GLASNIK MATEMATIČKI
Vol. 42(62)(2007), 291 – 299

ON LINEAR SUBSPACES OF $\mathcal{M}_n$ AND THEIR SINGULAR SETS RELATED TO THE CHARACTERISTIC MAP

Marcin Skrzyński
Cracow University of Technology, Poland

Abstract. We study linear subspaces $\mathcal{L} \subseteq \mathcal{M}_n$ (over an algebraically closed field $\mathbb{F}$ of characteristic zero) and their singular sets $\mathcal{S}(\mathcal{L})$ defined by $\mathcal{S}(\mathcal{L}) = \{A \in \mathcal{M}_n : \chi(A+\mathcal{L}) \text{ is not dense in } \mathbb{F}^n\}$, where $\chi : \mathcal{M}_n \longrightarrow \mathbb{F}^n$ is the characteristic map. We give a complete characterization of the subspaces $\mathcal{L} \subset \mathcal{M}_2$ such that $\emptyset \neq \mathcal{S}(\mathcal{L}) \neq \mathcal{M}_2$. We also provide a complete characterization of the singular sets $\mathcal{S}(\mathcal{L})$ in the case of $n = 2$. Finally, we give a characterization of the n-dimensional subspaces $\mathcal{L} \subset \mathcal{M}_n$ such that $\mathcal{S}(\mathcal{L}) = \emptyset$ by means of their intersections with conjugacy classes.

1. Preliminaries and introduction

We work throughout over an algebraically closed field $\mathbb{F}$ of characteristic zero. We define $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$. We denote by $\#E$ the cardinality of a finite set E. The set of all $(n \times n)$-matrices whose entries are elements of $\mathbb{F}$ is denoted by $\mathcal{M}_n$. (We assume throughout that $n \geq 2$.) The zero matrix and the unit matrix belonging to $\mathcal{M}_n$ are denoted by O and I, respectively. We define $\mathcal{GL}_n$ to be the full linear group of size n over the field $\mathbb{F}$, i. e. $\mathcal{GL}_n = \{U \in \mathcal{M}_n : \det(U) \neq 0\}$. The conjugacy class of a matrix $A \in \mathcal{M}_n$ is denoted by $\mathcal{O}(A)$. (In other words, $\mathcal{O}(A) = \{U^{-1}AU : U \in \mathcal{GL}_n\}$.) A subset $\mathcal{E} \subseteq \mathcal{M}_n$ is said to be triangularizable if there is a $U \in \mathcal{GL}_n$ such that $U^{-1}\mathcal{E}U := \{U^{-1}AU : A \in \mathcal{E}\}$ consists of upper triangular matrices. The subset $\mathcal{E}$ is said to be $\mathcal{GL}_n$-invariant if $U^{-1}\mathcal{E}U \subseteq \mathcal{E}$ for all $U \in \mathcal{GL}_n$.

We consider $\mathbb{F}^n$, $\mathcal{M}_n \cong \mathbb{F}^{n^2}$, and their subsets as topological spaces endowed with the Zariski topology. We say that a property holds for a generic

2000 *Mathematics Subject Classification.* 15A18, 14A10, 14L35.
Key words and phrases. Characteristic polynomial of a matrix, characteristic map, dominant map, linear space of matrices, triangularizable set of matrices, conjugacy class of a matrix.

matrix $A \in \mathcal{M}_n$ if there exists a nonempty Zariski open subset $\mathcal{W} \subseteq \mathcal{M}_n$ such that the property holds for all $A \in \mathcal{W}$. The Zariski closure of a set E contained either in $\mathbb{F}^n$ or in $\mathcal{M}_n$ is denoted by $\overline{E}$.

For an $A \in \mathcal{M}_n$ and a positive integer $j \leq n$ we define $\mathrm{s}_j(A)$ to be the sum of all principal minors of size j of the matrix A. (Therefore,

$$T^n + \sum_{j=1}^{n} (-1)^j \mathrm{s}_j(A) T^{n-j} \in \mathbb{F}[T]$$

is the characteristic polynomial of A.) The regular map $\chi : \mathcal{M}_n \longrightarrow \mathbb{F}^n$ defined by $\chi(A) = (\mathrm{s}_1(A), \ldots, \mathrm{s}_n(A))$ is referred to as the characteristic map. Notice that Helton, Rosenthal and Wang [3] define the characteristic map by $A \mapsto ((-1)^j \mathrm{s}_j(A))_{j=1}^n$.

For a linear subspace $\mathcal{L} \subseteq \mathcal{M}_n$ we define a singular set $\mathcal{S}(\mathcal{L})$ related to the characteristic map. Namely,

$$\mathcal{S}(\mathcal{L}) = \{A \in \mathcal{M}_n : \chi(A + \mathcal{L}) \text{ is not dense in } \mathbb{F}^n\}.$$

Observe that the condition which defines the set $\mathcal{S}(\mathcal{L})$ may be reformulated in the following way: the regular map $\mathcal{L} \ni B \mapsto \chi(A + B) \in \mathbb{F}^n$ is not dominant. We refer to [2] for all needed information about matrix theory, to [5] for algebra, and to [6, 4] for algebraic geometry and invariant theory.

In [3] Helton, Rosenthal and Wang proved that the image $\chi(A + \mathcal{L})$ is dense in $\mathbb{F}^n$ for a generic matrix $A \in \mathcal{M}_n$ if and only if the dimension of a linear subspace $\mathcal{L} \subseteq \mathcal{M}_n$ is not smaller than n and there is a $B \in \mathcal{L}$ such that $\mathrm{tr}(B) \neq 0$. (Notice that $\chi(A + \mathcal{L})$ is a constructible subset of $\mathbb{F}^n$.) Applying the above introduced language we can rephrase the Helton – Rosenthal – Wang result as follows: $\mathcal{S}(\mathcal{L}) \neq \mathcal{M}_n$ if and only if $\dim \mathcal{L} \geq n$ and tr does not identically vanish on $\mathcal{L}$; moreover, $\mathcal{S}(\mathcal{L})$ is a (Zariski) closed subset of $\mathcal{M}_n$. In [7] we studied basic set-theorical, geometrical and topological properties of the singular sets $\mathcal{S}(\mathcal{L})$. In particular, we derived a counterpart of the Helton – Rosenthal – Wang theorem in the case of $n = 2$ and obtained a characterization of the linear subspaces $\mathcal{L} \subseteq \mathcal{M}_n$ such that $\mathcal{S}(\mathcal{L}) = \emptyset$. The present note is a continuation of [7]. Our first goal is to complete the study of the linear subspaces of $\mathcal{M}_2$ and their singular sets. The second goal is to give a characterization of the n-dimensional linear subspaces of $\mathcal{M}_n$ whose singular set is empty by means of their intersections with conjugacy classes (the case of $n = 2$ being considered in a detailed way).

2. The case of $n = 2$

We start with a continuation of the study of linear subspaces of $\mathcal{M}_2$ originated in [7, Section 2]. Our purpose is to characterize the subspaces $\mathcal{L} \subset \mathcal{M}_2$ such that $\emptyset \neq \mathcal{S}(\mathcal{L}) \neq \mathcal{M}_2$.

For $\lambda \in \mathbb{F}$ we define

$$\mathcal{T}_\lambda = \left\{ \begin{bmatrix} t & s \\ 0 & \lambda t \end{bmatrix} : \ t,\, s \in \mathbb{F} \right\}.$$

Furthermore, we define

$$\mathcal{K} = \left\{ \begin{bmatrix} 0 & s \\ 0 & t \end{bmatrix} : \ t,\, s \in \mathbb{F} \right\}.$$

THEOREM 2.1. *Let $\mathcal{L}$ be a linear subspace of $\mathcal{M}_2$. Then the following conditions are equivalent:*

(1) $\emptyset \neq \mathcal{S}(\mathcal{L}) \neq \mathcal{M}_2$,
(2) *either $\mathcal{L} = U^{-1}\mathcal{T}_\lambda U$ for a $U \in \mathcal{GL}_2$ and a $\lambda \in \mathbb{F}\backslash\{-1\}$, or $\mathcal{L} = U^{-1}\mathcal{K}U$ for a $U \in \mathcal{GL}_2$.*

PROOF. If condition (2) is satisfied, then $\dim \overline{\chi(\mathcal{L})} = 1$ and

$$U^{-1} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} U \notin \mathcal{S}(\mathcal{L}).$$

Condition (1) follows.

Assume that (1) is satisfied. Since $\mathcal{S}(\mathcal{L}) \neq \mathcal{M}_2$, we have that $\dim \mathcal{L} \geq 2$ and that tr does not identically vanish on $\mathcal{L}$. The nonemptiness of $\mathcal{S}(\mathcal{L})$ yields $\dim \mathcal{L} < 3$ [7, Corollary 1.7]. Pick two matrices $A,\, B \in \mathcal{L}$ such that $\mathrm{tr}(A) \neq 0 \neq \mathrm{tr}(B)$ and $(A,\, B)$ is a basis for $\mathcal{L}$.

Consider first the case where A is not diagonalizable. Then there are a $V \in \mathcal{GL}_2$ and a $\mu \in \mathbb{F}^*$ such that

$$\widetilde{A} := V^{-1}(\mu A)V = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Define $\widetilde{\mathcal{L}} = V^{-1}\mathcal{L}V$ and $\widetilde{B} = V^{-1}BV$. It is easy to see that $\mathcal{S}(\widetilde{\mathcal{L}}) = V^{-1}\mathcal{S}(\mathcal{L})V$. Furthermore, $(\widetilde{A},\, \widetilde{B})$ is a basis for $\widetilde{\mathcal{L}}$. Put $\widetilde{B} = [\beta_{jk}]$. By implication (1) $\Rightarrow$ (2) in [7, Proposition 2.3], we obtain $\beta_{11} + \beta_{22} - \beta_{21} = \beta_{11} + \beta_{22}$ and $4(\beta_{11}\beta_{22} - \beta_{12}\beta_{21}) = (\beta_{11} + \beta_{22})(\beta_{11} + \beta_{22} - \beta_{21})$. These equalities yield $\beta_{21} = 0$ and $\beta_{11} = \beta_{22}$. Notice that $\beta_{12} \neq \beta_{11} \neq 0$ (because $\widetilde{A}$ and $\widetilde{B}$ are linearly independent and $\mathrm{tr}(\widetilde{B}) \neq 0$). Consequently,

$$\widetilde{\mathcal{L}} = \left\{ \begin{bmatrix} t + \beta_{11}s & t + \beta_{12}s \\ 0 & t + \beta_{11}s \end{bmatrix} : \ t,\, s \in \mathbb{F} \right\}.$$

Condition (2) follows (with $\lambda = 1$).

Now, consider the case where $A \notin \mathbb{F}I$ is a diagonalizable matrix. Then there are a $V \in \mathcal{GL}_2$ and a $\mu \in \mathbb{F}^*$ such that

$$\widetilde{A} = V^{-1}(\mu A)V = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}$$

with an $\alpha \in \mathbb{F} \setminus \{\pm 1\}$. Define $\widetilde{\mathcal{L}}$ and $\widetilde{B} = [\beta_{jk}] \in \widetilde{\mathcal{L}}$ as in the previous part of the proof. By implication (1) $\Rightarrow$ (2) in [7, Proposition 2.3], we get

$$(\bullet) \qquad (1+\alpha)(\beta_{22}+\alpha\beta_{11}) = 2\alpha(\beta_{11}+\beta_{22})$$

and

$$(\bullet\bullet) \qquad 2(1+\alpha)(\beta_{11}\beta_{22}-\beta_{12}\beta_{21}) = (\beta_{11}+\beta_{22})(\beta_{22}+\alpha\beta_{11}).$$

Observe that equality $(\bullet)$ yields $\beta_{11} \neq 0$. (If $\beta_{11} = 0$, then $\beta_{22} = 0$, because $\alpha \neq 1$. This contradicts the fact that $\operatorname{tr}(\widetilde{B}) \neq 0$.) Reformulating $(\bullet)$ we obtain $(\alpha-1)(\alpha\beta_{11}-\beta_{22}) = 0$. Therefore, $\alpha = \dfrac{\beta_{22}}{\beta_{11}}$. This means that the diagonal entries of the matrix $\beta_{11}\widetilde{A}$ coincide with the diagonal entries of $\widetilde{B}$. The linear independence of $\widetilde{A}$ and $\widetilde{B}$ implies now that at least one of the elements β_{12}, β_{21} is different from 0. On the other hand, substituting $\alpha = \dfrac{\beta_{22}}{\beta_{11}}$ into equality $(\bullet\bullet)$ we get $\beta_{12}\beta_{21} = 0$. Consequently, either

$$\widetilde{\mathcal{L}} = \left\{ \begin{bmatrix} t+s & \frac{\beta_{12}}{\beta_{11}}s \\ 0 & \alpha(t+s) \end{bmatrix} : \ t,\, s \in \mathbb{F} \right\}$$

with $\beta_{12} \neq 0$, or

$$\widetilde{\mathcal{L}} = \left\{ \begin{bmatrix} t+s & 0 \\ \frac{\beta_{21}}{\beta_{11}}s & \alpha(t+s) \end{bmatrix} : \ t,\, s \in \mathbb{F} \right\}$$

with $\beta_{21} \neq 0$. In the first case condition (2) follows in an obvious way. Define

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In the second case, $P^{-1}\widetilde{\mathcal{L}}P = \mathcal{T}_{\alpha^{-1}}$ whenever $\alpha \neq 0$, and $P^{-1}\widetilde{\mathcal{L}}P = \mathcal{K}$ whenever $\alpha = 0$.

Finally, consider the case of $A = I$. Let $V \in \mathcal{GL}_2$ be such that $\widetilde{B} = V^{-1}BV$ is an upper triangular matrix. Implication (1) $\Rightarrow$ (2) in [7, Proposition 2. 3] applied to the matrices I and $\widetilde{B}$ yields $(\operatorname{tr}(\widetilde{B}))^2 = 4\det(\widetilde{B})$, which means that $\widetilde{B}$ has the double eigenvalue. In virtue of the linear independence of I and $\widetilde{B}$ we have

$$\widetilde{B} = \begin{bmatrix} \xi & \nu \\ 0 & \xi \end{bmatrix}$$

for some ξ, $\nu \in \mathbb{F}^*$. Thus,

$$V^{-1}\mathcal{L}V = \left\{ \begin{bmatrix} t+\xi s & \nu s \\ 0 & t+\xi s \end{bmatrix} : \ t,\, s \in \mathbb{F} \right\}.$$

Condition (2) follows. ∎

We have proven that each linear subspace of $\mathcal{M}_2$ with "nontrivial" singular set is triangularizable. As a simple consequence we obtain a complete characterization of the singular sets $\mathcal{S}(\mathcal{L})$ in the case of $n = 2$.

COROLLARY 2.2. *Let $\mathcal{E}$ be a nonempty proper subset of $\mathcal{M}_2$ and let $\mathcal{T} \subset \mathcal{M}_2$ be the set of all upper triangular matrices. Then the following are equivalent:*

(1) *$\mathcal{E} = \mathcal{S}(\mathcal{L})$ for a linear subspace $\mathcal{L} \subset \mathcal{M}_2$,*
(2) *there is a $U \in \mathcal{GL}_2$ such that $\mathcal{E} = U^{-1}\mathcal{T}U$.*

PROOF. A direct calculation shows that $\mathcal{S}(U^{-1}\mathcal{T}_\lambda U) = U^{-1}\mathcal{S}(\mathcal{T}_\lambda)U = U^{-1}\mathcal{T}U = \mathcal{S}(U^{-1}\mathcal{K}U)$ for an arbitrary $U \in \mathcal{GL}_2$, an arbitrary $\lambda \in \mathbb{F} \setminus \{-1\}$, and the subspaces $\mathcal{T}_\lambda$ and $\mathcal{K}$ defined as at the beginning of the section. Now implication (2) $\Rightarrow$ (1) is obvious. Furthermore, if condition (1) is satisfied, then, by Theorem 2.1, either $\mathcal{L} = U^{-1}\mathcal{T}_\lambda U$ for a $U \in \mathcal{GL}_2$ and a $\lambda \in \mathbb{F} \setminus \{-1\}$ or $\mathcal{L} = U^{-1}\mathcal{K}U$ for a $U \in \mathcal{GL}_2$. Condition (2) follows. □

Notice that for each subspace $\mathcal{L} \subset \mathcal{M}_2$ such that $\emptyset \neq \mathcal{S}(\mathcal{L}) \neq \mathcal{M}_2$ there is a matrix $A \in \mathcal{M}_2$ such that $\chi(A+\mathcal{L}) = \mathbb{F}^2$. (To see this take into consideration

$$A = U^{-1} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} U$$

with a suitable $U \in \mathcal{GL}_2$, as at the beginning of the proof of Theorem 2.1.) In [7, Example 1.9] we considered a two-dimensional linear subspace $\mathcal{L}_0 \subset \mathcal{M}_2$ with $\mathcal{S}(\mathcal{L}_0) = \emptyset$ such that $\chi(A + \mathcal{L}_0) \neq \mathbb{F}^2$ for all $A \in \mathcal{M}_2$.

Let X be a finite-dimensional vector space over $\mathbb{F}$. Denote by $\mathbb{G}_k(X)$ the Grassmann variety of all k-dimensional linear subspaces of X. The full linear group $\mathcal{GL}_n$ acts on $\mathbb{G}_k(\mathcal{M}_n)$ by $\mathbb{G}_k(\mathcal{M}_n) \times \mathcal{GL}_n \ni (\mathcal{L}, U) \mapsto U^{-1}\mathcal{L}U \in \mathbb{G}_k(\mathcal{M}_n)$. It is obvious that the family of all linear subspaces $\mathcal{L} \subset \mathcal{M}_n$ such that $\dim \mathcal{L} = k$ and $\emptyset \neq \mathcal{S}(\mathcal{L}) \neq \mathcal{M}_n$ is invariant under that action. Let $\mathfrak{F}$ be the family of all linear subspaces $\mathcal{L} \subset \mathcal{M}_2$ whose singular sets $\mathcal{S}(\mathcal{L})$ are nontrivial. Theorem 2.1 implies that $\mathfrak{F} \subset \mathbb{G}_2(\mathcal{M}_2)$ contains infinitely many orbits of the above defined action of $\mathcal{GL}_2$ on $\mathbb{G}_2(\mathcal{M}_2)$. Furthermore, observe that the orbit of the subspace $\mathcal{K}$ is disjoint with the orbit of any subspace of the form $\mathcal{T}_\lambda$.

We conclude the section with an example of a linear subspace of $\mathcal{M}_3$ that is not triangularizable and whose singular set is nontrivial.

EXAMPLE 2.3. Define

$$\mathcal{L} = \left\{ \begin{bmatrix} s & 0 & t \\ u & s & 0 \\ 0 & t & s \end{bmatrix} : \ s, t, u \in \mathbb{F} \right\}.$$

It is easy to verify that $\mathcal{L}$ is not a triangularizable subspace of $\mathcal{M}_3$. Making use of the Jacobian determinant of the map

$$\mathbb{F}^3 \ni (s,\, t,\, u) \mapsto \chi\left(\begin{bmatrix} \alpha_{11}+s & \alpha_{12} & \alpha_{13}+t \\ \alpha_{21}+u & \alpha_{22}+s & \alpha_{23} \\ \alpha_{31} & \alpha_{32}+t & \alpha_{33}+s \end{bmatrix}\right) \in \mathbb{F}^3$$

(cf. the proof of [7, Theorem 2.1]) one can prove that a matrix $A = [\alpha_{jk}] \in \mathcal{M}_3$ is an element of the singular set $\mathcal{S}(\mathcal{L})$ if and only if $\alpha_{23}+\alpha_{31} = 0$ and $\alpha_{12} = 0$. Therefore, $\mathcal{S}(\mathcal{L})$ is a linear subspace of codimension 2 in $\mathcal{M}_3$.

3. Subspaces of dimension n whose singular set is empty

We begin with certain remarks on the set of all diagonal matrices.

Example 3.1. Let $\mathcal{D} \subset \mathcal{M}_n$ be the set of all diagonal matrices. Obviously, $\chi(\mathcal{D}) = \mathbb{F}^n$. Define

$$\mathcal{Z} = \{A \in \mathcal{M}_n \mid \text{the eigenvalues of } A \text{ are pairwise distinct}\}.$$

It is easy to see that $\#(\mathcal{D} \cap \mathcal{O}(A)) = n!$ for all $A \in \mathcal{Z}$. In [1] Friedland proved that the map $\mathcal{D} \ni A \mapsto \chi(B + A) \in \mathbb{F}^n$, where $B \in \mathcal{M}_n$ is a fixed matrix, is onto and that each fibre of this map has $n!$ elements (when counted with multiplicities). Friedland's result implies that $0 < \#((B+\mathcal{D}) \cap \mathcal{O}(A)) \le n!$ for an arbitrary $B \in \mathcal{M}_n$ and an arbitrary $A \in \mathcal{Z}$. (Notice that $(B+\mathcal{D}) \cap \mathcal{O}(A) = (B + \mathcal{D}) \cap \chi^{-1}(\chi(A))$.)

The above observations lead to a characterization of the n-dimensional subspaces $\mathcal{L} \subset \mathcal{M}_n$ with $\mathcal{S}(\mathcal{L}) = \emptyset$.

Theorem 3.2. *Let $\mathcal{L}$ be an n-dimensional linear subspace of $\mathcal{M}_n$. Then the following conditions are equivalent:*

(1) $\mathcal{S}(\mathcal{L}) = \emptyset$,
(2) *the image $\chi(\mathcal{L})$ is dense in $\mathbb{F}^n$,*
(3) *for each $B \in \mathcal{M}_n$ there is a nonempty open subset $\mathcal{W}_B \subseteq \mathcal{M}_n$ and an integer $q_B > 0$ such that $\#((B + \mathcal{L}) \cap \mathcal{O}(A)) = q_B$ for all $A \in \mathcal{W}_B$,*
(4) $0 < \#(\mathcal{L} \cap \mathcal{O}(A)) < \infty$ *for a generic $A \in \mathcal{M}_n$.*

Proof. Equivalence (1) $\Leftrightarrow$ (2) follows from [7, Theorem 1.5]. Implication (3) $\Rightarrow$ (4) is obvious.

Consider the set $\mathcal{Z}$ defined in Example 3.1. It is open in $\mathcal{M}_n$ and $\mathcal{GL}_n$-invariant. Furthermore, $\chi^{-1}(\chi(A)) = \mathcal{O}(A)$ for all $A \in \mathcal{Z}$.

Assume that condition (1) is satisfied, pick a $B \in \mathcal{M}_n$, and denote $\mathcal{L}_B = B+\mathcal{L}$. It follows from (1) that $\chi(\mathcal{L}_B)$ is dense in $\mathbb{F}^n$. Thus, $\mathcal{L}_B \cap \mathcal{Z} \neq \emptyset$. (If $\mathcal{L}_B \cap \mathcal{Z} = \emptyset$, then the discriminant of the characteristic polynomial of the matrix A vanishes for all $A \in \mathcal{L}_B$, which implies that $\chi(\mathcal{L}_B)$ is contained in a hypersurface in $\mathbb{F}^n$, a contradiction.) Since the restriction $\chi|_{\mathcal{L}_B} : \mathcal{L}_B \longrightarrow \mathbb{F}^n$ is a dominant map and $\dim \mathcal{L}_B = \dim \mathcal{L} = n$, we get that there is a nonempty open subset $Y \subseteq \mathbb{F}^n$ and an integer $q_B > 0$ such that $\#(\mathcal{L}_B \cap \chi^{-1}(y)) = q_B$ for all $y \in Y$. Define $\mathcal{W}_B = \chi^{-1}(Y) \cap \mathcal{Z}$. Then $\mathcal{W}_B$ is a nonempty open subset of $\mathcal{M}_n$. For an arbitrary $A \in \mathcal{W}_B$ we have

$$\#(\mathcal{L}_B \cap \mathcal{O}(A)) = \#(\mathcal{L}_B \cap \chi^{-1}(\chi(A))) = q_B,$$

because $A \in \mathcal{Z}$ and $\chi(A) \in Y$. Condition (3) follows.

Assume that (4) is satisfied. Denote by $\mathcal{W}$ a nonempty open subset of $\mathcal{M}_n$ such that $0 < \#(\mathcal{L} \cap \mathcal{O}(A)) < \infty$ for all $A \in \mathcal{W}$. Observe that $\mathcal{Z} \cap \mathcal{W} \neq \emptyset$ and that $\cup_{A\in\mathcal{W}}\mathcal{O}(A)$ is an open subset of $\mathcal{M}_n$. Thus,

$$\widetilde{\mathcal{W}} := \mathcal{L} \cap \mathcal{Z} \cap \bigcup_{A\in\mathcal{W}} \mathcal{O}(A)$$

is a nonempty open subset of $\mathcal{L}$. Pick an arbitrary $C \in \widetilde{\mathcal{W}}$. There is an $A \in \mathcal{W}$ such that $C \in \mathcal{O}(A)$. Since $C \in \mathcal{Z}$, we have $\mathcal{L} \cap \chi^{-1}(\chi(C)) = \mathcal{L} \cap \mathcal{O}(C) = \mathcal{L} \cap \mathcal{O}(A)$. Consequently, $0 < \#(\mathcal{L} \cap \chi^{-1}(\chi(C))) < \infty$. By the theorem on the dimension of fibres of a dominant map and by the openess of $\widetilde{\mathcal{W}}$, we obtain $\dim \overline{\chi(\mathcal{L})} = \dim \mathcal{L} - \dim(\mathcal{L} \cap \chi^{-1}(\chi(C_0))) = n - 0 = n$, where C_0 is a suitable element of $\widetilde{\mathcal{W}}$. Condition (2) follows. □

We conclude the note with a two-dimensional counterpart of Friedland's result.

THEOREM 3.3. *Let $\mathcal{L} \subset \mathcal{M}_2$ be a two-dimensional linear subspace with $\mathcal{S}(\mathcal{L}) = \emptyset$ and let $B \in \mathcal{M}_2$ be an arbitrary matrix. Then*

(i) $\#((B + \mathcal{L}) \cap \mathcal{O}(A)) = 2$ *for a generic $A \in \mathcal{M}_n$ provided there is no nilpotent matrix in $\mathcal{L} \setminus \{O\}$,*
(ii) $\#((B + \mathcal{L}) \cap \mathcal{O}(A)) = 1$ *for a generic $A \in \mathcal{M}_n$ provided there is a nilpotent matrix $N \in \mathcal{L} \setminus \{O\}$.*

PROOF. Let f, $g : \mathcal{M}_2 \longrightarrow \mathbb{F}$ be linearly independent linear forms such that $\mathcal{L} = f^{-1}(0) \cap g^{-1}(0)$. For $\lambda \in \mathbb{F}$ define

$$\mathcal{X}_\lambda = \{C \in \mathcal{M}_2 : \ f(C - B) = 0 = g(C - B),\ \mathrm{tr}(C) = \lambda\}.$$

Making use of the fact that tr does not identically vanish on $\mathcal{L}$ (because $\mathcal{S}(\mathcal{L}) = \emptyset$) and of elementary properties of systems of linear equations, we get that there is a matrix $C_0 \in \mathcal{L} \setminus \{O\}$ and a nonconstant affine map $\Phi : \mathbb{F} \longrightarrow \mathcal{M}_2$ such that $\mathrm{tr}(C_0) = 0$ and $\mathcal{X}_\lambda = \Phi(\lambda) + \mathbb{F}C_0$. Now, for an arbitrary $(\lambda, \mu) \in \mathbb{F}^2$ define $\mathcal{Y}_{(\lambda,\,\mu)} = \{C \in \mathcal{X}_\lambda : \ \det(C) = \mu\}$. Observe that

$$\det(\Phi(\lambda) + tC_0) = \det(C_0)t^2 + h(\lambda)t + \det(\Phi(\lambda)),$$

where $t \in \mathbb{F}$ and $h : \mathbb{F} \longrightarrow \mathbb{F}$ is an affine function. Consequently, $\#\mathcal{Y}_{(\lambda,\,\mu)} \leq 2$. Furthermore, if $A \in \mathcal{M}_2$ is a matrix with two different eigenvalues, then

$$\begin{aligned}&(B+\mathcal{L})\cap\mathcal{O}(A)\\ &\quad=\{C\in\mathcal{M}_2:\ f(C-B)=0=g(C-B),\ \mathrm{tr}(C)=\mathrm{tr}(A),\ \det(C)=\det(A)\}\\ &\quad=\mathcal{Y}_{\chi(A)}.\end{aligned}$$

Assume that there is no nilpotent matrix in $\mathcal{L} \setminus \{O\}$. Then $\det(C_0) \neq 0$. Therefore, the set $\mathcal{Y}_{(\lambda,\,\mu)}$ (with an arbitrary $(\lambda,\,\mu) \in \mathbb{F}^2$) has exactly two elements if and only if $\Delta(\lambda,\,\mu) := (h(\lambda))^2 - 4\det(C_0)(\det(\Phi(\lambda)) - \mu) \neq 0$. Consequently, $Y := \{(\lambda,\,\mu) \in \mathbb{F}^2 :\ \#\mathcal{Y}_{(\lambda,\,\mu)} = 2\}$ is a nonempty open subset of $\mathbb{F}^2$. Define

$$\mathcal{W} = \{A \in \chi^{-1}(Y) :\ A \text{ has two different eigenvalues}\}.$$

The set $\mathcal{W}$ is nonempty and open in $\mathcal{M}_2$. Moreover, for an arbitrary $A \in \mathcal{W}$ we have $\#((B+\mathcal{L}) \cap \mathcal{O}(A)) = \#\mathcal{Y}_{\chi(A)} = 2$. This completes the proof for case (i).

If there is a nilpotent matrix $N \in \mathcal{L} \setminus \{O\}$, then $C_0 = \alpha N$ for an $\alpha \in \mathbb{F}^*$. Consequently, $\det(\Phi(\lambda) + tC_0) = h(\lambda)t + \det(\Phi(\lambda))$. Thus, $\mathcal{Y}_{(\lambda,\,\mu)}$ has at most one element (for an arbitrary $(\lambda,\,\mu) \in \mathbb{F}^2$). Let $\mathcal{W}_B \subseteq \mathcal{M}_2$ be a nonempty open subset from condition (3) of Theorem 3.2. Recall that $\mathcal{W}_B$ consists of matrices with two different eigenvalues. Therefore,

$$1 \leq \#((B+\mathcal{L}) \cap \mathcal{O}(A)) = \#\mathcal{Y}_{\chi(A)} \leq 1$$

for all $A \in \mathcal{W}_B$. The proof is complete. □

Notice that the subspace

$$\mathcal{L}_0 := \left\{ \begin{bmatrix} t & t \\ s & t \end{bmatrix} :\ s,\, t \in \mathbb{F} \right\} \subset \mathcal{M}_2$$

considered in [7, Example 1.9] satisfies the assumptions of case (ii) in the above theorem.

References

[1] S. Friedland, *Inverse eigenvalue problems*, Linear Algebra Appl. **17** (1977), 15-51.
[2] F. R. Gantmacher, Théorie des matrices, Dunod, Paris, 1966.
[3] W. Helton, J. Rosenthal and X. Wang, *Matrix extensions and eigenvalue completions, the generic case*, Trans. Amer. Math. Soc. **349** (1997), 3401-3408.
[4] H.–P. Kraft, Geometrische Methoden in der Invariantentheorie, Friedr. Vieweg & Sohn, Braunschweig – Wiesbaden, 1984.
[5] S. Lang, Algebra, Addison Wesley, Reading, MA, 1993.
[6] I. R. Shafarevich, Basic Algebraic Geometry, Springer–Verlag, Berlin – Heidelberg – New York, 1977.
[7] M. Skrzyński, *On a singular set for the restriction of the characteristic map to a linear subspace of $\mathcal{M}_n$*, Math. Slovaca **54** (2004), 229-236.

M. Skrzyński
Institute of Mathematics
Cracow University of Technology
ul. Warszawska 24
31–155 Kraków
Poland
E-mail: pfskrzyn@cyf-kr.edu.pl

Received: 13.7.2005.