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THE DAVIS-GUT LAW FOR INDEPENDENT AND IDENTICALLY DISTRIBUTED BANACH SPACE VALUED RANDOM ELEMENTS

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Abstract. An analog of the Davis-Gut law for a sequence of independent and identically distributed Banach space valued random elements is obtained, which extends the result of Li and Rosalsky (A supplement to the Davis-Gut law. J. Math. Anal. Appl. 330 (2007), 1488–1493).

1. Introduction

Let $\{X, X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables. The following theorem, which is related to the classical Hartman-Wintner law of the iterated logarithm (see, Hartman and Wintner, [6]), is well known. As usual we let $\log t = \log_e \max\{e, t\}$ for $t \geq 0$.

Theorem 1.1. *The following three statements are equivalent:*

$$EX = 0 \text{ and } EX^2 = 1, \tag{1.1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left\{\left|\sum_{k=1}^{n} X_k\right| > (1+\varepsilon)\sqrt{2n\log\log n}\right\} \begin{cases} < \infty, & \text{if } \varepsilon > 0 \\ = \infty, & \text{if } \varepsilon < 0, \end{cases} \tag{1.2}$$

$$\sum_{n=1}^{\infty} \frac{\log\log n}{n} P\left\{\left|\sum_{k=1}^{n} X_k\right| > (1+\varepsilon)\sqrt{2n\log\log n}\right\} \begin{cases} < \infty, & \text{if } \varepsilon > 0 \\ = \infty, & \text{if } \varepsilon < 0. \end{cases} \tag{1.3}$$

This result is referred to as the Davis-Gut law. The implication "(1.1)⇒(1.2)" was formulated by Davis ([3]) with an invalid proof which was corrected by Li et al. ([11]). The implication "(1.2)⇒(1.1)" was obtained

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by Gut ([5]). The equivalence between (1.1) and (1.3) was established by Li ([9]). Necessary and sufficient conditions for (1.3) in a Banach space setting were obtained by Li ([9]). For moving average processes, the implications "(1.1)⇒(1.2)" and "(1.1)⇒(1.3)" were obtained by Chen and Wang ([1]).

Li and Rosalsky ([10]) provided the following supplement to the Davis-Gut law. When $h(t) \equiv 1$, it yields the equivalence between (1.1) and (1.2).

THEOREM 1.2. *Let $h(\cdot)$ be a positive nondecreasing function on $(0,\infty)$ such that $\int_1^\infty (th(t))^{-1}dt = \infty$. Write $\Psi(t) = \int_1^t (sh(s))^{-1}ds, t \geq 1$. Then (1.1) and*

$$\sum_{n=1}^{\infty} \frac{1}{nh(n)} P\left\{\left|\sum_{k=1}^{n} X_k\right| > (1+\varepsilon)\sqrt{2n\log\Psi(n)}\right\} \begin{cases} < \infty, & \text{if } \varepsilon > 0 \\ = \infty, & \text{if } \varepsilon < 0 \end{cases} \tag{1.4}$$

are equivalent.

Recently, Liu et al. ([12]) extended Theorem 1.2 to moving average processes which then extends the work of Chen and Wang ([1]) by establishing the implication "(1.2) ⇒ (1.1)" for moving average processes.

In this paper, we will extend Theorem 1.2 for a sequence of independent and identically distributed Banach space valued random elements.

2. Preliminaries and Lemmas

Let B be a real separable Banach space with norm $\|\cdot\|$ and let B^* denote the topological dual space of B. We let B_1^* denote the unit ball of B^*. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A *random element* X taking values in B is defined as an $\mathcal{F}$-measurable function from $(\Omega, \mathcal{F})$ into B equipped with the Borel sigma-algebra; we call it a B-valued random element for short. The *expected value* or *mean* of a B-valued random element X is defined to be the Bochner integral and is denoted by EX.

LEMMA 2.1. *Let $\{k_n, n \geq 1\}$ be a sequence of positive integers and $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ an array of rowwise independent B-valued random elements. Suppose that there exists $\delta > 0$ such that $\|X_{nk}\| \leq \delta$ a.s. for all $1 \leq k \leq k_n, n \geq 1$. If $\sum_{k=1}^{k_n} X_{nk} \to 0$ in probability, then $E\|\sum_{k=1}^{k_n} X_{nk}\| \to 0$ as $n \to \infty$.*

PROOF. Let $\{X'_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be an independent copy of $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$. Then by Lemma 2.2 in Chen and Wang ([2]), it suffices to show that

$$E\left\|\sum_{k=1}^{k_n}(X_{nk} - X'_{nk})\right\| \to 0 \text{ as } n \to \infty. \tag{2.1}$$

It is easy to show that

$$\sum_{k=1}^{k_n}(X_{nk} - X'_{nk}) \to 0 \text{ in probability}$$

and $\|X_{nk} - X'_{nk}\| \leq 2\delta$. Therefore by Lemma 2.1 in Hu et al. ([7]), (2.1) holds and the proof is completed. ∎

LEMMA 2.2. *Let $0 < b_n \uparrow \infty$, and $\{X, X_n, n \geq 1\}$ a sequence of independent and identically distributed B-valued random elements. If $b_n^{-1} \sum_{k=1}^n X_k \to 0$ in probability, then $E\|b_n^{-1} \sum_{k=1}^n X_k I(\|X_k\| \leq b_n)\| \to 0$ as $n \to \infty$.*

PROOF. Let $\{X, X'_n, n \geq 1\}$ be an independent copy of $\{X, X_n, n \geq 1\}$. Then

$$b_n^{-1} \sum_{k=1}^n (X_k - X'_k) \to 0 \text{ in probability.} \tag{2.2}$$

By Lévy's inequality (see display (2.7) in Ledoux and Talagrand [8, p. 47]), for every $t > 0$,

$$P\{\max_{1\leq k\leq n} \|X_k - X'_k\| > t\} \leq 2P\{\|\sum_{k=1}^n (X_k - X'_k)\| > t\},$$

which by (2.2) ensures that

$$P\{\max_{1\leq k\leq n} \|X_k - X'_k\| > b_n/2\} \to 0 \text{ as } n \to \infty. \tag{2.3}$$

By Lemma 2.6 of Ledoux and Talagrand [8, p. 51],

$$\begin{aligned} nP\{\|X - X'\| > b_n/2\} &= \sum_{k=1}^n P\{\|X_k - X'_k\| > b_n/2\} \\ &\leq 2P\{\max_{1\leq k\leq n} \|X_k - X'_k\| > b_n/2\} \end{aligned} \tag{2.4}$$

when n is sufficiently large. By display (6.1) in Ledoux and Talagrand [8, p. 150],

$$P\{\|X\| > b_n\} \leq 2P\{\|X - X'\| > b_n/2\} \tag{2.5}$$

when n is sufficiently large. Therefore by (2.3), (2.4), and (2.5),

$$nP\{\|X\| > b_n\} \to 0 \text{ as } n \to \infty. \tag{2.6}$$

Note that for any $\varepsilon > 0$

$$P\left\{\left\|\sum_{k=1}^n X_k I(\|X_k\| \leq b_n)\right\| > \varepsilon b_n\right\} \leq nP\{\|X\| > b_n\} + P\left\{\left\|\sum_{k=1}^n X_k\right\| > \varepsilon b_n\right\}.$$

Then by (2.6) and $b_n^{-1}\sum_{k=1}^n X_k \to 0$ in probability, it follows that

$$b_n^{-1}\sum_{k=1}^n X_k I(\|X_k\| \le b_n) \to 0$$

in probability. The conclusion then follows from Lemma 2.1. □

The following lemma is due to Einmahl and Li ([4]).

LEMMA 2.3. *Let $Z_1, \ldots, Z_n$ be independent B-valued random elements with mean zero such that for some $s > 2$, $E\|Z_k\|^s < \infty$, $1 \le k \le n$. Then we have for $0 < \eta \le 1$, $\delta > 0$, and $t > 0$,*

$$P\left\{\max_{1\le m\le n}\left\|\sum_{k=1}^m Z_k\right\| \ge (1+\eta)E\left\|\sum_{k=1}^n Z_k\right\| + t\right\}$$
$$\le \exp\left\{-\frac{t^2}{(2+\delta)\Lambda_n^2}\right\} + C\sum_{k=1}^n E\|Z_k\|^s/t^s,$$

where $\Lambda_n^2 = \sup\{\sum_{k=1}^n Ef^2(Z_k) : f \in B_1^*\}$ and C is a positive constant depending on η, δ and s.

LEMMA 2.4. *Let $h(t)$ and $\Psi(t)$ be as in Theorem 1.2. Suppose that X is a B-valued random element with*

$$\sum_{n=1}^\infty \frac{1}{h(n)} P\{\|X\| > \sqrt{n\log\Psi(n)}\} < \infty. \tag{2.7}$$

Then for any $s > 2$,

$$\sum_{n=1}^\infty \frac{1}{h(n)} \cdot \frac{1}{(n\log\Psi(n))^{s/2}} \cdot E\|X\|^s I(\|X\| \le \sqrt{n\log\Psi(n)}) < \infty.$$

PROOF. Set $b_0 = 0$ and $b_n = \sqrt{n\log\Psi(n)}, n \ge 1$. Note that $\Psi(n)\uparrow$ and therefore $b_n/\sqrt{n}\uparrow$. Then $b_k/b_n \le \sqrt{k/n}$ whenever $1 \le k \le n$. Hence,

$$\sum_{n=1}^\infty \frac{1}{h(n)} \cdot \frac{1}{(n\log\Psi(n))^{s/2}} \cdot E\|X\|^s I(\|X\| \le \sqrt{n\log\Psi(n)})$$
$$= \sum_{n=1}^\infty \frac{1}{h(n)b_n^s}\sum_{k=1}^n E\|X\|^s I(b_{k-1} < \|X\| \le b_k)$$
$$\le \sum_{n=1}^\infty \frac{1}{h(n)b_n^s}\sum_{k=1}^n b_k^s P\{b_{k-1} < \|X\| \le b_k\}$$
$$= \sum_{k=1}^\infty b_k^s P\{b_{k-1} < \|X\| \le b_k\}\sum_{n=k}^\infty \frac{1}{h(n)b_n^s}$$

$$\leq \sum_{k=1}^{\infty} k^{s/2} P\{b_{k-1} < \|X\| \leq b_k\} \sum_{n=k}^{\infty} \frac{1}{n^{s/2} h(n)}$$
$$\leq C \sum_{k=1}^{\infty} \frac{k}{h(k)} P\{b_{k-1} < \|X\| \leq b_k\}$$
$$\leq \frac{C}{h(1)} + C \sum_{k=1}^{\infty} \left[\frac{k+1}{h(k+1)} - \frac{k}{h(k)} \right] P\{\|X\| > b_k\}$$
$$\leq \frac{C}{h(1)} + C \sum_{k=1}^{\infty} \frac{1}{h(k)} P\{\|X\| > b_k\} < \infty,$$

where $C = (s/2-1)^{-1}$. The proof is completed. ∎

Lemma 2.5. *Let $h(n), \Psi(n)$ be as in Theorem 1.2. Then for any B-valued random element X, (2.7) is equivalent to*

$$\sum_{n=1}^{\infty} \frac{1}{h(n)} P\{\|X\| > M\sqrt{n \log \Psi(n)}\} < \infty \tag{2.8}$$

for some $M > 0$.

Proof. It suffices to prove that (2.7) implies (2.8) for all $0 < M < 1$. Set $b_n = \sqrt{n \log \Psi(n)}, n \geq 1$. Note that $\Psi(n) \uparrow$ and therefore $b_n/\sqrt{n} \uparrow$. Then $b_n \leq 2^{-1/2} b_{2n}$ for $n \geq 1$. Hence,

$$\frac{1}{h(2n)} P\{\|X\| > 2^{-1/2} b_{2n}\} \leq \frac{1}{h(n)} P\{\|X\| > b_n\}$$

and

$$\frac{1}{h(2n+1)} P\{\|X\| > 2^{-1/2} b_{2n+1}\} \leq \frac{1}{h(2n)} P\{\|X\| > 2^{-1/2} b_{2n}\}$$
$$\leq \frac{1}{h(n)} P\{\|X\| > b_n\},$$

which ensures that

$$\sum_{n=1}^{\infty} \frac{1}{h(n)} P\{\|X\| > 2^{-1/2} b_n\} = \frac{1}{h(1)} P\{\|X\| > 2^{-1/2} b_1\}$$
$$+ \sum_{n=1}^{\infty} \frac{1}{h(2n)} P\{\|X\| > 2^{-1/2} b_{2n}\} + \sum_{n=1}^{\infty} \frac{1}{h(2n+1)} P\{\|X\| > 2^{-1/2} b_{2n+1}\}$$
$$\leq \frac{1}{h(1)} P\{\|X\| > 2^{-1/2} b_1\} + 2 \sum_{n=1}^{\infty} \frac{1}{h(n)} P\{\|X\| > \sqrt{n \log \Psi(n)}\} < \infty.$$

Then by mathematical induction, for any integer $k \geq 1$,

$$\sum_{n=1}^{\infty} \frac{1}{h(n)} P\{\|X\| > 2^{-k/2} b_n\} < \infty.$$

The proof is completed. □

3. The Main Result and its Proof

We now state and prove the main result.

Theorem 3.1. *Let $h(t)$ and $\Psi(t)$ be as in Theorem 1.2. Let $\{X, X_n, n \geq 1\}$ be a sequence of independent and identically distributed B-valued random elements. Suppose that*

$$(\sqrt{n \log \Psi(n)})^{-1} \sum_{k=1}^{n} X_k \to 0 \text{ in probability.}$$

(i) Suppose that (2.7) holds and

$$EX = 0,\ Ef^2(X) < \infty \ \ \forall\, f \in B^*. \tag{3.1}$$

Then

$$\sum_{n=1}^{\infty} \frac{1}{nh(n)} P\left\{\left\|\sum_{k=1}^{n} X_k\right\| > (1+\varepsilon)\sqrt{2\sigma^2 n \log \Psi(n)}\right\} \begin{cases} < \infty, & \text{if } \varepsilon > 0 \\ = \infty, & \text{if } \varepsilon < 0, \end{cases} \tag{3.2}$$

where $\sigma^2 = \sup\{Ef^2(X) : f \in B_1^*\}$.

(ii) Conversely, suppose that

$$\sum_{n=1}^{\infty} \frac{1}{nh(n)} P\left\{\left\|\sum_{k=1}^{n} X_k\right\| > M\sqrt{n \log \Psi(n)}\right\} < \infty \tag{3.3}$$

holds for some $M > 0$. Then (2.7) and (3.1) hold.

Proof. Set $a_n = \sqrt{2\sigma^2 n \log \Psi(n)}, b_n = \sqrt{n \log \Psi(n)}, n \geq 1$ and

$$X_{nk} = X_k I(\|X_k\| \leq b_n),\ Z_{nk} = X_{nk} - EX_{nk},\ 1 \leq k \leq n,\ n \geq 1.$$

(i) Suppose that (2.7) and (3.1) hold. We first prove that

$$\sum_{n=1}^{\infty} \frac{1}{nh(n)} P\left\{\left\|\sum_{k=1}^{n} X_k\right\| > (1+\varepsilon)a_n\right\} < \infty \ \ \forall\, \varepsilon > 0. \tag{3.4}$$

Note that for any $\varepsilon > 0$,

$$P\left\{\left\|\sum_{k=1}^{n} X_k\right\| > (1+\varepsilon)a_n\right\} \leq nP\{\|X\| > b_n\} + P\left\{\left\|\sum_{k=1}^{n} X_{nk}\right\| > (1+\varepsilon)a_n\right\}.$$

Hence, by (2.7), to prove (3.4), it suffices to prove that

$$\sum_{n=1}^{\infty} \frac{1}{nh(n)} P\left\{\left\|\sum_{k=1}^{n} X_{nk}\right\| > (1+\varepsilon)a_n\right\} < \infty \quad \forall\, \varepsilon > 0. \tag{3.5}$$

By Lemma 2.2,

$$\frac{1}{b_n}\left\|\sum_{k=1}^{n} EX_{nk}\right\| \le \frac{1}{b_n} E\left\|\sum_{k=1}^{n} X_{nk}\right\| \to 0 \ \text{ as } \ n \to \infty$$

and

$$\frac{1}{b_n} E\left\|\sum_{k=1}^{n} Z_{nk}\right\| \le \frac{2}{b_n} E\left\|\sum_{k=1}^{n} X_{nk}\right\| \to 0 \ \text{ as } \ n \to \infty.$$

Then to prove (3.5), it suffices to prove that

$$\sum_{n=1}^{\infty} \frac{1}{nh(n)} P\left\{\left\|\sum_{k=1}^{n} Z_{nk}\right\| > 2E\left\|\sum_{k=1}^{n} Z_{nk}\right\| + (1+\varepsilon)a_n\right\} < \infty \quad \forall\, \varepsilon > 0. \tag{3.6}$$

By Lemma 2.3, for some $s > 2$ and any $\delta > 0$

$$\begin{aligned} &P\left\{\left\|\sum_{k=1}^{n} Z_{nk}\right\| > 2E\left\|\sum_{k=1}^{n} Z_{nk}\right\| + (1+\varepsilon)a_n\right\} \\ &\quad \le \exp\left\{-\frac{(1+\varepsilon)^2 a_n^2}{(2+\delta)\Lambda_n^2}\right\} + \frac{C}{b_n^s}\sum_{k=1}^{n} E\|Z_{nk}\|^s, \end{aligned} \tag{3.7}$$

where $\Lambda_n^2 = \sup\{\sum_{k=1}^n Ef^2(Z_{nk}) : f \in B_1^*\}$. Note that for all $f \in B_1^*$,

$$\begin{aligned} Ef^2(Z_{nk}) &= Ef^2(X_{nk}) - (Ef(X_{nk}))^2 \le Ef^2(X_{nk}) \\ &\le Ef^2(X),\ 1 \le k \le n,\ n \ge 1. \end{aligned}$$

Therefore $\Lambda_n^2 \le n\sigma^2, n \ge 1$. Choose $\delta > 0$ small enough so that $t = 2(1+\varepsilon)^2/(2+\delta) > 1$. Then

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{nh(n)} \exp\left\{-\frac{(1+\varepsilon)^2 a_n^2}{(2+\delta)\Lambda_n}\right\} &\le \sum_{n=1}^{\infty} \frac{1}{nh(n)} \exp\left\{-\frac{(1+\varepsilon)^2 a_n^2}{(2+\delta)\Lambda_n}\right\} \\ &\le \sum_{n=1}^{\infty} \frac{1}{nh(n)} \exp\{-t\log\Psi(n)\} \\ &\le \sum_{n=1}^{\infty} \frac{1}{nh(n)} \cdot \frac{1}{(\Psi(n))^t} < \infty, \end{aligned} \tag{3.8}$$

since $\int_1^\infty dx/[xh(x)\Psi^t(x)] < \infty$. By the C_r-inequality, Hölder's inequality, and Lemma 2.4,

$$\begin{aligned}(3.9)\quad &\sum_{n=1}^{\infty} \frac{1}{nh(n)} \cdot \frac{1}{b_n^s} \sum_{k=1}^{n} E\|Z_{nk}\|^s \\ &\leq \sum_{n=1}^{\infty} \frac{1}{h(n)} \cdot \frac{1}{(n\log\Psi(n))^{s/2}} \cdot E\|X\|^s I(\|X\| \leq \sqrt{n\log\Psi(n)}) < \infty.\end{aligned}$$

By (3.7), (3.8), and (3.9), (3.6) holds and hence (3.4) holds as was argued above.

Now we prove that

$$(3.10)\qquad \sum_{n=1}^{\infty} \frac{1}{nh(n)} P\left\{\left\|\sum_{k=1}^{n} X_k\right\| > (1+\varepsilon)a_n\right\} = \infty \ \ \forall\, \varepsilon < 0.$$

For any $f \in B^*$, by (3.1), $Ef(X) = 0$ and $Ef^2(X) < \infty$. Then by the implication "(1.1) ⇒ (1.4)" in Theorem 1.2, for all $\varepsilon < 0$

$$(3.11)\qquad \sum_{n=1}^{\infty} \frac{1}{nh(n)} P\left\{\left|\sum_{k=1}^{n} f(X_k)\right| > (1+\varepsilon)\sqrt{2Ef^2(X)n\log\Psi(n)}\right\} = \infty.$$

Note that for any $f \in B_1^*$, $|\sum_{k=1}^n f(X_k)| \leq \|\sum_{k=1}^n X_k\|$ and so it follows from (3.11) that for all $f \in B_1^*$, for all $\varepsilon < 0$

$$(3.12)\qquad \sum_{n=1}^{\infty} \frac{1}{nh(n)} P\left\{\left\|\sum_{k=1}^{n} X_k\right\| > (1+\varepsilon)\sqrt{2Ef^2(X)n\log\Psi(n)}\right\} = \infty.$$

Hence (3.10) holds by (3.12) and $\sigma^2 = \sup\{Ef^2(X) : f \in B_1^*\}$. Combining (3.4) and (3.10) yields (3.2).

(ii) Assume that (3.3) holds for some $M > 0$. Then for any $f \in B_1^*$,

$$\sum_{n=1}^{\infty} \frac{1}{nh(n)} P\left\{\left|\sum_{k=1}^{n} f(X_k)\right| > Mb_n\right\} < \infty.$$

Then by the implication "(2.3) ⇒ (2.4)" of Li and Rosalsky ([10]), it follows that $Ef(X) = 0$ and $Ef^2(X) < \infty$. Hence (3.1) holds.

Let $\{X', X_n', n \geq 1\}$ be an independent copy of $\{X, X_n, n \geq 1\}$. Then by the same argument as in the proof of Lemma 2.2,

$$\begin{aligned} nP\{\|X\| > 4Mb_n\} &\leq 8P\left\{\left\|\sum_{k=1}^{n} (X_k - X_k')\right\| > 2Mb_n\right\} \\ &\leq 16P\left\{\left\|\sum_{k=1}^{n} X_k\right\| > Mb_n\right\},\end{aligned}$$

which by (3.3) ensures that

$$\sum_{n=1}^{\infty} \frac{1}{h(n)} P\{\|X\| > 4Mb_n\} < \infty$$

and so (2.7) holds by Lemma 2.5. The proof is completed. ∎

Remark 3.2. A sufficient condition for (2.7) is $E\|X\|^2 < \infty$. Indeed,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{h(n)} P\{\|X\| > \sqrt{n \log \Psi(n)}\} &\leq \frac{1}{h(1)} \sum_{n=1}^{\infty} P\{\|X\| > \sqrt{n}\} \\ &\leq \frac{1}{h(1)} E\|X\|^2 < \infty.\end{aligned}$$

Remark 3.3. Some examples of moment conditions which are equivalent to (2.7) for various choices of $h(\cdot)$ will now be given.

Case (i). Set $h(t) = (\log\log t)^b$ where $b \geq 0$. Then $\log \Psi(t) \sim \log\log t$ as $t \to \infty$ and (2.7) is equivalent to $E\|X\|^2/(\log\log\|X\|)^{b+1} < \infty$.

Case (ii). Set $h(t) = (\log t)^r$ where $0 \leq r < 1$. Then $\log \Psi(t) \sim (r - 1)\log\log t$ as $t \to \infty$ and (2.7) is equivalent to $E\|X\|^2/[(\log\|X\|)^r \log\log\|X\|] < \infty$.

Case (iii). Set $h(t) = \log t$. Then $\log \Psi(t) \sim \log\log\log t$ as $t \to \infty$ and (2.7) is equivalent to $E\|X\|^2/[(\log\|X\|)\log\log\log\|X\|] < \infty$.

Case (iv). In Case (i), take $b = 0$, or in Case (ii), take $r = 0$. Then (2.7) is equivalent to $E\|X\|^2/\log\log\|X\| < \infty$.

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