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IDEALITY IN HILBERT C^*-MODULES: IDEAL SUBMODULES VS. TERNARY IDEALS

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Abstract. The definition of ideal submodules of Hilbert C^*-modules is known and classical. We introduce a definition of ternary ideals of Hilbert C^*-modules and show that in general the set of norm-closed ternary ideals is richer than the set of ideal submodules.

1. Introduction

Notion of ideal submodules of Hilbert C^*-modules first appeared in 1979 (H_m's in [7]). In [1] D. Bakić and B. Guljaš gave a formal definition of ideal submodules needed in a theory of extensions of Hilbert C^*-modules developed later in the series of papers ([2, 3]). Ideal submodules of Hilbert C^*-modules are generalisations of norm-closed, two-sided ideals of C^*-algebras. Here we give a definition of norm-closed ternary ideals of Hilbert C^*-modules and show that the set of norm-closed ternary ideals is richer than the set of ideal submodules.

The structure of this paper is the following. In Section 2 we give preliminary definitions of Hilbert C^*-modules and their ideal submodules. We also comment on a bimodule structure of a Hilbert C^*-module as a part of the linking C^*-algebra. Section 3 introduces two module maps that are equivalent to each other: morphisms of modules and ternary homomorphisms. Finally, there is a definition of ternary ideals in Section 4. The main theorem there (Theorem 4.3) claims that ideal submodules and closed ternary ideals are not the same.

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2. Hilbert C^*-modules and ideal submodules

Let $\mathcal{B}$ be a C^*-algebra. A Hilbert C^*-module E over a C^*-algebra $\mathcal{B}$ is a complex vector space and a right $\mathcal{B}$-module which is complete in the norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$ given for an inner product $\langle \cdot, \cdot \rangle : E \times E \to \mathcal{B}$ that satisfies:

1. $\langle x, \lambda y + z\rangle = \lambda\langle x, y\rangle + \langle x, z\rangle$,
2. $\langle x, ya\rangle = \langle x, y\rangle a$,
3. $\langle x, y\rangle^* = \langle y, x\rangle$,
4. $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ implies $x = 0$.

We will call E simply a Hilbert $\mathcal{B}$-module.

We denote by $\mathcal{B}_E = \overline{\text{span}}\,\langle E, E\rangle$ the *range ideal* in $\mathcal{B}$. If $\mathcal{B}_E = \mathcal{B}$, we say that a Hilbert $\mathcal{B}$-module E is *full*. Denote by $\mathbf{K}(E)$ the C^*-algebra of all "compact" operators on a Hilbert $\mathcal{B}$-module E, that is $\mathbf{K}(E) = \{xy^* : x, y \in E\}$ for a "rank one operator" xy^* given by its action $xy^*(z) = x\langle y, z\rangle$. A full right Hilbert $\mathcal{B}$-module E additionaly has a structure of a full left $\mathbf{K}(E)$-module. Namely, besides the right inner product $\langle \cdot, \cdot \rangle$ taking values in $\mathcal{B}$, one can naturally define the inner product ${}_{\mathbf{K}(E)}\langle x, y\rangle = xy^*$, with values in $\mathbf{K}(E)$. We have

$$ {}_{\mathbf{K}(E)}\langle x, y\rangle z = xy^*(z) = x\langle y, z\rangle. $$

This property gives E the structure of a $\mathbf{K}(E) - \mathcal{B}-$bimodule (cf. [5]). The same follows from the theory of linking C^*-algebras. The *linking C^*-algebra* $\mathcal{L}(E)$ of E was introduced in [4]. It is defined as the matrix algebra of the form

$$ \mathcal{L}(E) = \begin{bmatrix} \mathbf{K}(\mathcal{B}) & \mathbf{K}(E, \mathcal{B}) \\ \mathbf{K}(\mathcal{B}, E) & \mathbf{K}(E) \end{bmatrix} $$

i.e. it is isomorphic to $\mathbf{K}(\mathcal{B} \oplus E)$, the C^*-algebra of all "compact" operators on a Hilbert C^*-module $\mathcal{B} \oplus E$. After identifications of corresponding corners, the linking algebra of E can be written in its common form

$$ \mathcal{L}(E) = \begin{bmatrix} \mathcal{B} & E^* \\ E & \mathbf{K}(E) \end{bmatrix}. $$

If $\mathcal{A}$ is a norm-closed two-sided ideal in a C^*-algebra $\mathcal{B}$, the *ideal submodule I of E associated with* $\mathcal{A}$ is $I = E\mathcal{A}$, see [1]. More generally, we say $I \subset E$ is an *ideal submodule* of E if $I = E\mathcal{A}$ for some ideal $\mathcal{A}$ in $\mathcal{B}$. Further, $\mathcal{B}_I = \langle I, I\rangle$ is the unique smallest ideal in $\mathcal{B}$ for which I is an associated ideal submodule. Indeed, if $I = E\mathcal{A}$ for an ideal $\mathcal{A}$ of $\mathcal{B}$, then also $I = E\mathcal{A}\mathcal{B}_I$. So, $\mathcal{A} \cap \mathcal{B}_I$ is a smaller ideal with which I is associated. Now, if $\mathcal{A} \cap \mathcal{B}_I$ would be smaller than $\mathcal{B}_I$, then $E\mathcal{A}\mathcal{B}_I$ would be necessarily smaller than I. Let us emphasise the following three facts concerning ideal submodules:

(i) Any ideal submodule I of a given Hilbert C^*-module E is generated by a certain norm-closed two-sided ideal $\mathcal{A}$ of $\langle E, E\rangle$ as $I = E\mathcal{A}$ and therefore $\langle I, I\rangle = \mathcal{A} = \mathcal{B}_I$. In other words, there is a one-to-one correspondence between norm-closed two-sided ideals $\mathcal{A}$ of $\langle E, E\rangle$ and ideal submodules $I = E\mathcal{A}$ of E.

(ii) If I is a norm-closed ideal submodule of E, then $I\langle E, E\rangle \subset I$. Namely, if I is an ideal submodule associated to an ideal $\mathcal{A}$ in $\langle E, E\rangle$, then

$$I\langle E, E\rangle = E\mathcal{A}\langle E, E\rangle \subset E\mathcal{A} = I.$$

(iii) If there are two Hilbert C^*-modules E and F with $\langle E, E\rangle = \langle F, F\rangle$, then there is a one-to-one correspondence between ideal submodules of E and F.

3. Morphisms of modules and ternary homomorphisms

Let E be a Hilbert $\mathcal{B}$-module and F be a Hilbert $\mathcal{C}$-module. Morphisms of modules are special maps between Hilbert C^*-modules.

A map $\Phi : E \to F$ is called *a morphism of modules* if there is a $*$-homomorphism $\varphi : \mathcal{B} \to \mathcal{C}$ of underlying C^*-algebras such that $\langle \Phi(x), \Phi(y)\rangle = \varphi(\langle x, y\rangle)$ is satisfied for all $x, y \in E$. Sometimes module maps are also called generalized isometries for an obvious reason. Each morphism of modules is necessarily both linear and contractive. It is also a module map in the sense that $\Phi(va) = \Phi(v)\varphi(a)$ is valid for all $v \in E, a \in \mathcal{B}$. Indeed,

$$\begin{aligned}\langle \Phi(x), \Phi(ya)\rangle &= \varphi(\langle x, ya\rangle) = \varphi(\langle x, y\rangle a) = \varphi(\langle x, y\rangle)\varphi(a)\\ &= \langle \Phi(x), \Phi(y)\varphi(a)\rangle.\end{aligned}$$

A linear map $\Phi : E \to F$ such that $\Phi(x)\langle \Phi(y), \Phi(z)\rangle = \Phi(x\langle y, z\rangle)$ is satisfied for all $x, y, z \in E$ is called a *ternary homomorphism*. This definition originates from [6] but there the authors did not require Φ to be linear assuming it is a consequence of the defining property of a ternary homomorphism. There are, however, maps that satisfy a ternary property but are not linear. The simplest example of such ternary homomorphism is the homomorphism $\Phi : \mathcal{B} \to \mathcal{C}$ (on C^*-algebras considered as Hilbert C^*-modules over themselves) defined by $\Phi(x) := 1_{\mathcal{C}},\ x \in \mathcal{B}$, where $\mathcal{C}$ is supposed to have the identity $1_{\mathcal{C}}$.

The property of a morphism of modules to be a module map ensures that it is also a ternary homomorphism:

$$\Phi(x)\langle \Phi(y), \Phi(z)\rangle = \Phi(x)\varphi(\langle y, z\rangle) = \Phi(x\langle y, z\rangle).$$

The converse is also true for Φ defined on a full Hilbert $\mathcal{B}$-module E; this is proved in Theorem 2.1 of [6]. We repeat the proof here for the sake of completeness.

Theorem 3.1 (cf. Theorem 2.1, [6]). *A ternary homomorphism Φ from a full Hilbert $\mathcal{B}$-module E to a Hilbert $\mathcal{C}$-module F is also a generalized isometry.*

Proof. The authors define a homomorphism $\varphi : \mathcal{B} \to \mathcal{C}$ by a left action of $\varphi(b), b \in \mathcal{B}$ on the elements of the pre-C^*-algebra $\mathcal{C}_{\Phi(E)} :=$ span $\langle \Phi(E), \Phi(E)\rangle$

as follows

$$\varphi(b)\langle\Phi(x),\Phi(y)\rangle := \langle\Phi(xb^*),\Phi(y)\rangle.$$

They did not notice that φ is not a homomorphism because it fails to be linear due to the fact that ternary homomorphisms in [6] are not defined as linear maps satisfying the ternary property. Since we include the property of beeing linear into the definition of a ternary homomorphism, the proof from [6] is correct. Clearly, if well-defined, φ is multiplicative. So, firstly, one has to see that φ is well-defined and that it maps into $\mathbf{B}^a(\overline{\mathcal{C}_{\Phi(E)}})$. The decisive property which guarantees that $\varphi(b)$ is well-defined operator on the pre-C^*-algebra generated by $\langle\Phi(E),\Phi(E)\rangle$ is the property of possessing an adjoint. The authors show that $\varphi(b^*)$ is an adjoint of $\varphi(b)$ by observing first that for all $c \in \mathcal{C}_{\Phi(E)}$ the following is valid:

$$\langle c, \langle\Phi(x),\Phi(y)\rangle\rangle = c^*\langle\Phi(x),\Phi(y)\rangle = \langle\Phi(x)c,\Phi(y)\rangle.$$

Then, using this and the ternary property, they find

$$\begin{aligned}
\langle\langle\Phi(x),\Phi(y)\rangle,\varphi(b)\langle\Phi(x'),\Phi(y')\rangle\rangle &= \langle\langle\Phi(x),\Phi(y)\rangle,\langle\Phi(x'b^*),\Phi(y')\rangle\rangle\\
&= \langle\Phi((x'b^*)\langle x,y\rangle),\Phi(y')\rangle\\
&= \langle\Phi((x')\langle xb,y\rangle),\Phi(y')\rangle\\
&= \langle(\Phi(x')\langle\Phi(xb),\Phi(y)\rangle,\Phi(y')\rangle\\
&= \langle\langle\Phi(xb),\Phi(y)\rangle,\langle\Phi(x'),\Phi(y')\rangle\rangle\\
&= \langle\varphi(b^*)\langle\Phi(x),\Phi(y)\rangle,\langle\Phi(x'),\Phi(y')\rangle\rangle.
\end{aligned}$$

Next, like every homomorphism from a C^*-algebra into the adjointable operators on a pre-Hilbert C^*-module, φ maps into bounded operators and is also a contraction (like every homomorphism from a C^*-algebra into a pre-C^*-algebra). Further, calculating how $\varphi(\langle x,y\rangle)$ acts on $\mathcal{C}_{\Phi(E)}$

$$\begin{aligned}
\varphi(\langle x,y\rangle)\langle\Phi(x'),\Phi(y')\rangle &= \langle\Phi(x'(\langle x,y\rangle^*),\Phi(y')\rangle\\
&= \langle\Phi(x')\langle\Phi(y),\Phi(x)\rangle,\Phi(y')\rangle = \langle\Phi(x),\Phi(y)\rangle\langle\Phi(x'),\Phi(y')\rangle
\end{aligned}$$

we see that it is simply by multiplication with the element $\langle\Phi(x),\Phi(y)\rangle$ from the left. So, the subalgebra of $\varphi(\mathcal{B}_E)$ of $\mathbf{B}^a(\overline{\mathcal{C}_{\Phi(E)}})$ is $\mathcal{C}_{\Phi(E)}$ itself and it is faithfully contained in $\mathbf{B}^a(\overline{\mathcal{C}_{\Phi(E)}})$. Therefore, one can conclude that φ has the unique continuous extension from $\mathcal{B}_E$ to its completion $\mathcal{B}$ and so maps into $\overline{\mathcal{C}_{\Phi(E)}} \subset \mathcal{C}$ (and obviously turns Φ into a φ-isometry). ∎

4. Ternary ideals

Definition 4.1. *A linear subspace I of a Hilbert $\mathcal{B}$-module E is a ternary ideal in E if $E\langle I,E\rangle \subset I$.*

Example 4.2. For a ternary homomorphism $\Phi : E \to F$, $\operatorname{Ker}\Phi$ is a ternary ideal in E. Namely, by the ternary property for $x, z \in E, y \in \operatorname{Ker}\Phi$ we have

$$\Phi(x\langle y, z\rangle) = \Phi(x)\langle\Phi(y), \Phi(z)\rangle = 0$$

and so we see that $E\langle\operatorname{Ker}\Phi, E\rangle \subset \operatorname{Ker}\Phi$ as required.

On the other hand, let $\mathcal{B} = \mathbf{B}(l_2)$ and let $p \in \mathcal{B}$ be a non trivial projection onto a finite-dimensional subspace of l_2. Setting $E = \mathcal{B}$ and $I = p\mathcal{B}$ one obtains $E\langle I, E\rangle = \mathbf{B}(l_2)p\mathbf{B}(l_2) = \mathbf{K}(l_2) \not\subset I$. So I is really not a ternary ideal of E. The same is valid for $\mathcal{B}_1 = \mathbf{K}(l_2)$.

Theorem 4.3 claims that the set of norm-closed ternary ideals is richer than the set of ideal submodules.

Theorem 4.3. *An ideal submodule I of a Hilbert $\mathcal{B}$-module E is also a norm-closed ternary ideal of E. The converse is not true.*

Proof. If I is an ideal submodule, i.e. $I = E\mathcal{B}_I$, it is sure a norm-closed $\mathcal{B}$-submodule of E. (To show it is a linear space, we make use of an approximate unit for $\mathcal{B}$.) Since for each submodule I, $\langle I, E\rangle \subset \mathcal{B}_I$, we get $E\langle I, E\rangle \subset E\mathcal{B}_I = I$.

As a counterexample to the converse take $\mathcal{B}$ to be the bounded linear diagonal operators on the Hilbert space (direct sum) $l_2^{(1)} \oplus l_2^{(2)}$, i.e. $\mathcal{B} = \{(h, g) : h \in \mathbf{B}(l_2^{(1)}), g \in \mathbf{B}(l_2^{(2)})\}$. Then the inclusion hierarchy of norm-complete two-sided ideals in $\mathcal{B}$ is not a linear graph: e.g. we have $A_1 = \{(h, g) : h \in \mathbf{B}(l_2^{(1)}), g \in \mathbf{K}(l_2^{(2)})\}$ and $A_2 = \{(h, g) : h \in \mathbf{K}(l_2^{(1)}), g \in \mathbf{K}(l_2^{(2)})\}$. Set $E = \mathcal{B} \oplus \mathcal{B}$. Consequently, the Hilbert $\mathcal{B}$-module $I := A_1 \oplus A_2$ (direct orthogonal sum) is a closed ternary ideal of E, but it is not an ideal submodule of E. ∎

Remark 4.4. In fact, already $\mathcal{B} = \mathbf{B}(l_2) \oplus \mathbf{B}(l_2)$ and $I = \mathbf{K}(l_2) \oplus \mathbf{B}(l_2)$ give a counterexample. So the hierarchy of closed two-sided ideals of $\mathcal{B}$ may even be a linear graph. The necessary additional condition on closed ternary ideals might be that every Hilbert $\mathcal{B}$-submodule of I which is an orthogonal summand of I has the same maximal range equal to $\langle I, I\rangle$. (This is not true for submodules which are not direct orthogonal summands, like proper ideals.)

Proposition 4.5. *Let I be a closed ternary ideal in a Hilbert $\mathcal{B}$-module E. Then $E\langle E, I\rangle \subset I$. If $\Phi : E \to F$ is a surjective ternary homomorphism that maps E onto a Hilbert $\mathcal{C}$-module F, then $\Phi(I)$ is a ternary ideal in F.*

Proof. The first claim follows from the fact that $\langle E, I\rangle = \langle I, E\rangle$ is valid. If I is a closed ternary ideal, then

$$\langle E, E\rangle\langle I, E\rangle \subset \langle E, I\rangle.$$

Making use of an approximate unit for $\mathcal{B}_E$, we get $\langle I, E\rangle \subset \langle E, I\rangle$, and by taking adjoints $\langle E, I\rangle \subset \langle I, E\rangle$. So $E\langle E, I\rangle \subset I$ as claimed. The second claim is a simple consequence of the ternary property of Φ. ∎

REMARK 4.6. Inclusion $E\langle E, I\rangle \subset I$ implies also $\mathbf{K}(E)I \subseteq I$. This reveals that ternary ideals are left ideals in $\mathbf{K}(E)$.

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