GLASNIK MATEMATIČKI
Vol. 52(72)(2017), 241 – 246

ON CERTAIN EQUATION RELATED TO DERIVATIONS ON STANDARD OPERATOR ALGEBRAS AND SEMIPRIME RINGS

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Abstract. In this paper we prove the following result, which is related to a classical result of Chernoff. Let X be a real or complex Banach space, let $\mathcal{A}(X)$ be a standard operator algebra on X and let $\mathcal{L}(X)$ be an algebra of all bounded linear operators on X. Suppose we have a linear mapping $D : \mathcal{A}(X) \to \mathcal{L}(X)$ satisfying the relation $D(A^{m+n}) = D(A^m)A^n + A^m D(A^n)$ for all $A \in \mathcal{A}(X)$ and some fixed integers $m \geq 1, n \geq 1$. In this case there exists $B \in \mathcal{L}(X)$, such that $D(A) = AB - BA$ holds for all $A \in \mathcal{F}(X)$, where $\mathcal{F}(X)$ denotes the ideal of all finite rank operators in $\mathcal{L}(X)$. Besides, $D(A^m) = A^m B - BA^m$ is fulfilled for all $A \in \mathcal{A}(X)$.

Throughout, R will represent an associative ring with center $Z(R)$. Given an integer $n > 1$, a ring R is said to be $n-$torsion free, if $nx = 0$, $x \in R$ implies $x = 0$. Recall that a ring R is prime if for $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime in case $aRa = (0)$ implies $a = 0$. Let A be an algebra over the real or complex field and let B be a subalgebra of A. A linear mapping $D : B \to A$ is called a linear derivation in case $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in B$. In case we have a ring R an additive mapping $D : R \to R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. A derivation D is inner in case there exists $a \in R$, such that $D(x) = ax - xa$ holds for all $x \in R$. Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein ([5]) asserts that any Jordan derivation on a $2-$torsion free prime ring is a derivation. A brief proof

2010 *Mathematics Subject Classification.* 16N60, 39B05, 46K15.
Key words and phrases. Prime ring, semiprime ring, Banach space, standard operator algebra, derivation, Jordan derivation.

of Herstein theorem can be found in [1]. Cusack ([4]) generalized Herstein theorem to 2−torsion free semiprime rings (see also [2] for an alternative proof). Let X be a real or complex Banach space and let $\mathcal{L}(X)$ and $\mathcal{F}(X)$ denote the algebra of all bounded linear operators on X and the ideal of all finite rank operators in $\mathcal{L}(X)$, respectively. An algebra $\mathcal{A}(X) \subset \mathcal{L}(X)$ is said to be standard in case $\mathcal{F}(X) \subset \mathcal{A}(X)$. Let us point out that any standard algebra is prime, which is a consequence of Hahn-Banach theorem.

Let us start with the following result proved by Chernoff ([3]) (see also [6, 7]).

THEOREM A. *Let X be a real or complex Banach space and let $\mathcal{A}(X)$ be a standard operator algebra. Let $D : \mathcal{A}(X) \to \mathcal{L}(X)$ be a linear derivation. In this case D is of the form $D(A) = AB - BA$ for all $A \in \mathcal{A}(X)$ and some $B \in \mathcal{L}(X)$.*

It is our aim in this paper to prove the following result which is related to the result mentioned above.

THEOREM 1. *Let X be a real or complex Banach space and let $\mathcal{A}(X)$ be a standard operator algebra on X. Suppose we have a linear mapping $D : \mathcal{A}(X) \to \mathcal{L}(X)$ satisfying the relation*

$$D(A^{m+n}) = D(A^m)A^n + A^m D(A^n)$$

for all $A \in \mathcal{A}(X)$ and some fixed integers $m \geq 1, n \geq 1$. In this case there exists $B \in \mathcal{L}(X)$, such that the following statements are fulfilled

(i) $D(A) = AB - BA$ *for all* $A \in \mathcal{F}(X)$,
(ii) $D(A^m) = A^m B - BA^m$ *for all* $A \in \mathcal{A}(X)$.

In the proof of the result above we shall use Herstein theorem and Theorem A.

PROOF. We have the relation

$$D(A^{m+n}) = D(A^m)A^n + A^m D(A^n). \tag{1}$$

Let A be from $\mathcal{F}(X)$ and let $P \in \mathcal{F}(X)$ be a projection with $AP = PA = A$. From the above relation one obtains

$$D(P) = D(P)P + PD(P).$$

Right multiplication of the above relation by P gives

$$PD(P)P = 0. \tag{2}$$

Setting $A+\alpha P$ for A, $\alpha \in \mathcal{F}$, in the relation (1) we obtain

$$\begin{aligned}\sum_{i=0}^{m+n}&\binom{m+n}{i}D\left(A^{m+n-i}(\alpha P)^i\right)\\ &=D\left(\sum_{i=0}^{m}\binom{m}{i}A^{m-i}(\alpha P)^i\right)\left(\sum_{i=0}^{n}\binom{n}{i}A^{n-i}(\alpha P)^i\right)\\ &\quad+\left(\sum_{i=0}^{m}\binom{m}{i}A^{m-i}(\alpha P)^i\right)D\left(\sum_{i=0}^{n}\binom{n}{i}A^{n-i}(\alpha P)^i\right).\end{aligned}$$

Collecting all expressions with coefficient α^{m+n-2} and all expressions with coefficient α^{m+n-1} from the above relation gives

$$\begin{aligned}\binom{m+n}{m+n-2}&D(A^2)-\binom{m}{m-2}\binom{n}{n}D(A^2)P-\binom{m}{m-1}\binom{n}{n-1}D(A)A\\ &-\binom{m}{m}\binom{n}{n-2}D(P)A^2-\binom{m}{m-2}\binom{n}{n}A^2D(P)\\ &-\binom{m}{m-1}\binom{n}{n-1}AD(A)-\binom{m}{m}\binom{n}{n-2}PD(A^2)=0,\end{aligned}$$

and

$$\begin{aligned}\binom{m+n}{m+n-1}&D(A)-\binom{m}{m-1}\binom{n}{n}D(A)P-\binom{m}{m}\binom{n}{n-1}D(P)A\\ &-\binom{m}{m-1}\binom{n}{n}AD(P)-\binom{m}{m}\binom{n}{n-1}PD(A)=0,\end{aligned}$$

respectively. The above equations reduce into

$$\begin{aligned}&(m+n)(m+n-1)D(A^2)=m(m-1)D(A^2)P+n(n-1)PD(A^2)\\ &\quad+2mn(D(A)A+AD(A))+m\,(m-1)\,A^2D(P)+n(n-1)D(P)A^2,\end{aligned}\tag{3}$$

and

$$(m+n)\,D(A)=mD(A)P+nD(P)A+mAD(P)+nPD(A),\tag{4}$$

respectively. Applying the relation (2) and the fact that $AP=PA=A$, we have

$$PD(P)A=(PD(P)P)A=0.\tag{5}$$

Similarly one obtains that

$$AD(P)P=0.\tag{6}$$

Right multiplication of the relation (4) by P and using (6) we obtain

$$D(A)P=D(P)A+PD(A)P.\tag{7}$$

Similarly one obtains using (5)

$$PD(A) = AD(P) + PD(A)P. \tag{8}$$

Subtracting the relation (8) from the relation (7) we obtain

$$D(A)P - PD(A) - D(P)A + AD(P) = 0. \tag{9}$$

Using the relation (9) in the relation (4) we obtain

$$D(A) = D(P)A + PD(A) = D(A)P + AD(P). \tag{10}$$

From the above relation one obtains

$$D(A^2) = D(P)A^2 + PD(A^2) = D(A^2)P + A^2D(P). \tag{11}$$

Using the above relation in the relation (3) we obtain after some calculation

$$D(A^2) = D(A)A + AD(A) \tag{12}$$

for any $A \in \mathcal{F}(X)$. From the relation (4) one can conclude that $D(A) \in \mathcal{F}(X)$ for any $A \in \mathcal{F}(X)$. It means that D is a mapping which maps $\mathcal{F}(X)$ into itself. According to the relation (12) we have therefore a Jordan derivation on $\mathcal{F}(X)$. As we have mentioned at the beginning of the paper any standard algebra is prime, which is a consequence of Hahn-Banach theorem. Since $\mathcal{F}(X)$ is prime all the assumptions of Herstein theorem are fulfilled and one can conclude that D is a derivation. Applying Theorem A one can conclude that D is of the form

$$D(A) = AB - BA, \tag{13}$$

for all $A \in \mathcal{F}(X)$ and some $B \in \mathcal{L}(X)$. This proves the statement (i) of the theorem. It remains to prove the statement (ii). Let us introduce $D_1 : \mathcal{A}(X) \to \mathcal{L}(X)$ by $D_1(A) = AB - BA$ and consider $D_0 = D - D_1$. The mapping D_0 is, obviously, linear and satisfies the relation (1). Besides D_0 vanishes on $\mathcal{F}(X)$. Let $A \in \mathcal{A}(X)$, let $P \in \mathcal{F}(X)$, be a one-dimensional projection and $S = A + PAP - (AP + PA)$. Since, obviously, $S - A \in \mathcal{F}(X)$, we have $D_0(S) = D_0(A)$. Besides $SP = PS = 0$. We have therefore the relation

$$D_0(S^{m+n}) = D_0(S^m)S^n + S^mD_0(S^n). \tag{14}$$

Applying the relation (14) and the fact that $SP = PS = 0$, $D_0(P) = 0$ we obtain

$$\begin{aligned} D_0(S^m)S^n + S^mD_0(S^n) &= D_0(S^{m+n}) = D_0(S^{m+n} + P) \\ &= D_0((S+P)^{m+n}) = D_0((S+P)^m)(S+P)^n + (S+P)^mD_0((S+P)^n) \\ &= D_0((S^m + P)(S^n + P) + (S^m + P)D_0(S^n + P) \\ &= D_0(S^m)S^n + D_0(S^m)P + S^mD_0(S^n) + PD_0(S^n). \end{aligned}$$

We have therefore

$$D_0(A^m)P + PD_0(A^n) = 0. \tag{15}$$

Putting $2A$ for A in the above relation one obtains

$$2^{m-n}D_0(A^m)P + PD_0(A^n) = 0. \tag{16}$$

Combining the relations (15) and (16) we arrive at

$$D_0(A^m)P = 0.$$

Since P is an arbitrary one-dimensional projection, it follows from the above relation that $D_0(A^m) = 0$ for any $A \in \mathcal{A}(X)$. Thus we have

$$0 = D_0(A^m) = D(A^m) - D_1(A^m),$$

which means that

$$D(A^m) = D_1(A^m) = A^m B - BA^m,$$

for all $A \in \mathcal{A}(X)$. The proof of the theorem is therefore complete. □

We proceed with the following conjecture.

Conjecture 2. *Let R be a semiprime ring with suitable torsion restrictions and let $D : R \to R$ be an additive mapping satisfying the relation*

$$D(x^{m+n}) = D(x^m)x^n + x^m D(x^n)$$

for all $x \in R$ and some fixed integers $m \geq 1, n \geq 1$. In this case D is a derivation.

We are going to prove the above conjecture in case a ring has the identity element.

Theorem 3. *Let $m \geq 1$, $n \geq 1$ be some fixed integers and let R be a $2, m$ and $n-$torsion free semiprime ring with the identity element. Suppose we have an additive mapping $D : R \to R$ satisfying the relation*

$$D(x^{m+n}) = D(x^m)x^n + x^m D(x^n)$$

for all $x \in R$. In this case D is a derivation.

Proof. We have the relation

$$D(x^{m+n}) = D(x^m)x^n + x^m D(x^n), \quad x \in R. \tag{17}$$

From the above relation it follows immediately that $D(e) = 0$, where e stands for the identity element. With the same approach as in the proof of Theorem 1 we obtain from the relation (17)

$$\binom{m+n}{m+n-2}D(x^2) = \binom{m}{m-2}\binom{n}{n}D(x^2) + \binom{m}{m-1}\binom{n}{n-1}D(x)x + \binom{m}{m-1}\binom{n}{n-1}xD(x) + \binom{m}{m}\binom{n}{n-2}D(x^2), \quad x \in R.$$

From the above relation we obtain after some calculation

$$D(x^2) = D(x)x + xD(x), \quad x \in R. \tag{18}$$

In the procedure mentioned above we used the fact that R is 2, m and $n-$torsion free. The relation (18) means that D is a Jordan derivation. According to Cusack's generalization of Herstein theorem D is a derivation. The proof of the theorem is complete. □

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Received: 23.11.2016.
Revised: 15.3.2017.