solution* $u \in S_{m+d}^{(d)}(Z_N)$ *defined by (13),(17) is locally superconvergent at the mesh points*

$$\max_{1 \le n \le N} |y(t_n) - u(t_n)| \le Ch^{m+p}. \qquad (61)$$

Proof. The collocation equation (7) (applied to the linear delay equation (20)) may be written in the form

$$u(t) = -\delta(t) + g(t) + \int_0^t K_1(t,s)u(s)ds + \int_0^{t-\tau} K_2(t,s)u(s)ds,\ t \in I,$$

where the defect δ vanishes on X_N:

$$\delta(t_{n,j}) = 0,\ \ j = 1,...,m;\ n = 0,...,N-1; \qquad (62)$$

we also have $\delta(t) = 0$ for $t < 0$. The collocation error $e := y - u$ solves the integral equation

$$e(t) = \delta(t) + \int_0^t K_1(t,s)e(s)ds + F(t), \quad t \in I, \quad \text{where} \tag{63}$$

$$F(t) := \int_0^{t-\tau} K_2(t,s)e(s)ds \text{ if } \quad t \in [\tau, T]. \tag{64}$$

For $t \in [0, \tau]$, we have $F(t) = 0$ (by our assumption on $\Phi(t)$), and so the error equation (63) reduces to a "classical" Volterra equation (unique) solution given by

$$e(t) = \delta(t) + \int_0^1 R_1(t,s)\delta(s)ds \tag{65}$$

where R_1 denotes the resolvent kernel associated with the given kernel K_1 (see, [3, pp. 11-13]). As $T = M\tau$ for some positive integer M, we may set $\xi_\mu := \mu\tau$ ($\mu = 0, ..., M$), and then for $t \in [\xi_\mu, \xi_{\mu+1}]$ $(1 \le \mu \le M-1)$ the collocation error $e(t)$ governed by (63) can be expressed in the form

$$e(t) = \delta(t) + \sum_{i=0}^{\mu} \int_0^{t-i\tau} Q_{\mu,i}(t,s)\delta(s)ds, \tag{66}$$

where the functions $Q_{\mu,i}(t,s)$ depend on the given kernel functions $K_i(t,s)$ $(i = 1,2)$. This result is the crucial part of the proof of *Theorem 4* (proved by H. Brunner, see [5]), and may be viewed as a generalisation of the resolvent representation of the collocation error for the nonlinear case (see, e.g. [3, 4]).

Let $t = t_n \in [\xi_\mu + h, \xi_{\mu+1}]$. Because $t_n - i\tau = t_n - irh = (n - ir)h$, we obtain

$$e(t_n) = \delta(t_n) + \sum_{i=0}^{\mu} \int_0^{t_n - i\tau} Q_{n,i}(t_n,s)\delta(s)ds = \delta(t_n) + \sum_{i=0}^{\mu} \sum_{k=0}^{n-ir-1} \Psi_{n,i}^{[n]}(t_k + vh)dv, \tag{67}$$

with $\Psi_{n,i}^{[n]}(t_k + vh) := Q_{n,i}(t_n, t_k + vh)\delta(t_k + vh)$.

We now replace each of the integrals in the above expression by the sum of its interpolatory m-point quadrature formula (with the abscissas coinciding with the collocation points $t_k + c_j h$, $j = 1, ..., m$), and the corresponding quadrature error $E_{\mu,i}^{[n]}$. Since $\delta(t) = 0$ for $t \in X_N$ (cf. (62)), we obtain $\Psi_{\mu,i}^{[n]}(t_k + c_j h) = 0$,

$$e(t_n) = \delta(t_n) + h \sum_{i=0}^{\mu} \sum_{k=0}^{n-ir-1} E_{\mu,i}^{[n]} (0 \le \mu < n \le \mu + 1 \le M), \text{ where } M\tau = T. \tag{68}$$

Since the integrands $\Psi_{n,i}^{[n]}(t_k + vh)$ are smooth for all $v \in [0,1]$, and by the orthogonality conditions (60), the quadrature errors in (68) can be bounded by $|E_{\mu,i}^{[n]}| \le Ch^{m+p}$ with some finite constant C not depending on h. Finally, because $M\tau = Mrh = T = Nh$, we obtain $|e(t_n)| \le |\delta(t_n)| + Ch^{m+p}$. Since $c_m = 1$, $t_n \in X_N$, we obtain $|e(t_n)| \le Ch^{m+p}$ $(n = 1, ..., N)$. □

The maximum value of p in the orthogonality condition (60) occurs iff the collocation parameters are the Gauss-Legendre points in $(0,1)$. We have fixed $c_m = 1$, thus the degree of precision in (61) cannot exceed $2m-1$.

The proofs of *Theorems 2, 3* and *4* suggest that the local superconvergence results are also true for discretized collocation solution $\hat{u} \in S_{m+d}^{(d)}(Z_N)$ defined by (14)-(16), (18) and (19) and characterized by $\mu_0 = \mu_1 = m$. The quadrature approximation to the delay integral

$$\begin{aligned} \Phi(t_n) &= -\int_{t_{n-r}}^{0} k_2(t_n, s, \phi(s))ds \quad \text{is given by} \\ \hat{\Phi}(t_n) &= -h \sum_{i=n-r}^{-1} \sum_{l=0}^{m} w_l k_2(t_n, t_i + d_l, \phi(t_i + d_l)) \quad (0 \le n < r). \end{aligned} \tag{69}$$

Theorem 5. *If the assumptions of* Theorem 4 *hold, assume that the approximations to the delay integrals* $\Phi(t_{n,j})$ *and* $\Phi(t_n)$ *(where* $0 \le n < r$*) are given by the quadrature formulas (18) and (69), respectively, if* $h = \tau/r$ *is sufficiently small and if the orthogonality conditions (60) hold, then the solution* $\hat{u}$ *given by ((14)-(16), (18), and (19)) has the property*

$$\max_{1 \le n \le N} |y(t_n) - \hat{u}(t_n)| \le C h^{m+p}. \tag{70}$$

With the same techniques as in the proof of *Theorem 4* we can obtain the statement of the above theorem. We leave these details to the reader.

Finally, we comment on the extension of the results in *Theorems 4* and *5* to the nonlinear case. Instead of (63), the equation for the collocation error e now has the form

$$\begin{aligned} e(t) &= \delta(t) + \int_0^t \{k_1(t, s, y(s)) - k_1(t, s, u(s))\}ds + F(t), \quad \text{where} \\ F(t) &:= \int_0^{t-\tau} \{k_2(t, s, y(s)) - k_2(t, s, u(s))\}ds. \end{aligned}$$

Under appropriate differentiability and boundedness conditions for k_1 and k_2, we obtain

$$k_i(t, s, y(s)) - k_i(t, s, u(s)) = \frac{\partial k_i}{\partial y}(t, s, y(s)) \cdot e(s) + \frac{1}{2}\frac{\partial^2 k_i}{\partial y^2}(t, s, z_i(s)) \cdot e^2(s),$$

where z_i is between y and u. The global convergence of u and $\hat{u}$ has already been established (see *Section 3*). So, we know that

$$\|e^2\|_\infty = O(h^{2(m+d+1)}) \text{ for any } \{c_j\}.$$

The remaining part of the proofs (both u and $\hat{u}$) once more makes use of the techniques described before.

5. Numerical examples

We applied the collocation method to some model Volterra integral equations with constant delay.

Example 1. $y(t) = \lambda e^{\tau - t} - (\lambda - 1)e^{-t} - \lambda \int_{t-\tau}^{t} y(s)ds$ *with*

$$\begin{cases} y(t) = \phi(t) = e^{-t}, & t \in [-\tau, 0) \\ y(t) = e^{-t}, & t \geq 0. \end{cases}$$

d	N	h	*Gauss*		*Radau II*		*Gauss* $c_m = 1$	
			$\|e_N\|_\infty$	$\|e\|_\infty$	$\|e_N\|_\infty$	$\|e\|_\infty$	$\|e_N\|_\infty$	$\|e\|_\infty$
0	*10*	*0.1*	*2.23E-7*	*8.50E-8*	*2.15E-9*	*2.77E-8*	*1.45E-8*	*1.66E-8*
	20	*0.05*	*1.48E-8*	*5.91E-9*	*6.51E-11*	*1.78E-9*	*9.10E-10*	*1.03E-9*
	40	*0.025*	*9.24E-9*	*3.70E-10*	*2.00E-12*	*1.13E-10*	*5.70E-11*	*6.40E-11*
1	*10*	*0.1*	*4.59E+4*	*5.19E+3*	*5.88E-7*	*1.71E-5*	*1.79E-10*	*6.56E-10*
	20	*0.05*	*2.07E+17*	*2.44E+16*	*2.87E-3*	*3.36E-1*	*1.26E-12*	*1.03E-11*
	40	*0.025*	*1.28E+44*	*1.53E+43*	*8.05E+4*	*3.77E+7*	*2.08E-14*	*3.05E-13*

Table 1. Errors for *Example 1*

Example 2. $y(t) = \frac{1}{4}\left(\sin(2(t-\tau)) - \sin 2\tau\right) + \cos t - \frac{1}{2}\tau + \int_{t-\tau}^{t} y^2(s)ds$ *with*

$$\begin{cases} y(t) = \phi(t) = \cos t, & t \in [-\tau, 0) \\ y(t) = \cos t, & t \geq 0. \end{cases}$$

d	N	h	*Gauss*		*Radau II*		*Gauss* $c_m = 1$	
			$\|e_N\|_\infty$	$\|e\|_\infty$	$\|e_N\|_\infty$	$\|e\|_\infty$	$\|e_N\|_\infty$	$\|e\|_\infty$
0	*10*	*0.1*	*2.03E-7*	*6.42E-8*	*2.81E-9*	*2.96E-8*	*4.65E-8*	*5.83E-8*
	20	*0.05*	*1.28E-8*	*4.14E-9*	*8.97E-11*	*1.84E-9*	*2.92E-9*	*3.63E-9*
	40	*0.025*	*8.09E-10*	*2.63E-10*	*2.82E-12*	*1.15E-10*	*1.83E-10*	*2.27E-10*
1	*10*	*0.1*	-	-	*8.70E-10*	*1.33E-7*	*2.42E-11*	*3.65E-11*
	20	*0.05*	-	-	-	-	*3.80E-13*	*1.66E-12*
	40	*0.025*	-	-	-	-	*6.22E-15*	*5.95E-14*

Table 2. Errors for *Example 2*

The *Examples 1* and *2* were taken from [1]. The exact solutions of these equations we approximated by the exact collocation method, the integrals occurring in the collocation equation (13) being evaluated analytically. In case of *Example 2* the obtained nonlinear equations were solved by the Newton's method.

We choose $m = 3$ and $d \in \{0, 1\}$, and for the collocation parameters Gauss points, Radau II points and Gauss points with $c_m = 1$, respectively.

We found that the collocation method for $m = 3$ and $d = 1$ exhibits an unstable behaviour for Gauss and Radau II points. In the table 2 the symbol '-' indicates that Newton method could not solve the system of nonlinear algebraic equations on the whole interval [0,T] for Gauss and Radau II points.

The stability properties of the collocation methods with $d \geq 1$ will be studied elsewhere.

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