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INTRINSIC STRONG SHAPE FOR PARACOMPACTA

Beti Andonovic and Nikita Shekutkovski
Ss Cyril and Methodius University, Macedonia

Dedicated to Sibe Mardešić

Abstract. In this paper an intrinsic definition of strong shape for paracompact topological spaces is presented. At first a coherent proximate net $\underline{f} : X \to Y$ is defined, indexed by finite subsets of normal coverings of Y, and then there is a homotopy between two coherent proximate nets defined. A definition of composition of classes of homotopies between two coherent proximate nets $\underline{f} : X \to Y$ and $\underline{g} : Y \to Z$ is given. Then it is proved that for any other choice of corresponding coverings, a function is obtained that is in the same class with the previously defined composition. The strong shape category is obtained, with paracompacta as objects, and the homotopy classes of coherent proximate nets as morphisms.

1. Introduction

The shape theory has shown to be more appropriate tool than homotopy theory when study of spaces with bad local behavior is involved ([2, 6–8, 10]). The strong shape theory is a strengthening of shape theory ([3, 6]). The first definition of strong shape for compact metric spaces is given in [9] by embedding metric compacta in Hilbert cube. In [6, 8] strong shape is described for topological spaces approximating the space by inverse system of polyhedra (ANRs). Both approaches follow the corresponding approaches in shape theory. For equivalence of different approaches for metric compacta we refer to [6].

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The intrinsic approach to shape does not use any approximation of spaces. The intrinsic definition of strong shape for compact metric spaces is presented in [11].

The definition of strong shape in [11] is based on the notion of strong proximate sequence.

The sequence of pairs $(f_n, f_{n,n+1})$ of functions $f_n : X \to Y$ and $f_{n,n+1} : X \times I \to Y$, is a *strong proximate sequence* from X to Y, if there exists a cofinal sequence of finite coverings, $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \cdots$ of Y, such that for each natural number n, $f_n : X \to Y$, is a $\mathcal{V}_n$-continuous function and $f_{n,n+1} : X \times I \to Y$ is a homotopy connecting $\mathcal{V}_n$-continuous functions $f_n : X \to Y$ and $f_{n,n+1} : X \times I \to Y$.

We say that $(f_n, f_{n,n+1})$ is a strong proximate sequence over $(\mathcal{V}_n)$.

If $(f_n, f_{n,n+1})$ and $(f'_n, f'_{n,n+1})$ are strong proximate sequences from X to Y, than there exists a cofinal sequence of finite coverings $(\mathcal{V}_n)$ such that $(f_n, f_{n,n+1})$ and $(f'_n, f'_{n,n+1})$ are strong proximate sequences over $(\mathcal{V}_n)$.

In compact metric space, the existence of cofinal sequence of coverings $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \cdots$, allows to define strong shape theory using only homotopies of second order.

In more general case of paracompact spaces, homotopies of all orders must be considered. In [13] the construction for (strongly) paracompact spaces is described. We form all finite sets of coverings of Y, $a = \{\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_n\}$, having a maximal element (i.e. a covering that refines all other coverings of that finite set, and is not refined by any other covering of that finite set). The maximal element is denoted by $\max a$.

Finite sets of coverings with a maximal element are ordered by inclusion, and this ordering is cofinite, i.e. each a has only finite number of predecessors. This fact allows composition of coherent proximate nets to be defined, although such definition is technically rather complex. In this way category of strong shape is obtained for paracompact spaces. In [1] it is shown that strong shape category of metric compacta is a subcategory of the last category.

2. Coherent proximate nets

Let $\Delta^n \subseteq R^n$, $\Delta^n = \{(t_1, t_2, \ldots, t_n) | 1 \geq t_1 \geq t_2 \geq \ldots \geq t_n \geq 0\}$ be the *non standard* n-simplex.

It is important to note that by a function $f : X \to Y$ we do not necessarily mean continuous function.

Let Y be a paracompact space. We form all finite sets of coverings of Y, $a = \{\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_n\}$, having a maximal element (a covering that refines all other coverings of that finite set, and is not refined by any other covering of that finite set). The maximal element is denoted by $\max a$. If $a \subseteq b$, then $\max a \succ \max b$.

We define an ordering "$<$" by $a < b$ if $a \subset b$.

DEFINITION 2.1. *A coherent proximate net $\underline{f}: X \to Y$ consists of functions*

$$\underline{f} = \{f_{\underline{a}} | \underline{a} = (a_0, a_1, \ldots, a_n), a_0 < \cdots < a_n\}$$

such that each $f_{\underline{a}} : X \times \Delta^n \to Y$ is $st^n \max a_0$-continuous and is $st^{n-1} \max a_0$-continuous on $X \times \partial\Delta^n$, and the following coherence condition is satisfied:

$$f_{\underline{a}}(x, t_1, t_2, \ldots, t_n) = \begin{cases} f_{a_1 \ldots a_n}(x, t_2, \ldots, t_n), & t_1 = 1 \\ f_{a_0 \ldots \hat{a}_i \ldots a_n}(x, t_1, \ldots, \hat{t}_i, \ldots, t_n), & t_i = t_{i+1} \\ f_{a_0 \ldots a_{n-1}}(x, t_1, \ldots, t_{n-1}), & t_n = 0 \end{cases}.$$

The coherent proximate net will be shortly denoted by $\underline{f} = (f_{\underline{a}})$. Next we explain the definition in special cases $n = 0$ and $n = 1$. If $n = 0$, for each a_0, there exists $f_{a_0} : X \to Y$, so that f_{a_0} is $\max a_0$- continuous. If $n = 1$, for each a_0, there exists $f_{a_0} : X \to Y$, so that f_{a_0} is $\max a_0$- continuous and for every a_0, a_1, there exists $f_{a_0 a_1} : \Delta^1 \times X \to Y$, such that $f_{a_0 a_1}$ is $st(\max a_0)$-continuous and $f_{\underline{a}}$ is $\max a_0$- continuous on $\partial\Delta^1 \times X$, and also

$$f_{a_0 a_1}(0, x) = f_{a_0}(x), \ f_{a_0 a_1}(1, x) = f_{a_1}(x).$$

DEFINITION 2.2. *Coherent proximate nets $\underline{f}, \underline{g} : X \to Y$ are homotopic (notation: $\underline{f} \approx \underline{g}$), if there exists a coherent proximate net $\underline{H} = (H_{\underline{a}})$, such that $H_{\underline{a}} : X \times \Delta^n \times I \to Y$ is $st^{n+1} \max a_0$- continuous, $H_{\underline{a}}$ is $st^n \max a_0$-continuous on $X \times \partial(\Delta^n \times I)$, and the following conditions are satisfied:*

$$H_{\underline{a}}(x, \underline{t}, 0) = f_{\underline{a}}(x, \underline{t}),$$
$$H_{\underline{a}}(x, \underline{t}, 1) = g_{\underline{a}}(x, \underline{t}).$$

The relation $f \approx g$ is an equivalence relation. This is shown in [13].

3. COMPOSITION OF COHERENT PROXIMATE NETS - EXISTENCE AND UNIQUENESS

The following two Theorems are needed for the definition of composition of coherent proximate nets. Theorem 3.2 is proved in [9] for the case $k = 1$.

THEOREM 3.1. *If $f : x \to Y$ is W-continuous, then $id \times f : K \times X \to K \times Y$ is $K \times \mathcal{W}$-continuous, where K is compact and $K \times \mathcal{W} = \{K \times W | W \in \mathcal{W}\}$.*

PROOF. $f : X \to Y$ is W-continuous, and therefore it follows $\forall x \in X, \exists U$-neighborhood of x and $\exists W \in \mathcal{W}$, so that $f(U) \subseteq W$.

As usually $id \times f$ is defined by $id \times f(k, x) = (k, f(x))$. Hence, for $(k, x) \in K \times X$, there exists U, U being a neighborhood of x, and there exists $K \times W \in K \times \mathcal{W}$, so that $id \times f(K \times U) \subseteq K \times W$. □

THEOREM 3.2. *Let $\mathcal{W}$ be a covering of Z and $G : \Delta^k \times Y \to Z$ be a $st^k(\mathcal{W})$-continuous function and $st^{k-1}(\mathcal{W})$- continuous on $\partial\Delta^k \times Y$. Then there exists $\mathcal{V}$, a covering of Y, such that for each function $f : X \to Y$ that is $\mathcal{V}$-continuous, $G(id \times f) : \Delta^k \times X \to Z$ is $st^k(\mathcal{W})$-continuous, and $G(id \times f)$ is $st^{k-1}(\mathcal{W})$ - continuous on $\partial\Delta^k \times X$.*

PROOF. Let $y \in Y$ be a fixed point. For each $s \in \Delta^k \backslash \partial\Delta^k$, there exists $J_{\underline{s}} \subseteq \Delta^k \backslash \partial\Delta^k$, a neighborhood of $\underline{s}$, and a neighborhood $V_y^{\underline{s}}$ of y, so that $G(J_{\underline{s}} \times V_y^{\underline{s}}) \subseteq W_y^{\underline{s}}$, for some element $W_y^{\underline{s}} \in st^k(\mathcal{W})$. For each $\underline{s} \in \partial\Delta^k$, there exists $J_{\underline{s}} \subseteq N$, a neighborhood of $\underline{s}$, and a neighborhood $V_y^{\underline{s}}$ of y, so that $G(J_{\underline{s}} \times V_y^{\underline{s}}) \subseteq W_y^{\underline{s}}$, for some element $W_y^{\underline{s}} \in st^{k-1}(\mathcal{W})$.

Then $\{J_{\underline{s}} | \underline{s} \in \Delta^k\}$ is an open covering of Δ^k. There exists a finite subcovering $J_{\underline{s_1}}, J_{\underline{s_2}}, \ldots, J_{\underline{s_p}}$.

Let $J_{\underline{s}} y = V_y^{\underline{s_1}} \cap \ldots V_y^{\underline{s_p}}$. Then $G(J_{\underline{s}_i} \times V_y) \subseteq W_* \frac{s_i}{y}$, for $\underline{s}_i \in \Delta^k \setminus \partial\Delta^k$, and $G(J_{\underline{s}_i} \times V_y) \subseteq W \frac{s_i}{y}$, for $\underline{s}_i \in \partial\Delta^k$. $\mathcal{V} = \{V_y | y \in Y\}$ is a covering of Y. Let $V \in \mathcal{V}$. Then the following holds: $G(J_{\underline{s}_i} \times V_y) \subseteq W_* \frac{s_i}{y}$, for $\underline{s}_i \in \Delta^k \backslash \partial\Delta^k$, and $G(J_{\underline{s}_i} \times V_y) \subseteq W \frac{s_i}{y}$, for $\underline{s}_i \in \partial\Delta^k$. $J_{\underline{s}_i \times V | i=1,\ldots,p, V \in \mathcal{V}}$ is a covering of $\Delta^k \times Y$. If $f : X \to Y$ is a $\mathcal{V}$- continuous function, then there exists a covering $\mathfrak{V}$ of X, so that $f(\mathfrak{V}) \prec \mathcal{V}$. Now, if we define $H : \Delta^k \times X \to Z$ by:

$$H(\underline{t}, x) = G(\underline{t}, f(x)),$$

then H is a $st^k(\mathcal{W})$- continuous function and H is $st^{k-1}(\mathcal{W})$- continuous on $\partial\Delta^k \times X$. ∎

We will now define a partitioning of the simplex

$$\Delta^n = \{(t_1, t_2, \ldots, t_n) | 1 \geq t_1 \geq \ldots \geq t_n \geq 0\},$$

by defining the sets $K_m, 0 \leq m \leq n$ in the following way:

$$K_m = \{(t_1, \ldots, t_n) | t_m \geq \frac{1}{2} \geq t_{m+1}\}.$$

Let B be the denotation of the finite sets of coverings of Y with a maximal element and C be the denotation of the finite sets of coverings of Z with a maximal element. Let $\underline{f} = (f_{\underline{b}}) : X \to Y$ and $\underline{g} = (g_{\underline{c}}) : Y \to Z$ be coherent proximate nets. In order to proceed and define the composition $\underline{h} = (h_{\underline{c}}) : X \to Z$ of $\underline{f}$ and $\underline{g}$, an induction by the height of the element $c \in C$ is performed.

DEFINITION 3.3. *Let $c \in C, h(c) = 0$ be an ordered cofinite set. Then the height of a is defined as follows:*

$$h(a) = \max\{n | a_0 < a_1 < \cdots < a_{n-1} < a\}.$$

A strictly increasing function $g : C \to B$ is constructed as follows:

Case 0. Let $c \in C$, $h(c) = 1$. We choose an element $g(c)$, such that $g(st \max b) \prec \max c$. Let $g(c) = b$. Now $h_c : X \to Z$ may be defined by $h_c = g_c f_{g(c)}$. Then h_c is max c-continuous.

Case 1. Let $c \in C$, $h(c) = 1$. We define $g(c)$, choosing $b \in B$, so that the following conditions hold:

1. $g(st \max b) \subseteq \max c$;
2. $g(c_0) < b$, for all possible choices of $c_0, c_0 < c$;

3. $g_{c_0c}(id \times f_b)$ is $st \max c$-continuous and $g_{c_0c}(id \times f_b)$ is $\max c$-continuous on $\partial\Delta^1 \times X$.

Let $g(c) = b$. The functions h_c and h_{c_0c} are defined as follows:

$$h_c = g_c f_{g(c)}.$$

Then h_c is $\max c$- continuous. The function $h_{c_0c} : \Delta^1 \times X \to Z$ is defined by:

$$h_{c_0c}(t,x) = \begin{cases} g_{c_0c}(2t-1, f_b(x)), & t \in K_0 \\ g_{c_0} f_{b_0b}(2t, x), & t \in K_1 \end{cases} .$$

Theorem 3.1 provides that $g_{c_0} f_{b_0b}$ is $st \max c_0$- continuous, and Theorem 2.2 and the condition 3 provide that $g_{c_0c}(id \times f_b)$ is $st \max c_0$- continuous and $g_{c_0c}(id \times f_b)$ is $maxc_0$- continuous on $\partial\Delta^1 \times X$. Then h_{c_0c} is $st \max c_0$- continuous and h_{c_0c} is $\max c_0$- continuous on $\partial\Delta^1 \times X$. On Figure 1 below there is a given review of the mapping h_{c_0c}.

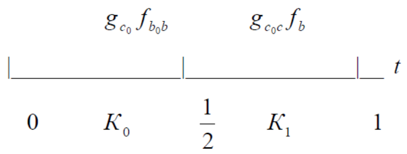

FIGURE 1

Case $n-1$ (inductive assumption). We assume that for each c having a height $h(c) \leq n-1, g(c) = b$ is defined, so that the following conditions hold:

1) $g(st \max b) \prec \max c$;
2) $g(c_0) < g(c_1) < \cdots < g(c_{n-2}) < g(c) = b$, for all possible choices of indices $C_0 < c_1 < \cdots < c_{n-2} < c$;
3) The following mappings are defined: $h_c, h_{c_0c}, h_{c_0c_1c}, \ldots, h_{c_0c_1\ldots c_{n-2}c}$, $0 \leq i \leq n-2$, such that $h_{c_0c_1\ldots c_ic} : \Delta^{i+1} \times X \to Z$ is $st^{i+1} \max c_0$-continuous and $h_{c_0c_1\ldots c_ic}$ is $st^i \max c_0$- continuous on $\partial\Delta^{i+1} \times X$.

On Figure 2 below there is a given review of the mapping $h_{c_0c_1c}$, and the conditions 1-3 below hold.

1. $g(st \max b) \prec \max c$;
2. $g(c_0) < b$, for all possible choices of c_0, $c_0 < c$;
3. $g_{c_0c}(id \times f_b)$ is $st \max c$- continuous and $g_{c_0c}(id \times f_b)$ is $\max c$- continuous on $\partial\Delta^1 \times X$.

Case n. Let $c \in C, h(c) = n$. Herein it is important to mention that there exists a linear homeomorphism between the sets K_i and $\Delta^i \times \Delta^{n-i}$ mapping vertices to vertices. We choose b, so that:

1. $g(st \max b) \prec \max c$;

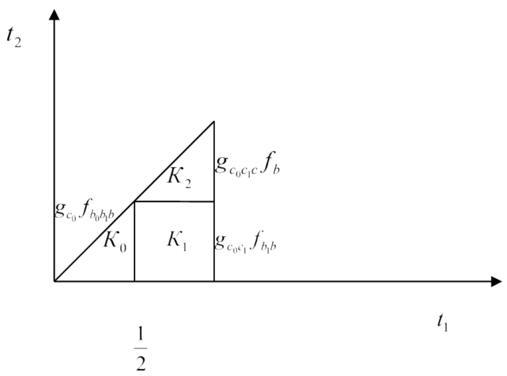

FIGURE 2

2. $g(c') < b$, for all possible choices of c', $c' < c$;
3. $g_{c_0\dots c_k}(id \times f_{b_{k+1}\dots b_n}) : \Delta^k \times \Delta^{n-k} \times X \to Z$ is $st^k \max c$- continuous, for $k = 1, 2, \dots n$ and for $\max b, g_{c_0\dots c_k}(id \times f_{b_{k+1}\dots b_n})$ is $st^{k-1} \max c$-continuous on $\partial\Delta^k \times \Delta^{n-k} \times X$.

Let $g(c) = b$. Because of the inductive assumption, the following holds: $g(c_0) < g(c_1) < \dots < g(c_{n-1}) < g(c) = b$ for each $c_0 < c_1 < \dots < c_{n-2} < c_{n-1} < c$, and the following mappings: $h_c, h_{c_0c}, h_{c_0c_1c}, \dots, h_{c_0c_1\dots c_{n-2}c}$ are defined. We define the function $h_{c_0c_1\dots c_{n-1}c} : \Delta^n \times X \to Z$ by:

$$h_{c_0c_1\dots c_{n-1}c}(\underline{t}, x) =$$

$$\begin{cases} g_{c_0}f_{g(c_0c_1\dots c_{n-1}c)}(2t_1, \dots, 2t_n, x), & t_1 \leq \frac{1}{2} \\ g_{c_0\dots c_i}(2t-1, \dots, 2t_i - 1, f_{g(c_ic_{i+1}\dots c_{n-1}c)}(2t_{i+1}, \dots, 2t_n, x)), & t_i \geq \frac{1}{2} \geq t_{i+1}. \\ g_{c_0c_1\dots c_{n-1}c}(2t_1 - 1, \dots, 2t_i - 1, f_{g(c)}(x)), & t_n \geq \frac{1}{2} \end{cases}$$

Theorem 3.1, Theorem 3.2 and the condition 3 provide that $h_{c_0c_1\dots c_{n-1}c}$ is $st^n \max c_0$- continuous and $h_{c_0c_1\dots c_{n-1}c}$ is $st^{n-1} \max c_0$- continuous on $\partial\Delta^n \times X$. We check that $h_{c_0c_1\dots c_{n-1}c}$ is well defined. As in previous cases, $g(c_0) = b_0, g(c_1) = b_1, \dots, g(c_{n-1}) = b_{n-1}, g(c) = b$. Let $t_1 = \frac{1}{2}$.

$$\begin{aligned} g_{c_0}f_{b_0b_1\dots b_{n-1}b}(2 \cdot \frac{1}{2}, 2t_2, \dots, 2t_n, x) &= g_{c_0}f_{b_0b_1\dots b_{n-1}b}(1, 2t_2, \dots, 2t_n, x) \\ &= g_{c_0}f_{b_0b_1\dots b_{n-1}b}(2t_2, \dots, 2t_n, x). \\ g_{c_0c_1}(2 \cdot \frac{1}{2}, f_{b_0b_1\dots b_{n-1}b}(2t_2, \dots, 2t_n, x)) &= g_{c_0c_1}(0, f_{b_1\dots b_{n-1}b}(2t_2, \dots, 2t_n, x)) \\ &= g_{c_0}(f_{b_1\dots b_{n-1}b}(2t_2, \dots, 2t_n, x)) \\ &= g_{c_0}(f_{b_1\dots b_{n-1}b}(2t_2, \dots, 2t_n, x)). \end{aligned}$$

Let $t_i = t_{i+1} = \frac{1}{2}$. $t_i = t_{i+1}$ implicates the following:

$$g_{c_0 \ldots c_i}(2t_1 - 1, 2t_2 - 1, \ldots, 2t_i - 1, f_{b_i b_{i+1} \ldots b_{n-1} b}(2t_{i+1}, \ldots, 2t_n, x))$$
$$= g_{c_0 \ldots c_{i-1}}(2t_1 - 1, 2t_2 - 1, \ldots, 2t_i - 1, f_{b_i b_{i+1} \ldots b_{n-1} b}(2t_{i+1}, \ldots, 2t_n, x)),$$

whereas $t_i = \frac{1}{2}$ implicates:

$$g_{c_0 \ldots c_i}(2t_1 - 1, 2t_2 - 1, \ldots, 2t_i - 1, f_{b_i b_{i+1} \ldots b_{n-1} b}(2t_{i+1}, \ldots, 2t_n, x))$$
$$= g_{c_0 \ldots c_i}(2t_1 - 1, 2t_2 - 1, \ldots, 2\frac{1}{2} - 1, f_{b_i b_{i+1} \ldots b_{n-1} b}(2t_i, \ldots, 2t_n, x))$$
$$= g_{c_0 \ldots c_i}(2t_1 - 1, 2t_2 - 1, \ldots, 0, f_{b_i b_{i+1} \ldots b_{n-1} b}(2t_{i+1}, \ldots, 2t_n, x))$$
$$= g_{c_0 \ldots c_{i-1}}(2t_1 - 1, 2t_2 - 1, \ldots, 2t_{i-1} - 1, f_{b_i b_{i+1} \ldots b_{n-1} b}(2t_{i+1}, \ldots, 2t_n, x)).$$

Let $t_n = \frac{1}{2}$.

$$g_{c_0 \ldots c_{n-1} c}(2t_1 - 1, 2t_2 - 1, \ldots, 2t_n - 1)$$
$$= g_{c_0 \ldots c_{n-1} c}(2t_1 - 1, 2t_2 - 1, \ldots, 2\frac{1}{2} - 1, f_b(x)),$$
$$g_{c_0 \ldots c_{n-1} c}(2t_1 - 1, 2t_2 - 1, \ldots, 0, f_b(x))$$
$$= g_{c_0 \ldots c_{n-1}}(2t_1 - 1, 2t_2 - 1, \ldots, 2t_{n-1} - 1, f_b(x)).$$

$$g_{c_0 \ldots c_i}(2t_1 - 1, 2t_2 - 1, \ldots, 2t_i - 1, f_{b_i b_{i+1} \ldots b_{n-1} b}(2t_{i+1}, \ldots, 2t_n, x))$$
$$= g_{c_0 \ldots c_{n-1}}(2t_1 - 1, 2t_2 - 1, \ldots, 2t_n - 1, f_{b_{n-1} b}(2 \cdot \frac{1}{2}, x))$$
$$= g_{c_0 \ldots c_{n-1}}(2t_1 - 1, 2t_2 - 1, \ldots, 2t_n - 1, f_{b_{n-1} b}(1, x))$$
$$= g_{c_0 \ldots c_{n-1}}(2t_1 - 1, 2t_2 - 1, \ldots, 2t_n - 1, f_b(x)).$$

Therefore it is obtained that the function is well defined. Consequently it is obtained that the composition of the coherent proximate nets $\underline{f} = (f_{\underline{b}}) : X \to Y$ and $\underline{g} = (g_{\underline{b}}) : Y \to Z$ is a coherent proximate net $\underline{h} = (h_{\underline{c}}) : Y \to Z$. Let this composition be denoted by $\underline{f} \circ \underline{g} : X \to Y$.

4. Composition of homotopy classes of coherent proximate nets

We define a composition of homotopy classes of coherent proximate nets by $[(g_c)][(f_b)] = [(g_c f_{g(c)})]$. In order for this definition to be valid, it is necessary to be proved that it does not depend on the choice of strictly increasing function $g : C \to B$. It is enough to show that for another choice of a strictly increasing function $g' : C \to B$, such that it satisfies the same required conditions 1 - 3 from the definition of composition in 2, the corresponding coherent proximate net $\underline{h}' = (h'_{\underline{c}}) : X \to Z$ is in the same homotopy class with $\underline{h} = (h_{\underline{c}}) : X \to Z$. In fact, as the way $g : C \to B$, and $(h_{\underline{c}}) : X \to Z$, and $g' : C \to B$, $(h'_{\underline{c}}) : X \to Z$ are obtained, similarly we may obtain another strictly increasing function $g'' : C \to B$, with additional condition $g''(c) > g(c), g'(c)$ for all $c \in C$, and a coherent proximate net $(h''_{\underline{c}}) : X \to Z$.

Now, by induction, a homotopy $\underline{H} = (H_{\underline{c}}) : X \times I \to Z$, connecting $(h_{\underline{c}})$ and $(h''_{\underline{c}})$, is constructed.

Case 0. If $c \in C, h(c) = 1, H_c : I \times X \to Z$, is defined by $H_c(t,x) = g_c f_{bg(c")}(t,x)$. This homotopy connects $h_c = g_c f_{g(c)}$ and $h"_c = g_c f_{g(c")}$.

Case 1. Let $c \in C, h(c) = 1$. The homotopy $H_c : I \times X \to Z$ is defined by $H_c(t,x) = g_c f_{bg(c")}(t,x)$. The following step is to show that h_{c_0c} is homotopic to $h"_{c_0c}$. Therefore, $H_{c_0c} : \Delta^1 \times I \times X \to Z$ is defined in the following way (Figure 3):

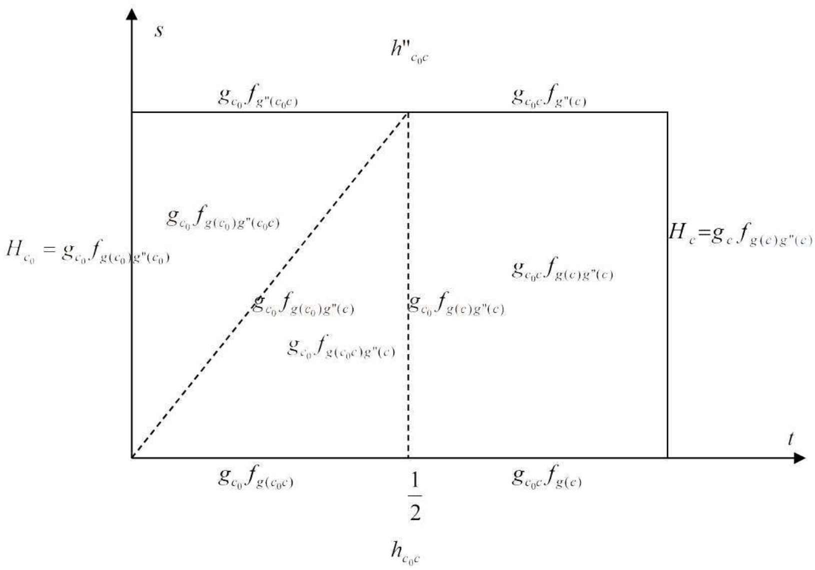

FIGURE 3

$$H_{c_0c}(t,s,x) = \begin{cases} g_{c_0} f_{g(c_0)g"(c_0c)}(s, 2t, x), & 0 \le t \le \frac{s}{2} \\ g_{c_0} f_{g(c_0c)g"(c)}(s, 2t, x), & \frac{s}{2} \le t \le \frac{1}{2} \\ g_{c_0c}(2t-1, f_{g(c)g"(c)}(s,x)), & \frac{1}{2} \le t \le 1 \end{cases} .$$

H_{c_0c} is well defined on the edges, and it is shown as follows: - If $t = \frac{s}{2}$, then

$$g_{c_0} f_{b_0b_0"b"}(s, 2 \cdot \frac{s}{2}, x) = g_{c_0} f_{b_0b_0"b"}(s,s,x) = g_{c_0} f_{b_0b"}(s,x).$$

On the other hand,

$$g_{c_0} f_{b_0bb"}(2 \cdot \frac{s}{2}, s, x) = g_{c_0} f_{b_0bb"}(s,s,x) = g_{c_0} f_{b_0b"}(s,x).$$

If $t = \frac{1}{2}$, then

$$g_{c_0} f_{b_0 bb"}(2 \cdot \frac{1}{2}, s, x) = g_{c_0} f_{b_0 bb"}(1, s, x) = g_{c_0} f_{bb"}(s, x).$$

On the other hand,

$$g_{c_0 c}(2 \cdot \frac{1}{2} - 1, f_{bb"}(s, x)) = g_{c_0 c}(0, f_{bb"}(s, x)) = g_{c_0} f_{bb"}(s, x).$$

The following also holds: If $s = 0$, then

$$\begin{aligned}
H_{c_0 c}(t, 0, x) &= \begin{cases} g_{c_0} f_{b_0 b_0" b"}(0, 2t, x), & t = 0 \\ g_{c_0} f_{b_0 bb"}(2t, 0, x), & 0 \le t \le \frac{1}{2} \\ g_{c_0 c}(2t - 1, f_{bb"}(0, x)), & \frac{1}{2} \le t \le 1 \end{cases} \\
&= \begin{cases} g_{c_0} f_{b_0 b_0" b"}(0, 0, x), & t = 0 \\ g_{c_0} f_{b_0 bb"}(2t, 0, x), & 0 \le t \le \frac{1}{2} \\ g_{c_0 c}(2t - 1, f_{bb"}(0, x)), & \frac{1}{2} \le t \le 1 \end{cases} \\
&= \begin{cases} g_{c_0} f_{b_0 b_0"}(0, x), & t = 0 \\ g_{c_0} f_{b_0 b}(2t, x), & 0 \le t \le \frac{1}{2} \\ g_{c_0 c}(2t - 1, f_b(x)), & \frac{1}{2} \le t \le 1 \end{cases} \\
&= \begin{cases} g_{c_0} f_{b_0}(x), & t = 0 \\ g_{c_0} f_{b_0 b}(2t, x), & 0 \le t \le \frac{1}{2} \\ g_{c_0 c}(2t - 1, f_b(x)), & \frac{1}{2} \le t \le 1 \end{cases} \\
&= \begin{cases} g_{c_0} f_{b_0 b}(2t, x), & 0 \le t \le \frac{1}{2} \\ g_{c_0 c}(2t - 1, f_b(x)), & \frac{1}{2} \le t \le 1 \end{cases} = h_{c_0 c}(t, x).
\end{aligned}$$

If $s = 1$, then

$$\begin{aligned}
H_{c_0 c}(t, 1, x) &= \begin{cases} g_{c_0} f_{b_0 b_0" b"}(1, 2t, x), & 0 \le t \le \frac{1}{2} \\ g_{c_0} f_{b_0 bb"}(2t, 1, x), & t = \frac{1}{2} \\ g_{c_0 c}(2t - 1, f_{bb"}(1, x)), & \frac{1}{2} \le t \le 1 \end{cases} \\
&= \begin{cases} g_{c_0} f_{b_0 b_0" b"}(1, 2t, x), & 0 \le t \le \frac{1}{2} \\ g_{c_0} f_{b_0 bb"}(1, 1, x), & t = \frac{1}{2} \\ g_{c_0 c}(2t - 1, f_{bb"}(1, x)), & \frac{1}{2} \le t \le 1 \end{cases} \\
&= \begin{cases} g_{c_0} f_{b_0 b_0" b"}(1, 2t, x), & 0 \le t \le \frac{1}{2} \\ g_{c_0} f_{bb"}(1, x), & t = \frac{1}{2} \\ g_{c_0 c}(2t - 1, f_{b"}(x)), & \frac{1}{2} \le t \le 1 \end{cases} \\
&= \begin{cases} g_{c_0} f_{b_0" b"}(2t, x), & 0 \le t \le \frac{1}{2} \\ g_{c_0} f_{b"}(x), & t = \frac{1}{2} \\ g_{c_0 c}(2t - 1, f_{b"}(x)), & \frac{1}{2} \le t \le 1 \end{cases} \\
&= \begin{cases} g_{c_0} f_{b_0" b"}(2t, x), & 0 \le t \le \frac{1}{2} \\ g_{c_0 c}(2t - 1, f_{b"}(x)), & \frac{1}{2} \le t \le 1 \end{cases} = h"_{c_0 c}(t, x).
\end{aligned}$$

Hence it is proved that $H_{c_0 c}$ is a homotopy between $h_{c_0 c}$ and $h"_{c_0 c}$.

Case n. The homotopy $H_{c_0c_1\ldots c_{n-1}c} : \Delta^n \times I \times X \to Z$ is defined in the following way: We have defined the partitioning of the non-standard simplex $\Delta^n = \{(t_1, t_2, \ldots, t_n) | 1 \geq t_1 \geq t_2 \ldots \geq t_n \geq 0\}$ by the sets $K_i, 0 \leq i \leq n$ where $K_i = \{(t_1, t_2, \ldots, t_n) | t_i \geq \frac{1}{2} \geq t_{i+1}\}$. Now we need a partitioning of the sets, and therefore we define the sets for each, in the following way:

$$\begin{aligned} K_i^j =& \{(t_1, \ldots, t_i, \ldots, t_n, s) |\, (t_1, \ldots, t_i, \ldots, t_n, s) \in K_i \times I \\ & \text{so that } \forall m, i \leq m \leq n-j, \frac{s}{2} \leq t_m \leq \frac{1}{2}, \forall l, n-j < l \leq n, 0 \leq t_l \leq \frac{1}{2}\}. \end{aligned}$$

Now $H_{c_0c_1\ldots c_{n-1}c} : \Delta^n \times I \times X \to Z$ is defined on $K_i \times I$, $\forall i = 0, 1, \ldots, n$, by:

$$\begin{aligned} & H_{c_0c_1\ldots c_{n-1}c}(t_1, \ldots, t_n, s, x) \\ & \quad = g_{c_0c_1\ldots c_i}(2t_1 - 1, \ldots, 2t_i - 1, f_{g(c_i, \ldots, c_{n-j})g\text{"}(c_{n-j}c_{n-j+1}\ldots c)} \\ & \quad\ (2t_{i+1}, \ldots, s, 2t_{n-j+1}, \ldots, 2t_n, x)) \end{aligned}$$

for $(t_1, \ldots, t_n, s) \in K_i^j$. Next we show that $H_{c_0c_1\ldots c_{n-1}c}$ is well defined. In fact, we need to observe several cases, and for each we will show that $H_{c_0c_1\ldots c_{n-1}c}$ is well defined:

1. For $(t_1, \ldots, t_n, s, x) \in K_{i-1}^j \cap K_i^j, j = 1, \ldots, n-i$, we will show that the definition of $H_{c_0c_1\ldots c_{n-1}c}$ is unique.
2. For $(t_1, \ldots, t_n, s, x) \in K_i^{j-1} \cap K_i^j, j = 1, \ldots, n-i$, we will show that the definition of $H_{c_0c_1\ldots c_{n-1}c}$ is unique.
3. We will show that the definition of $H_{c_0c_1\ldots c_{n-1}c}$ on the edges of $\Delta^n \times I$ coincides with the corresponding homotopies on $\Delta^{n-1} \times I$, i.e., with $H_{c_0c_1\ldots \hat{c_i}\ldots c_{n-1}c}, i = 1, \ldots, n$.

1. Let $(t_1, \ldots, t_n, s, x) \in K_{i-1}^j \cap K_i^j$, i.e. $t_i = \frac{1}{2}$. Because of $(t_1, \ldots, t_n, s, x) \in K_{i-1}^j$ and $t_i = \frac{1}{2}$, we obtain the following:

$$\begin{aligned} & H_{c_0\ldots c}(t_1, \ldots, s, x) \\ & \quad = g_{c_0\ldots c_{i-1}}(2t_1 - 1, \ldots, 2t_{i-1} - 1, \\ & \quad\quad f_{b_{i-1}\ldots b_{n-j}\text{"}b_{n-j+1}\text{"}\ldots b\text{"}}(2t_i, \ldots, s, 2t_{n-j+1}, \ldots, 2t_n, x)) \\ & \quad = g_{c_0\ldots c_{i-1}}(2t_1 - 1, \ldots, 2t_{i-1} - 1, \\ & \quad\quad f_{b_{i-1}\ldots b_{n-j}\text{"}b_{n-j+1}\text{"}\ldots b\text{"}}(2 \cdot \frac{1}{2}, \ldots, s, 2t_{n-j+1}, \ldots, 2t_n, x)) \\ & \quad = g_{c_0\ldots c_{i-1}}(2t_1 - 1, \ldots, 2t_{i-1} - 1, \\ & \quad\quad f_{b_{i-1}\ldots b_{n-j}\text{"}b_{n-j+1}\text{"}\ldots b\text{"}}(1, \ldots, s, 2t_{n-j+1}, \ldots, 2t_n, x)) \\ & \quad = g_{c_0\ldots c_{i-1}}(2t_1 - 1, \ldots, 2t_{i-1} - 1, \\ & \quad\quad f_{b_{i-1}\ldots b_{n-j}\text{"}b_{n-j+1}\text{"}\ldots b\text{"}}(2t_{i+1}, \ldots, s, 2t_{n-j+1}, \ldots, 2t_n, x)). \end{aligned}$$

On the other hand, because of $(t_1,\dots,t_n,s,x)\in K_i^j$ and $t_i=\frac{1}{2}$, we obtain:

$$\begin{aligned}
&H_{c_0\dots c}(t_1,\dots,s,x)\\
&\quad= g_{c_0\dots c_i}(2t_1-1,\dots,2\cdot\frac{1}{2}-1,\\
&\qquad f_{b_i\dots b_{n-j}"b_{n-j+1}"\dots b"}(2t_{i+1},\dots,s,2t_{n-j+1},\dots,2t_n,x))\\
&\quad= g_{c_0\dots c_{i-1}}(2t_1-1,\dots,0,\\
&\qquad f_{b_i\dots b_{n-j}"b_{n-j+1}"\dots b"}(2t_{i+1},\dots,s,2t_{n-j+1}\dots,2t_n,x))\\
&\quad= g_{c_0\dots c_{i-1}}(2t_1-1,\dots,2t_{i-1}-1,\\
&\qquad f_{b_i\dots b_{n-j}"b_{n-j+1}"\dots b"}(2t_{i+1},\dots,s,2t_{n-j+1}\dots,2t_n,x))
\end{aligned}$$

by which we show case 1.

2. Let $(t_1,\dots,t_n,s,x)\in K_i^{j-1}\cup K_i^j$, i.e. $t_{n-j+1}=\frac{s}{2}$. Because of $(t_1,\dots,t_n,s,x)\in K_i^{j-1}$ and $t_{n-j+1}=\frac{s}{2}$, we obtain:

$$\begin{aligned}
&H_{c_0\dots c}(t_1,\dots t_n,s,x)\\
&\quad= g_{c_0\dots c_i}(2t_1-1,\dots,2t_i-1,\\
&\qquad f_{b_i\dots b_{n-j+1}b_{n-j+1}"\dots b"}(2t_{i+1},\dots,2\cdot\frac{s}{2},s,2t_{n-j+2},\dots,2t_n,x))\\
&\quad= c_0\dots c_i(2t_1-1,\dots,2t_i-1,\\
&\qquad f_{b_i\dots b_{n-j+1}b_{n-j+1}"\dots b"}(2t_{i+1},\dots,s,s,2t_{n-j+2},\dots,2t_n,x))\\
&\quad= g_{c_0\dots c_i}(2t_1-1,\dots,2t_i-1,\\
&\qquad f_{b_i\dots b_{n-j}b_{n-j+1}"\dots b"}(2t_{i+1},\dots,2t_{n-j},s,2t_{n-j+2}\dots,2t_n,x)).
\end{aligned}$$

On the other hand, because of $(t_1,\dots,t_n,s,x)\in K_i^j$ and $t_{n-j+1}=\frac{s}{2}$, we obtain:

$$\begin{aligned}
&H_{c_0\dots c}(t_1,\dots t_n,s,x)\\
&\quad= g_{c_0\dots c_i}(2t_1-1,\dots,2t_i-1,\\
&\qquad f_{b_i\dots b_{n-j+1}b_{n-j+1}"\dots b"}(2t_{i+1},\dots,\frac{2s}{2},s,2t_{n-j+2},\dots,2t_n,x))\\
&\quad= g_{c_0\dots c_i}(2t_1-1,\dots,2t_i-1,\\
&\qquad f_{b_i\dots b_{n-j}b_{n-j}",b_{n-j+1},\dots b"}(2t_{i+1},\dots,s,\frac{2s}{2},2t_{n-j+2},\dots,2t_n,x))\\
&\quad= g_{c_0\dots c_i}(2t_1-1,\dots,2t_i-1,\\
&\qquad f_{b_i\dots b_{n-j}b_{n-j+1}"\dots b"}(2t_{i+1},\dots,2t_{n-j},s,2t_{n-j+2}\dots,2t_n,x))
\end{aligned}$$

by which case 2 is completed.

3. The edges of $\Delta^n \times I$, because of the definition of Δ^n, are of a type:

$$\partial_0^n = \{(1, t_2, \dots, t_n, s) | (1, t_2, \dots, t_n) \in \Delta^n\},$$
$$\partial_l^n = \{(1, t_2, \dots, t_n, s) | (t_1, t_2, \dots, t_n) \in \Delta^n, t_l = t_{l+1}\}, l = 1 \dots, n-1,$$
$$\partial_n^n = \{(t_1, \dots, t_{n-1}, 0, t, s) | (t_1, t_2, \dots, t_{n-1}, 0) \in \Delta^n\}.$$

On ∂_0^n there are the sets $K_1, K_2, \dots, K_n$. On ∂_l^n there are the sets $K_0, \dots, K_{i-1}$, $K_{i+1}, \dots, K_n$. On ∂_n^n there are the sets $K_0, K_1, \dots, K_{n-1}$. For ∂_0^n, we obtain:

$$\begin{aligned}
&H_{c_0 \dots c}(t_1, \dots t_n, s, x) \\
&\quad = g_{c_0 \dots c_i}(2 \cdot 1 - 1, \dots, 2t_i - 1, \\
&\qquad f_{b_i \dots b_{n-j} b_{n-j}" b_{n-j+1}" \dots b"}(2t_{i+1}, \dots, s, 2t_{n-j+1}, \dots, 2t_n, x)) \\
&\quad = g_{c_0 \dots c_i}(1, \dots, 2t_i - 1, \\
&\qquad f_{b_i \dots b_{n-j} b_{n-j}", b_{n-j+1}", \dots b"}(2t_{i+1}, \dots, s, 2t_{n-j+1}, \dots, 2t_n, x)) \\
&\quad = g_{c_0 \dots c_i}(2t_2 - 1, \dots, 2t_i - 1, \\
&\qquad f_{b_i \dots b_{n-j} b_{n-j}" b_{n-j+1}" \dots b"}(2t_{i+1}, \dots, s, 2t_{n-j+1} \dots, 2t_n, x)) \\
&\quad = H_{c_1 \dots c}(t_2, \dots, t_n, s, x),
\end{aligned}$$

for $(t_2, \dots, t_n, s) \in K_i^j, \forall i = 1, \dots, n$.

For ∂_i^n, there are two cases, either $l < i$ or $l > i$ in the general formula of $H_{c_0 c_1 \dots c_{n-1} c}$. It is impossible that $l = i$, because on the edge ∂_i^n there is no set K_i. For $l < i$, we obtain:

$$\begin{aligned}
&H_{c_0 \dots c}(t_1, \dots t_n, s, x) \\
&\quad = g_{c_0 \dots c_i}\big(2t_1 - 1, \dots, 2t_l - 1, 2t_{l+1} - 1, \dots 2t_i - 1, \\
&\qquad f_{b_l \dots b_{n-j} b_{n-j}" \dots b"}(2t_{i+1}, \dots, s, 2t_{n-j+1}, \dots, 2t_n, x)\big) \\
&\quad = g_{c_0 \dots c_i}\big(2t_1 - 1, \dots, 2t_l - 1, 2t_i - 1, \\
&\qquad f_{b_i \dots b_{n-j} b_{n-j}", b_{n-j+1}", \dots b"}(2t_{i+1}, \dots, s, 2t_{n-j+1}, \dots, 2t_n, x)\big) \\
&\quad = g_{c_0 \dots \hat{c}_l \dots c_i}\big(2t_2 - 1, \dots, 2t_i - 1, \\
&\qquad f_{b_i \dots b_{n-j} b_{n-j}" b_{n-j+1}" \dots b"}(2t_{i+1}, \dots, s, 2t_{n-j+1}, \dots, 2t_n, x)\big) \\
&\quad = H_{c_0 c_1 \dots \hat{c}_l \dots c_{n-1} c}(t_1, \dots, \hat{t}_{l+1}, t_{l+2}, \dots, t_i, \dots, t_n, s, x),
\end{aligned}$$

for $(t_1, \dots, \hat{t}_{l+1}, t_{l+2}, \dots, t_i, \dots, t_n, s) \in K_i^j, i = 1, \dots, \hat{l}, \dots, n$. For $l > i$, there are also two cases possible: either both t_l and t_{l+1} are before s, or both t_l and t_{l+1} are after s. We will show the case when both t_l and t_{l+1} are before

s, and the other can be obtained similarly:

$$
\begin{aligned}
&H_{c_0c_1\ldots c_{n-1}c}(t_1,\ldots t_n,s,x)\\
&\quad= g_{c_0\ldots c_i}(2t_1-1,\ldots,2t_i-1,\\
&\qquad f_{b_i\ldots b_{n-j}b_{n-j}"\ldots b"}(2t_{i+1},\ldots,2t_l,2t_{l+1},\ldots,s,2t_{n-j+1},\ldots,2t_n,x))\\
&\quad= g_{c_0\ldots c_i}(2t_1-1,\ldots,2t_l-1,2t_i-1,\\
&\qquad f_{b_i\ldots b_{n-j}b_{n-j}",b_{n-j+1}",\ldots b"}\\
&\qquad\qquad (2t_{i+1},\ldots,2t_l,2t_l,\ldots,s,2t_{n-j+1},\ldots,2t_n,x))\\
&\quad= g_{c_0\ldots c_i}(2t_1-1,\ldots,2t_i-1,\\
&\qquad f_{b_i\ldots\hat{b}_l\ldots b_{n-j}b_{n-j}"b_{n-j+1}"\ldots b"}\\
&\qquad\qquad (2t_{i+1},\ldots,2t_l,2t_{l+2},\ldots,s,2t_{n-j+1},\ldots,2t_n,x))\\
&\quad= H_{c_0c_1\ldots\hat{c}_l\ldots c_{n-1}c}(t_1,\ldots,t_i,\ldots,\hat{t}_{l+1},\ldots,t_n,s,x),
\end{aligned}
$$

for $(t_1,\ldots,t_i,\ldots\hat{t}_{l+1},\ldots,t_n,s)\in K_i^j, i=1,\ldots,\hat{l},\ldots,n$. For ∂_n^n, we obtain:

$$
\begin{aligned}
&H_{c_0c_1\ldots c_{n-1}c}(t_1,\ldots t_n,s,x)\\
&\quad= g_{c_0\ldots c_i}(2t_1-1,\ldots,2t_i-1,\\
&\qquad f_{b_i\ldots b_{n-j}b_{n-j}"\ldots b"}(2t_{i+1},\ldots,s,2t_l,2t_{n-j+1},\ldots,0,x))\\
&\quad= g_{c_0\ldots c_i}(2t_1-1,\ldots,2t_i-1,\\
&\qquad f_{b_i\ldots b_{n-j}b_{n-j}",b_{n-j+1}",\ldots b"}(2t_{i+1},\ldots,s,2t_{n-j+1},\ldots,0,x))\\
&\quad= g_{c_0\ldots c_i}(2t_1-1,\ldots,2t_i-1,\\
&\qquad f_{b_i\ldots b_{n-j}b_{n-j}"b_{n-j+1}"\ldots b"}(2t_{i+1},\ldots,s,2t_{n-j+1},\ldots,0,x))\\
&\quad= H_{c_0c_1\ldots c_{n-1}}(t_1,\ldots,t_i,\ldots,t_{n-1},s,x),
\end{aligned}
$$

$(t_1,\ldots,t_i,\ldots,t_{n-1},s)\in K_i^j, i=0,\ldots,n-1$. Showing case 3 is in fact showing the coherence condition for $H_{c_0c_1\ldots c_{n-1}c}$. We have also shown that the homotopy $H_{c_0c_1\ldots c_{n-1}c}$ is well defined. Next we show that $H_{c_0c_1\ldots c_{n-1}c}$ connects $h_{c_0c_1\ldots c_{n-1}c}$ and $h"_{c_0c_1\ldots c_{n-1}c}$: For $s=0$, by the definition of the sets K_i^j, it follows that for each $n-j<l\le n, t_l=0$, and therefore we obtain:

$$
\begin{aligned}
&H_{c_0c_1\ldots c_{n-1}c}(t_1,\ldots t_n,0,x)\\
&\quad= g_{c_0\ldots c_i}(2t_1-1,\ldots,2t_i-1,\\
&\qquad f_{b_i\ldots b_{n-j}b_{n-j}"b_{n-j+1}\ldots b"}(2t_{i+1},\ldots,s,2t_l,2t_{n-j+1},0,\ldots,0,x))\\
&\quad= g_{c_0\ldots c_i}(2t_1-1,\ldots,2t_i-1,\\
&\qquad f_{b_i\ldots b_{n-j}}(2t_{i+1},\ldots,s,2t_{n-j+1},x)),
\end{aligned}
$$

for $(t_1,\ldots,t_n,0)\in K_i^j$, which corresponds to the definition of $h_{c_0c_1\ldots c_{n-1}c}$.

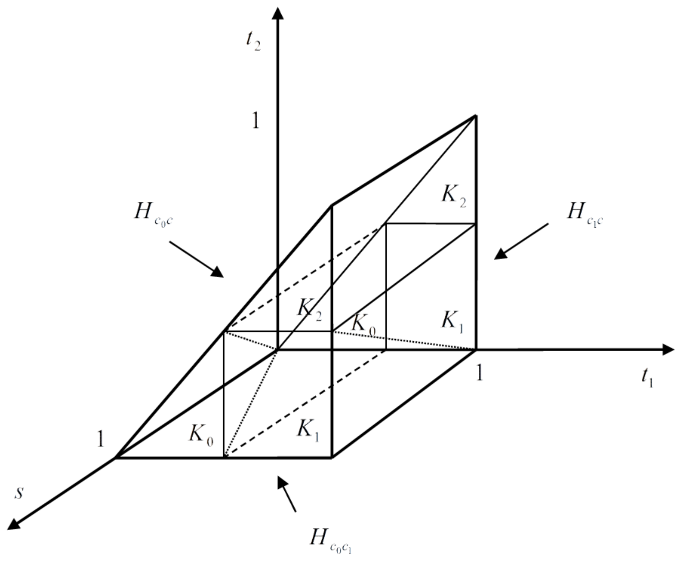

FIGURE 4

For $s = 1$, by the definition of the sets K_i^j, it follows that $\forall m, i < m \leq n - j, t_m = \frac{1}{2}$, and therefore we obtain:

$$\begin{aligned}
&H_{c_0c_1\ldots c_{n-1}c}(t_1, \ldots t_n, 1, x)\\
&\quad = g_{c_0\ldots c_i}(2t_1 - 1, \ldots, 2t_i - 1,\\
&\qquad f_{b_i\ldots b_{n-j}b_{n-j}"b_{n-j+1}\ldots b"}(2 \cdot \frac{1}{2}, \ldots, 2 \cdot \frac{1}{2}, 2t_{n-j+1}, \ldots, x))\\
&\quad = g_{c_0\ldots c_i}(2t_1 - 1, \ldots, 2t_i - 1,\\
&\qquad f_{b_i\ldots b_{n-j}b_{n-j}"b_{n-j+1}b"}(1, \ldots, 1, s, 2t_{n-j+1}, 2t_n, x)),\\
&\quad = g_{c_0\ldots c_i}(2t_1 - 1, \ldots, 2t_i - 1,\\
&\qquad f_{b_{n-i}"\ldots b_{n-j+1}"\ldots b"}(2t_{n-j+1}, \ldots, 2t_n, x)),
\end{aligned}$$

for $(t_1, \ldots, t_n, 1) \in K_i^j$, which corresponds to the definition of $h"_{c_0c_1\ldots c_{n-1}c}$. Next we observe the example when $n = 2$, in order to illustrate the above. $h_{c_0c_1c}$ and $h_{c_0c_1c}"$ were previously defined in the following way (Figure 2):

$$h_{c_0c_1c}(t_1, t_2, x) = \begin{cases} g_{c_0}f_{b_0b_1b}(2t_1, 2t_2, x), & t_1 \leq \frac{1}{2} \ \ ((t_1, t_2) \in K_0) \\ g_{c_0c_1}(2t_1 - 1, f_{b_1b}(2t_2, x)), & t_1 \geq \frac{1}{2} \geq t_2((t_1, t_2) \in K_1) \\ g_{c_0c_1c}(2t_1, 2t_2, f_b(x)), & t_2 \leq \frac{1}{2} \ \ ((t_1, t_2) \in K_2) \end{cases},$$

$$h"_{c_0c_1c}(t_1,t_2,x)=\begin{cases} g_{c_0}f_{b_0"b_1"b"}(2t_1,2t_2,x), & t_1\le\frac{1}{2}\ \ ((t_1,t_2)\in K_0)\\ g_{c_0c_1}(2t_1-1,f_{b_1"b"}(2t_2,x)), & t_1\ge\frac{1}{2}\ge t_2((t_1,t_2)\in K_1)\\ g_{c_0c_1c}(2t_1,2t_2,f_{b"}(x)), & t_2\le\frac{1}{2}\ \ ((t_1,t_2)\in K_2)\end{cases}$$

The homotopy $H_{c_0c_1c}:\Delta^2\times I\times X\to Z$, which connects $h_{c_0c_1c}$ and $h"_{c_0c_1c}$, is defined on $K_i\times I$, for each $i=0,1,2$, by:

$$H_{c_0c_1c}(t_1,t_2,s,x)$$
$$=g_{c_0\dots c_i}(2t_1-1,\dots,2t_i-1,f_{b_i\dots b_{2-j}b_{2-j}"\dots b"}(2t_{i+1},s,2t_{2-j+1},2t_2,x)),$$

for $(t_1,t_2,s)\in K_i^j, j=0,\dots,2-i$. More specifically, $H_{c_0c_1c}$ is defined as follows, for each $K_i\times I, i=0,1,2$: - For $K_0\times I$:

$$H_{c_0c_1c}(t_1,t_2,s,x)=\begin{cases} g_{c_0}f_{b_0b_1bb"}(2t_1,2t_2,s,x), & (t_1,t_2,s)\in K_0^0\\ g_{c_0}f_{b_0b_1b_1"b"}(2t_1,s,2t_2,x), & (t_1,t_2,s)\in K_0^1\\ g_{c_0}f_{b_0b_0"b_1"b"}(s,2t_1,2t_2,x), & (t_1,t_2,s)\in K_0^2\end{cases}.$$

- For $K_1\times I$:

$$H_{c_0c_1c}(t_1,t_2,s,x)=\begin{cases} g_{c_0c_1}(2t_1,f_{b_1bb"}(2t_2,s,x)), & (t_1,t_2,s)\in K_1^0\\ g_{c_0c_1}(2t_1,f_{b_1b_1"b"}(2t_2,s,x)), & (t_1,t_2,s)\in K_1^1\end{cases}.$$

- For $K_1\times I$:

$$H_{c_0c_1c}(t_1,t_2,s,x)=g_{c_0c_1c}(2t_1,2t_2-1,f_{bb"}(s,x)),(t_1,t_2,s)\in K_2^0.$$

On Figure 4 we see the way $\Delta^2\times I$ divides itself into three prisms $K_0\times I$, $K_1\times I$, and $K_2\times I$. On Figures 5, 6 and 7 below we can see the corresponding partitioning of $K_i\times I$ into $K_i^j, i=0,1,2$. -$K_0\times I$ divides into three parts, K_0^0, K_0^1 and K_0^2: - $K_1\times I$ divides into two parts, K_1^0 and K_1^1: - $K_2\times I$ does not divide and the one part is K_2^0:

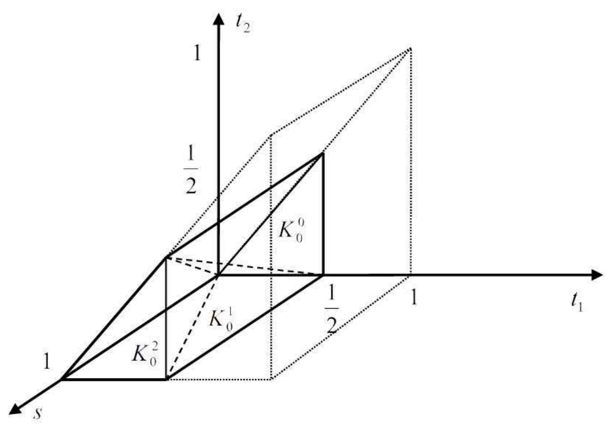

FIGURE 5

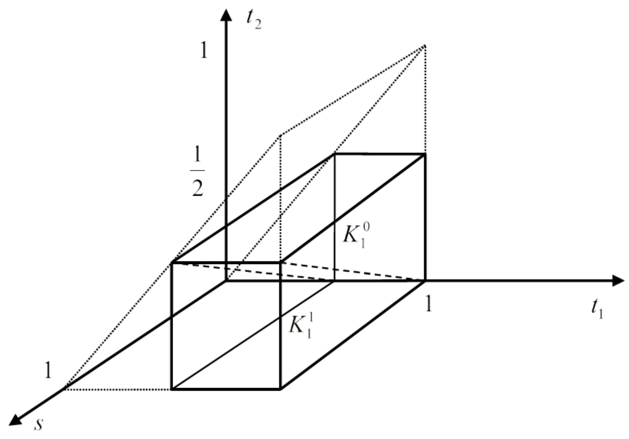

FIGURE 6

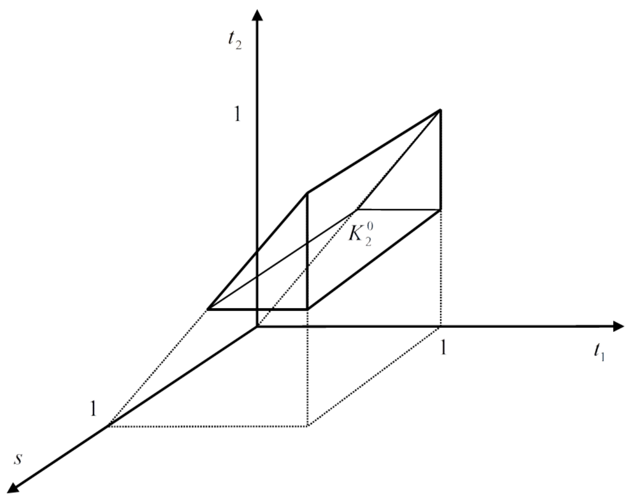

FIGURE 7

Hence it is shown that (h_c) and $(h"_c)$ are homotopic. In an analogue way we prove that (h_c) and $(h"_c)$ are homotopic. The relation of homotopy of coherent nets is an equivalence relation, therefore $(h_c) = (g_c f_{g(c)})$ and $(h"_c) = (g_c f_{g'(c)})$ and are in the same homotopy class. Now we may define a composition of homotopy classes of coherent proximate nets by

$$[(g_c)][(f_b)] = [(g_c f_{g(c)})],$$

and this definition does not depend on the choice of strictly increasing function $g : C \to B$.

5. The category of strong shape

Theorem 5.1. *If $f : X \to Y$, $g : Y \to Z$, $h : Z \to W = (W_d, w_{\underline{d}}, D)$ are proximate coherent nets, then the proximate coherent nets $h(gf)$ and $(hg)f$ are homotopic.*

Proof. Suppose $f = (f_{\underline{b}}), g = (g_{\underline{c}})$, and $h = (h_{\underline{d}})$. In order to obtain an explicit formula for the proximate coherent net $h(gf)$, we define a decomposition of Δ^n into subpolyhedra $K_{l,m}$ for any pair of integers l, m such that $0 \le l \le m \le n$, $K_{l,m} = \{(t_1, t_2, \dots, t_n)|t_l \ge \frac{1}{2} \ge t_{l+1}, t_m \ge \frac{1}{4} \ge t_{m+1}\}$. By applying the composition formula twice, for $(t_1, \dots, t_n) \in K_{l,m}$ we have

$$\begin{aligned}&(h(gf))_{\underline{d}}(t, x)\\&\quad = h_{d_0 \dots d_l}(2t_1 - 1, \dots, 2t_l - 1, g_{h(d_1 \dots d_m)}(4_{l+1} - 1, \dots, 4t_m - 1,\\&\qquad f_{gh(d_m \dots d_k)}(4t_{m+1}, \dots, 4t_k, x))).\end{aligned}$$

Similarly, to obtain an explicit formula for the proximate coherent net $(hg)f$, we define a decomposition of Δ^n into subpolyhedra $Q_{l,m}$ for any pair of integers l, m such that $0 \le l \le m \le n$, $Q_{l,m} = \{(t_1, \dots, t_n)|t_l \ge \frac{3}{4} \ge t_{l+1}, t_m \ge \frac{1}{2} \ge t_{m+1}\}$. Then, for $(t_1, \dots, t_k) \in Q_{l,m}$ we have

$$\begin{aligned}&(h(gf))_{\underline{d}}(t, x)\\&\quad = h_{d_0 \dots d_l}(4t_1 - 3, \dots, 4t_l - 3, g_{h(d_1 \dots d_m)}(4_{l+1} - m, \dots, 4t_m - 2,\\&\qquad f_{gh(d_m \dots d_k)}(2t_{m+1}, \dots, 2t_k, x))).\end{aligned}$$

We define a partition of $\Delta^n \times I$ into subpolyhedra $M_{l,m}$ for any pair of integers l, m such that $0 \le l \le m \le n$,

$$M_{l,m} = \{(t_1, \dots, t_n, s)|t_l \ge \frac{2+s}{4} \ge t_{l+1}, t_m \ge \frac{1+s}{2} \ge t_{m+1}\}.$$

We define a homotopy $H : I \times X \to W$ which connects $h(gf)$ and $(hg)f$. This map will be given by the function $fgh : D \to A$ and by the maps $H_{\underline{d}} : \Delta^n \times I \times X_{fgh(d_n)} \to W_{d_0}$ defined in the following way:

$$\begin{aligned}&H_{\underline{d}}(t, s, x)\\&\quad = h_{d_0 \dots d_l}\Big(\frac{4t_1 - 2 - s}{2 - s}, \dots, \frac{4t_l - 2 - s}{2 - s},\\&\qquad g_{h(d_1 \dots d_m)}(4t_{l+1} - 2, \dots, 4t_m - 2, f_{gh(d_m \dots d_n)(2t_{m+1}, \dots 2t_n, x)})\Big).\end{aligned}$$

We mention that $K_{l,m} \times 0 = \{(t_1, \dots, t_n, 0)|(t_1, \dots, t_n, 0) \in M_{l,m}\}$ and then, for $(t_1, \dots, t_n) \in K_{l,m}$, it is easily checked that $H_{\underline{d}}(t, 0, x) = (h(gf))^{j}_{\underline{d}}(t, x)$. Also, $Q_{l,m} \times 1 = \{(t_1, \dots, t_n, 1)|(t_1, \dots, t_n, 1) \in M_{l,m}\}$ and for $(t_1, \dots, t_n) \in Q_{l,m}$, $H_{\underline{d}}(t, 1, x) = (h(gf))_{\underline{d}}(t, x)$. To complete the proof we will check the well defining and the coherence conditions of the map $H^j_{\underline{d}}$. To check that the definition is well, suppose that $(t_1,\dots,t_n,s)\in M_{l,m}\cap M_{l-1,m}$, i.e., $t_i=\frac{2+s}{4}$. For these points $H^j_{\underline{d}}$ is defined in two ways. If we compute the formula for $t=(t_1,\dots,t_n,s)\in M_{l,m}$ and $t_l=\frac{2+s}{4}$, then we have:

$$\begin{aligned}
&H_{\underline{d}}(t,s,x)\\
&=h_{d_0\dots d_{l-1}}\Bigg(\frac{4t_1-2-s}{2-s},\dots,\frac{4t_{l-1}-2-s}{2-s},\\
&g_{h(d_l\dots d_m)}(4t_{l+1}-1-s,\dots,4t_m-1-s,f_{gh(d_m\dots d_n)}(\frac{4t_{m+1}}{1+s},\dots,\frac{4t_n}{1+s},x))\Bigg).
\end{aligned}$$

The same expression is obtained if we compute the formula for $(t_1,\dots,t_n,s)\in M_{l-1,m}$ and $t_l=\frac{2+s}{4}$. Similarly, we can check that the definition is well for $(t_1,\dots,t_n,s)\in M_{l,m}\cap M_{l,m-1}$, and the other cases can be deduced to one of these two cases. To check the coherence conditions of the homotopy $H_{\underline{d}}$, suppose that $(t_1,\dots,t_n,s)\in M_{l-1,m}$ and $t_n=0$. Then $f_{gh(d_m\dots d_n)}(\frac{4t_{m+1}}{1+s},\dots,\frac{4t_n}{1+s},x)$, and it follows that for $t_n=0, H_{\underline{d}}=H_{d_0\dots d_{n-1}}(t_1,\dots,t_{n-1},x)$. The case when $t_1=0$ is treated similarly. If $t_i=t_{i+1}$, and $i<l$, then

$$\begin{aligned}
&H_{\underline{d}}(t,s,x)\\
&=h_{d_0\dots d_i}\Bigg(\frac{4t_1-2-s}{2-s},\dots,\frac{4t_{i-1}-2-s}{2-s}\frac{4t_{i+1}-2-s}{2-s},\dots,\frac{4t_{l-1}-2-s}{2-s},\\
&g_{h(d_i\dots d_m)}(4t_{l+1}-1-s,\dots,4t_m-1-s,f_{gh(d_m\dots d_n)}(\frac{4t_{m+1}}{1+s},\dots,\frac{4t_n}{1+s},x))\Bigg)\\
&=H_{d_0\dots\hat{d}_i\dots d_n}(t_1,\dots,\hat{t}_i,\dots,t_n,s,x).
\end{aligned}$$

The cases $l<i<m$ and $m<i<n$ are treated similarly. ∎

THEOREM 5.2. *If proximate coherent nets $f,f':X\to Y$ are homotopic, and coherent maps $g,g':Y\to Z$ are level homotopic, then the coherent maps $gf,g'f':X\to Y$ are level homotopic.*

PROOF. Let $f,f':X\to Y$ be homotopic by a homotopy $F:I\times X\to Y$ given by a strictly increasing function $g:B\to A$. Then the proximate coherent nets $gf,gf':X\to Z$ are homotopic by the homotopy $gF:I\times X\to Z$ given by a strictly increasing function $fg:C\to A$. Let $g,g':Y\to Z$ be homotopic by a homotopy $G:I\times Y\to Z$ given by the strictly increasing function $g:C\to B$. Then the proximate coherent nets $gf',g'f':Z\to Z$ are homotopic by the homotopy $G(1\times f'):I\times X\to Z$ given by strictly increasing function $fg:C\to A$. It follows that the proximate coherent nets $gf,g'f':X\to Z$ are homotopic. ∎

THEOREM 5.3. *The proximate coherent nets f and $f1_X$ are homotopic; f and $1_Y f$ are homotopic.*

PROOF. We will prove that f and 1_Y are homotopic, and the other statement is treated in the similar way. First we define a partition of $\Delta^n \times I$ into subpolyhedra $L_l, l = 0, 1, \ldots, n$, by $L_l = \{(t_1, \ldots, t_n, s) | t_l \geq \frac{s}{2} + \frac{1}{2} \geq t_{l+1}\}$. We define a homotopy $F : I \times X \to Y$. This map will be given by the function $f : B \to A$ and by the maps $F_{\underline{b}} : \Delta^n \times I \times X_{f(b_n)} \to Y_{b_0}$ defined for $(t, s) \in L_l$ by $F_{\underline{b}}(t, s, x) = f_{b_1 \ldots b_n}(\frac{2t_l}{1+s}, \ldots, \frac{2t_n}{1+s}, x)$. We mention that $K_l \times 0 = \{(t_1 \ldots, t_n, 0) | (t_1 \ldots, t_n, 0) \in L_l\}$ and then, for $t = (t_1 \ldots, t_n) \in K_l$, we have $F_{\underline{b}}(t, 0, x) = (1_Y f)_{\underline{b}}(t, x)$. Also, $\{(t_1, \ldots, t_n, 1) | (t_1, \ldots, t_n) \in \Delta^n\} = L_0$ and $F_{\underline{b}}(t, 1, x) = f_{\underline{b}}(t, x)$. Category of strong shape is obtained. The objects are paracompact topological spaces, and the morphisms are the classes of the coherent proximate nets. For the isomorphic objects in this category we say they have the *same strong shape.* □

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B. Andonovic
Faculty of Technology and Metallurgy
Ss Cyril and Methodius University
1000 Skopje
Macedonia
E-mail: beti@tmf.ukim.edu.mk

N. Shekutkovski
Faculty of Mathematics and Natural Sciences
Ss Cyril and Methodius University
1000 Skopje
Macedonia
E-mail: nikita@pmf.ukim.mk

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