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# ORTHOGONALITY, SATURATION AND SHAPE

Luciano Stramaccia
Università di Perugia, Italy

*To Sibe Mardešić on his 80th birthday:*
AMICUS ET VITAE MAGISTER

Abstract. The class of shape equivalences for a pair $(\mathcal{C}, \mathcal{K})$ of categories is the orthogonal of $\mathcal{K}$, that is $\Sigma = \mathcal{K}^{\perp}$. Then $\Sigma$ is internally saturated ($\Sigma = \Sigma^{\top\perp}$). On the other hand, every internally saturated class of morphisms $\Sigma \subset \mathrm{Mor}(\mathcal{C})$, is the class of shape equivalences for some pair $(\mathcal{C}, \mathcal{K})$. Moreover, every class of shape equivalences $\Sigma$ enjoys a calculus of left fractions and such a fact allows one to use techniques from categories of fractions to obtain conditions for $\Sigma^{\top}$ to be reflective or proreflective in $\mathcal{C}$.

## 1. Introduction

Shape equivalences can be defined for any pair $(\mathcal{C}, \mathcal{K})$ of categories. They form the class $\Sigma$ of morphisms of $\mathcal{C}$ that are orthogonal to the class of objects of $\mathcal{K}$, in symbols $\Sigma = \mathcal{K}^{\perp}$. $\Sigma$ is always internally saturated, that is it coincides with its double orthogonal, moreover it is true that every internally saturated class $\Sigma \subset \mathrm{Mor}(\mathcal{C})$ is the class of shape equivalences for some pair $(\mathcal{C}, \mathcal{K})$. It is worth noting that every internally saturated class of morphisms, hence every class of shape equivalences enjoys a calculus of left fractions. This suggests connections between the shape category of $(\mathcal{C}, \mathcal{K})$ and the category of left fraction $\mathcal{C}[\Sigma^{-1}]$. On the other hand one is faced with the problem of the possible existence of other objects in $\mathcal{C}$, out of $\mathcal{K}$, which every shape equivalence for $(\mathcal{C}, \mathcal{K})$ is orthogonal to. This amounts to determine the internal saturation of the class of objects of $\mathcal{K}$. We do this when $\mathcal{K}$ is reflective or proreflective in $\mathcal{C}$, then we consider the general case. We conclude giving

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conditions for an internally saturated class $\mathcal{K} \subset \mathrm{Ob}(\mathcal{C})$ to be reflective or proreflective in $\mathcal{C}$.

## 2. Orthogonality and Saturation

All categories considered are finitely complete and cocomplete. $\mathtt{T}$ will denote the terminal object.

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. There are a category $\mathcal{C}_F$ and functors $F_0 : \mathcal{C} \to \mathcal{C}_F$, $F_1 : \mathcal{C}_F \to \mathcal{D}$ with the following properties:

(i) $F_0$ is the identity on objects and $F_1$ is fully faithful,

(ii) $F = F_1 \circ F_0$ and such a factorization is uniquely determined, up to isomorphisms, among all factorizations of $F$ by a functor bijective on objects followed by a fully faithful functor.

$\mathcal{C}_F$ is the *full image* of $F$. It has the same objects as $\mathcal{C}$ while $\mathcal{C}_F(X, Y)$ is identified with $\mathcal{D}(F(X), F(Y))$, for all $X, Y \in \mathrm{Ob}(\mathcal{C})$ ([10], 21.2).

The following easy lemma gives some important and useful properties of the full image.

Lemma 2.1. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors.*

(i) *There is a unique functor $T : \mathcal{C}_F \to \mathcal{C}_{G\circ F}$ such that $T \circ F_0 = (G \circ F)_0$ and $(G \circ F)_1 \circ T = G \circ F_1$. If $G$ is fully faithful, then $T$ is an isomorphism.*

(ii) *There is a unique functor $V : \mathcal{C}_{G\circ F} \to \mathcal{D}_G$ such that $V \circ (G \circ F)_0 = G_0 \circ F$ and $G_1 \circ V = (G \circ F)_1$. If $F$ is bijective on objects, then $V$ is an isomorphism.*

If $\Sigma \subset \mathrm{Mor}(\mathcal{C})$, then $\mathcal{C}[\Sigma^{-1}]$ denotes the *category of left fractions* of $\mathcal{C}$ and $P_\Sigma : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ the canonical functor. $\Sigma$ is *externally saturated* when $\Sigma = \mathcal{S}(P_\Sigma)$, that is, $\Sigma$ is the class of all morphisms that are turned into isomorphisms by $P_\Sigma$. Equivalently, $\Sigma$ is externally saturated iff $\Sigma = \mathcal{S}(F)$, for some functor $F : \mathcal{C} \to \mathcal{D}$. We refer to [6] as a classical source on such matters.

Proposition 2.2 ([10, page 267]). *A class $\Sigma \subset \mathrm{Mor}(\mathcal{C})$ has a calculus of left fractions* (CLF) *whenever*

(i) *$\Sigma$ contains all isomorphisms,*

(ii) *if two of the morphisms $s$, $t$, $t \circ s$ are in $\Sigma$, the third is also in $\Sigma$. If $v \circ s$ and $t \circ v$ are in $\Sigma$, then $v$ also belongs to $\Sigma$,*

(iii) *$\Sigma$ is pushout closed,*

(iv) *if $f \circ s = g \circ s$, $s \in \Sigma$ then* $\mathrm{coeq}(f, g) \in \Sigma$.

*$\Sigma$ has a terminal calculus of left fractions* (TCLF) *if, besides (i)-(iv), the comma category $X \downarrow \Sigma$ has a terminal object, for every $X \in \mathrm{Ob}(\mathcal{C})$.*

When $\Sigma$ has a CLF each morphism $\varphi \in \mathcal{C}[\Sigma^{-1}](X,Y)$ can be represented as $\varphi = P_\Sigma(s)^{-1} \circ P_\Sigma(f)$, where $s \in \Sigma$. In presence of a TCLF, $\mathcal{C}[\Sigma^{-1}]$ is also a legitimate category.

From now on $(\mathcal{C}, \mathcal{K})$ will be a pair of categories, where $\mathcal{K}$ is a full subcategory of $\mathcal{C}$ and $E : \mathcal{K} \to \mathcal{C}$ is the inclusion functor.

Let $\mathcal{Y}_\mathcal{K} : \mathcal{K} \to [\mathcal{K}^o, \mathtt{SET}]$ denote the Yoneda embedding and let

$$\gamma_E : \mathcal{C} \to [\mathcal{K}^o, \mathtt{SET}]$$

be its natural extension to $\mathcal{C}$, that is:

- $\gamma_E(X) = \mathcal{C}(X, E(-))$,
- $\gamma_E(f : X \to Y) = f^* : \mathcal{C}(Y, E(-)) \to \mathcal{C}(X, E(-))$ is the natural transformation defined by composition with $f$.

The *shape category* $Sh(\mathcal{C}, \mathcal{K})$ of the pair $(\mathcal{C}, \mathcal{K})$ [11] is the full image of the functor $\gamma_E$, as described by the commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\gamma_E"] \arrow[dr, "Sh"'] & & {[\mathcal{K}^o, \mathtt{SET}]} \\ & Sh(\mathcal{C}, \mathcal{K}) \arrow[ur, "(\gamma_E)_1"'] & \end{tikzcd}$$

Then $Sh(\mathcal{C}, \mathcal{K})$ has the same objects as $\mathcal{C}$ and morphisms given by

$$Sh(\mathcal{C}, \mathcal{K})(X, Y) = Nat(\mathcal{C}(Y, E(-)), \mathcal{C}(X, E(-))).$$

The *shape functor* $Sh : \mathcal{C} \to Sh(\mathcal{C}, \mathcal{K})$ is the identity on objects and $Sh(f) = f^*$.

The class $\Sigma = \mathcal{S}(Sh)$ of all morphisms in $\mathcal{C}$ that are turned into isomorphisms by $Sh$, is called the class of *shape equivalences* for $(\mathcal{C}, \mathcal{K})$.

Note that, by the very definition of shape equivalences, one has

$$\Sigma = \mathcal{K}^\perp = \{s \in \mathrm{Mor}(\mathcal{C}) |\ f_P^* : \mathcal{C}(Y, P) \to \mathcal{C}(X, P) \text{ is bijective for all } P \in \mathcal{K}\},$$

where $f_P^*$ is the map defined by composition with $f$. By a standard abuse of notation we often denote by $\mathcal{K}$ the class of objects and the full subcategory of $\mathcal{C}$ with these objects.

$\mathcal{K}^\perp$ is called the *orthogonal* of $\mathcal{K}$. Symmetrically, the *orthogonal* of $\Sigma \subset \mathrm{Mor}(\mathcal{C})$ is given by

$$\Sigma^\top = \{P \in \mathrm{Ob}(\mathcal{C}) |\ s_P^* : \mathcal{C}(Y, P) \to \mathcal{C}(X, P) \text{ is bijective for all } s \in \Sigma\}.$$

$\mathcal{K}$ (resp. $\Sigma$) will be called *internally saturated* whenever $\mathcal{K} = \mathcal{K}^{\perp\top}$ (resp. $\Sigma = \Sigma^{\top\perp}$) [1, 2]. It is worth noting that $(-)^{\perp\top}$ and $(-)^{\top\perp}$ are both closure operators for the classes of morphisms and for the classes of objects of $\mathcal{C}$, respectively.

Given morphisms $f : X \to Y$ and $g : V \to Z$ in $\mathcal{C}$, we write $f \uparrow g$ to mean that every commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "r"'] & Y \arrow[d, "s"] \arrow[dl, "d"'] \\ V \arrow[r, "g"'] & Z \end{tikzcd}$$

has a unique diagonal $d : Y \to V$ such that $g \circ d = s$ and $d \circ f = r$.
If $\Sigma$ is a class of morphisms in $\mathcal{C}$, then

$$\Sigma^{\uparrow} = \{f \in Mor(\mathcal{C})|\ f \uparrow g,\ for\ all\ g \in \Sigma\},$$
$$\Sigma^{\downarrow} = \{g \in Mor(\mathcal{C})|\ g \uparrow f,\ for\ all\ f \in \Sigma\}.$$

The following properties are easy verified, for $\mathcal{K}, \mathcal{H} \subset \mathrm{Ob}(\mathcal{C})$ and $\Sigma, \Gamma \subset \mathrm{Mor}(\mathcal{C})$:

1. $\mathcal{K} \subset \mathcal{K}^{\perp\top}$ and $\mathcal{K}^{\perp} = \mathcal{K}^{\perp\top\perp}$,
2. $\Sigma \subset \Sigma^{\top\perp}$ and $\Sigma^{\top} = \Sigma^{\top\perp\top}$,
3. $\mathcal{K} \subset \mathcal{H} \Rightarrow \mathcal{H}^{\perp} \subset \mathcal{K}^{\perp}$,
4. $\Sigma \subset \Gamma \Rightarrow \Gamma^{\top} \subset \Sigma^{\top}$,
5. $(\Sigma \cup \Gamma)^{\top} = \Sigma^{\top} \cap \Gamma^{\top}$ and $\Sigma^{\top} \cup \Gamma^{\top} \subset (\Sigma \cap \Gamma)^{\top}$,
6. $(\mathcal{K} \cup \mathcal{H})^{\perp} = \mathcal{K}^{\perp} \cap \mathcal{H}^{\perp}$ and $\mathcal{K}^{\perp} \cup \mathcal{H}^{\perp} \subset (\mathcal{K} \cap \mathcal{H})^{\perp}$,
7. $(\mathrm{Ob}(\mathcal{C}))^{\perp} = \mathrm{Iso}(\mathcal{C})$ and $(\mathrm{Mor}(\mathcal{C}))^{\top} = \mathtt{T}$.

PROPOSITION 2.3. *Let* $\Sigma = \mathcal{K}^{\perp}$*, then*

(i) $\Sigma$ *is internally saturated,*
(ii) $\Sigma$ *has a* CLF*,*
(iii) $\Sigma^{\top}$ *is closed under finite colimits.*

PROOF. (i) is immediate. (ii) is an easy verification of the properties of Proposition 2.2, considering the fact that contravariant representable functors take colimits into limits. For (iii) see [2]. □

THEOREM 2.4. *Let* $\Sigma \subset \mathrm{Mor}(\mathcal{C})$*.* $\Sigma$ *is the class of shape equivalences for a pair* $(\mathcal{C}, \mathcal{K})$ *if and only if it is an internally saturated class of morphisms.*

PROOF. The class of shape equivalences for $(\mathcal{C}, \mathcal{K})$ is given by $\Sigma = \mathcal{K}^{\perp}$, so that the assertion follows from (i) of the proposition above. Conversely, if $\Sigma$ is internally saturated, then it is the class of shape equivalences for the pair $(\mathcal{C}, \Sigma^{\top})$. □

Hence every shape theory determines and is determined by its class of shape equivalences.

The following result was obtained in [1].

COROLLARY 2.5. *An internally saturated class of morphisms is also externally saturated.*

Let now $\mathcal{C} \subset \mathcal{E}$ be two categories and let $\mathcal{K} \subset \mathrm{Ob}(\mathcal{E})$. It makes sense to consider the orthogonal of $\mathcal{K}$ with respect to $\mathcal{E}$, written $\mathcal{K}^{\perp_{\mathcal{E}}}$, and with respect to $\mathcal{C}$, written $\mathcal{K}^{\perp_{\mathcal{C}}}$. Symmetrically $\Sigma^{\top_{\mathcal{E}}}$ and $\Sigma^{\top_{\mathcal{C}}}$, for a class $\Sigma \subset \mathrm{Mor}(\mathcal{E})$. It is clear that

- $\mathcal{K}^{\perp_{\mathcal{C}}} \subset \mathcal{K}^{\perp_{\mathcal{E}}}$ and $\Sigma^{\top_{\mathcal{C}}} \subset \Sigma^{\top_{\mathcal{E}}}$,
- $\mathcal{K}^{\perp_{\mathcal{C}}} = \mathcal{K}^{\perp_{\mathcal{E}}} \cap \mathrm{Mor}(\mathcal{C})$ and $\Sigma^{\top_{\mathcal{C}}} = \Sigma^{\top_{\mathcal{E}}} \cap \mathrm{Ob}(\mathcal{C})$,

moreover the following proposition holds.

Proposition 2.6.
(i) $\mathcal{K}^{\perp_{\mathcal{C}}\top_{\mathcal{C}}} = \mathcal{K}^{\perp_{\mathcal{E}}\top_{\mathcal{E}}} \cap \mathrm{Ob}(\mathcal{C})$, *when* $\mathtt{T} \in \mathcal{K}$
(ii) $\Sigma^{\top_{\mathcal{C}}\perp_{\mathcal{C}}} = \Sigma^{\top_{\mathcal{E}}\perp_{\mathcal{E}}} \cap \mathrm{Mor}(\mathcal{C})$,

Proof. Consider that

$$\mathcal{K}^{\perp_{\mathcal{C}}\top_{\mathcal{C}}} = \mathcal{K}^{\perp_{\mathcal{C}}\top_{\mathcal{E}}} \cap \mathrm{Ob}(\mathcal{C}) = (\mathcal{K}^{\perp_{\mathcal{E}}} \cap \mathrm{Mor}(\mathcal{C}))^{\top_{\mathcal{E}}} \cap \mathrm{Ob}(\mathcal{C}) \supset$$
$$\supset (\mathcal{K}^{\perp_{\mathcal{E}}\top_{\mathcal{E}}} \cup \mathtt{T}) \cap \mathrm{Ob}(\mathcal{C}) = \mathcal{K}^{\perp_{\mathcal{E}}\top_{\mathcal{E}}} \cap \mathrm{Ob}(\mathcal{C}),$$

and

$$\mathcal{K}^{\perp_{\mathcal{C}}} \subset \mathcal{K}^{\perp_{\mathcal{E}}} \Rightarrow \mathcal{K}^{\perp_{\mathcal{E}}\top_{\mathcal{C}}} \subset \mathcal{K}^{\perp_{\mathcal{C}}\top_{\mathcal{C}}} \Rightarrow \mathcal{K}^{\perp_{\mathcal{E}}\top_{\mathcal{E}}} \cap \mathrm{Ob}(\mathcal{C}) \subset \mathcal{K}^{\perp_{\mathcal{C}}\top_{\mathcal{C}}}.$$

This proves (i). The proof of (ii) goes along the same lines. ∎

Let us denote by $\mathrm{Pro}\,\mathcal{C}$ the category of inverse systems in $\mathcal{C}$. Objects of $\mathrm{Pro}\,\mathcal{C}$ are covariant functors of type $\underline{X} : I \to \mathcal{C}$, where $I$ is an essentially small cofiltering category, denoted $\underline{X} = (X_i)_{i\in I}$. The category of direct systems in $\mathcal{C}$ is defined by $\mathrm{Ind}\,\mathcal{C} = (\mathrm{Pro}\,\mathcal{C}^{op})^{op}$. Its objects are covariant functors $\overline{X} : I \to \mathcal{C}$, where $I$ is an essentially small filtering category, also denoted $\overline{X} = (X_i)_{i\in I}$. A direct system $\overline{X} : I \to \mathcal{C}$ gives an inverse system $\underline{X} : I^{op} \to \mathcal{C}$, and vice-versa. We refer to [8, 7, 5], for constructions and terminology concerning inverse and direct systems.
Let $\Sigma \subset \mathrm{Mor}(\mathcal{C})$. A $\Sigma-$*level morphism* in $\mathrm{Pro}\,\mathcal{C}$ (resp. $\mathrm{Ind}\,\mathcal{C}$) is a morphism $\underline{f} : \underline{X} \to \underline{Y}$ (resp. $\overline{f} : \overline{X} \to \overline{Y}$) which belongs levelwise to $\Sigma$. Here we assume, as it is possible [7], that $\underline{X}$, $\underline{Y}$ are indexed over the same set and that $\underline{f}$ (resp. $\overline{f}$) is a natural transformation. Let us denote by $\mathrm{Pro}\,\Sigma$ (resp. $\mathrm{Ind}\,\Sigma$) the class of $\Sigma-$level morphisms in $\mathrm{Pro}\,\mathcal{C}$ (resp. $\mathrm{Ind}\,\mathcal{C}$) .

Lemma 2.7.
(i) $(\mathrm{Pro}\,\mathcal{K})^{\perp_{\mathrm{Pro}\,\mathcal{C}}} = \mathcal{K}^{\perp_{\mathrm{Pro}\,\mathcal{C}}}$.
(ii) $(\mathrm{Pro}\,\mathcal{K})^{\perp_{\mathcal{C}}} = \mathcal{K}^{\perp_{\mathcal{C}}}$.
(iii) $(\mathrm{Ind}\,\mathcal{K})^{\perp_{\mathrm{Ind}\,\mathcal{C}}} = \mathcal{K}^{\perp_{\mathrm{Ind}\,\mathcal{C}}}$.
(iv) $(\mathrm{Ind}\,\mathcal{K})^{\perp_{\mathcal{C}}} = \mathcal{K}^{\perp_{\mathcal{C}}}$.

Proof. (i) From $\mathcal{K} \subset \mathrm{Pro}\,\mathcal{K}$ it follows $(\mathrm{Pro}\,\mathcal{K})^{\perp_{\mathrm{Pro}\,\mathcal{C}}} \subset \mathcal{K}^{\perp_{\mathrm{Pro}\,\mathcal{C}}}$. Conversely, let $\underline{f} : \underline{X} \to \underline{Y}$, $\underline{f} \in \mathcal{K}^{\perp_{\mathrm{Pro}\,\mathcal{C}}}$, and let $\underline{P} = (P_\lambda)_{\lambda\in\Lambda}$ be an object of $\mathrm{Pro}\,\mathcal{K}$. $\underline{f}^*_{P_\lambda}$ is bijective for each $\lambda \in \Lambda$, hence $\underline{f}^*_{\underline{P}} = \varinjlim_\lambda \underline{f}^*_{P_\lambda}$ is bijective, so that $\underline{f} \in (\mathrm{Pro}\,\mathcal{K})^{\perp_{\mathrm{Pro}\,\mathcal{C}}}$. (ii) and (iv) are immediate, while the proof of (iii) is similar to that of (i). ∎

PROPOSITION 2.8.

(i) *$\mathcal{K}$ is internally saturated in $\mathcal{C}$ iff* $\mathrm{Pro}\,\mathcal{K}$ *(resp.* $\mathrm{Ind}\,\mathcal{K}$*) is internally saturated in* $\mathrm{Pro}\,\mathcal{C}$ *(resp.* $\mathrm{Ind}\,\mathcal{C}$*),*
(ii) *$\Sigma$ is internally saturated in $\mathcal{C}$ iff* $\mathrm{Pro}\,\Sigma$ *(resp.* $\mathrm{Ind}\,\Sigma$*) is internally saturated in* $\mathrm{Pro}\,\mathcal{C}$ *(resp.* $\mathrm{Ind}\,\mathcal{C}$*),*

PROOF. We only prove that

$$\mathcal{K} = \mathcal{K}^{\perp_{\mathcal{C}}\top_{\mathcal{C}}} \Leftrightarrow \mathrm{Pro}\,\mathcal{K} = \mathcal{K}^{\perp_{\mathrm{Pro}\,\mathcal{C}}\top_{\mathrm{Pro}\,\mathcal{C}}},$$

the other statements are proved similarly.
First of all, notice that $\mathcal{K}^{\perp_{\mathrm{Pro}\,\mathcal{C}}\top_{\mathcal{C}}} \supset \mathcal{K}^{\perp_{\mathrm{Pro}\,\mathcal{C}}\top_{\mathrm{Pro}\,\mathcal{C}}}$ and, since $\mathcal{K} \subset \mathcal{K}^{\perp_{\mathrm{Pro}\,\mathcal{C}}\top_{\mathrm{Pro}\,\mathcal{C}}}$, then $\mathcal{K} \subset \mathcal{K}^{\perp_{\mathrm{Pro}\,\mathcal{C}}\top_{\mathcal{C}}}$. Moreover, from $\mathcal{K}^{\perp_{\mathrm{Pro}\,\mathcal{C}}\top_{\mathcal{C}}} \subset \mathcal{K}^{\perp_{\mathcal{C}}\top_{\mathcal{C}}}$, it follows $\mathcal{K} = \mathcal{K}^{\perp_{\mathrm{Pro}\,\mathcal{C}}\top_{\mathcal{C}}}$. Finally

$$\mathrm{Pro}\,\mathcal{K} = \mathrm{Pro}\,(\mathcal{K}^{\perp_{\mathrm{Pro}\,\mathcal{C}}\top_{\mathcal{C}}}) =$$

$$= \mathrm{Pro}\,(\mathcal{K}^{\perp_{\mathrm{Pro}\,\mathcal{C}}\top_{\mathrm{Pro}\,\mathcal{C}}} \cap \mathrm{Ob}(\mathcal{C})) = \mathcal{K}^{\perp_{\mathrm{Pro}\,\mathcal{C}}\top_{\mathrm{Pro}\,\mathcal{C}}} \cap \mathrm{Ob}(\mathrm{Pro}\,\mathcal{C}) = \mathcal{K}^{\perp_{\mathrm{Pro}\,\mathcal{C}}\top_{\mathrm{Pro}\,\mathcal{C}}}.$$

The converse follows from Proposition 2.6(i). ∎

In the sequel we shall need the following [6, 10, 1, 3]:

THEOREM 2.9. *Let $R : \mathcal{C} \to \mathcal{K}$ be left adjoint to the inclusion $E : \mathcal{K} \to \mathcal{C}$ and let $\Sigma = \mathcal{S}(R)$. The following properties hold*

(i) *$\Sigma$ has a* TCLF,
(ii) *the unique functor $R' : \mathcal{C}[\Sigma^{-1}] \to \mathcal{K}$, such that $R' \circ P_\Sigma = R$, is an equivalence of categories,*
(iii) $\mathcal{K} = \Sigma^{\top}$ *and* $\Sigma = \mathcal{K}^{\perp} = (\mathrm{Mor}(\mathcal{K}))^{\uparrow}$.
(iv) $\mathcal{K} = \mathcal{K}^{\perp\top}$, $\Sigma = \Sigma^{\top\perp}$
(v) $\Sigma = \Sigma^{\downarrow\uparrow}$.

Notice that (ii) implies that $(\Sigma, \mathcal{K})$ is an orthogonal pair [2], while (v) says that $(\Sigma, \Sigma^{\downarrow})$ is a prefactorization system in $\mathcal{C}$ [3].

## 3. CATEGORICAL SHAPE

Let $(\mathcal{C}, \mathcal{K})$ be a pair of categories as in the previous section. For each $X \in \mathrm{Ob}(\mathcal{C})$ let $X \downarrow \mathcal{K}$ be the comma category of $X$ over $\mathcal{K}$ and let $\sigma_X : X \downarrow \mathcal{K} \to \mathcal{K}$ be the codomain functor. Then one can form the category $\mathcal{C} \downarrow \mathcal{K}$ whose objects are the comma categories $X \downarrow \mathcal{K}$, $X \in \mathrm{Ob}(\mathcal{C})$, and morphisms the functors $t : X \downarrow \mathcal{K} \to Y \downarrow \mathcal{K}$ such that $\sigma_Y \circ t = \sigma_X$. There is an evident functor

$\sigma : \mathcal{C} \to \mathcal{C} \downarrow \mathcal{K}$ and an isomorphism $\Phi : Sh(\mathcal{C}, \mathcal{K}) \to \mathcal{C} \downarrow \mathcal{K}$ which makes the following diagram commutative

$$\begin{tikzcd} \mathcal{C} \arrow[r, "Sh"] \arrow[dr, "\sigma"'] & Sh(\mathcal{C}, \mathcal{K}) \arrow[d, "\Phi"] \\ & \mathcal{C} \downarrow \mathcal{K} \end{tikzcd}$$

See [4] for a complete explanation of this fact.

If $E : \mathcal{K} \to \mathcal{C}$ has left adjoint $R : \mathcal{C} \to \mathcal{K}$, then there is, up to isomorphisms, a decomposition

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\gamma_E"] \arrow[dr, "R"'] & & {[\mathcal{K}^o, \mathtt{SET}]} \\ & \mathcal{K} \arrow[ur, "\mathcal{Y}_{\mathcal{K}}"'] & \end{tikzcd}$$

in fact, for each $X \in \mathrm{Ob}(\mathcal{C})$, there is a natural isomorphisms $\mathcal{C}(X, E(-)) \cong \mathcal{K}(RX, -)$. Since $\mathcal{Y}_{\mathcal{K}}$ is fully faithful, it follows that $Sh(\mathcal{C}, \mathcal{K})$ can be obtained as the full image of $R$. By Lemma 2.1 there is an equivalence

$$Sh(\mathcal{C}, \mathcal{K}) \simeq \mathcal{K}$$

and $\Sigma = \mathcal{S}(R) = \mathcal{S}(\gamma_E)$ has a TCLF.

Note that the unique functor $R' : \mathcal{C}[\Sigma^{-1}] \to K$ such that $R = R' \circ P_\Sigma$, being an equivalence, gives the decomposition $\gamma_E = \mathcal{Y}_E \circ R' \circ P_\Sigma$, with $\mathcal{Y}_E \circ R'$ fully faithful, from which one gets an isomorphism

$$Sh(\mathcal{C}, \mathcal{K}) \cong \mathcal{C}[\Sigma^{-1}].$$

A more interesting case, related to classical topological shape, is when $E$ has a *proadjoint* $P : \mathcal{C} \to \mathrm{Pro}\,\mathcal{K}$ [5]. This means that, if $\tilde{E} : \mathrm{Pro}\,\mathcal{K} \to \mathrm{Pro}\,\mathcal{C}$ denotes the inclusion of the categories of inverse systems, then $\tilde{E}$ has a left adjoint $\tilde{P} : \mathrm{Pro}\,\mathcal{C} \to \mathrm{Pro}\,\mathcal{K}$ and $P$ is its restriction to $\mathcal{C}$. $\tilde{P}$ can be recovered from $P$ as $\tilde{P} = \varprojlim \mathrm{Pro}\,P$ [11]. $P$ is proadjoint to $E$ iff there is a natural isomorphism

$$\mathrm{Pro}\,\mathcal{K}(P(X), -) \cong \mathcal{C}(X, E(-))$$

for every $X \in \mathrm{Ob}(\mathcal{C})$.

Let $L : \mathrm{Pro}\,\mathcal{K} \to [\mathcal{K}^o, \mathtt{SET}]$ be the Grothendieck functor which sends an inverse system $(X_i)_{i \in I}$ to the colimit of representables $\varinjlim_i \mathcal{K}(X_i, -)$. In this case $\gamma_E$ can be decomposed as shown

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\gamma_E"] \arrow[dr, "P"'] & & {[\mathcal{K}^o, \mathtt{SET}]} \\ & \mathrm{Pro}\,\mathcal{K} \arrow[ur, "L"'] & \end{tikzcd}$$

and, since $L$ is fully faithful, it turns out that $Sh(\mathcal{C},\mathcal{K})$ can be obtained as the full image of $P$.

If $\tilde{\Sigma} = \mathcal{S}(\tilde{P})$ then $\Sigma = \mathcal{S}(P) = \mathcal{S}(\gamma_E)$. From the previous observations, one has that

- $\tilde{\Sigma}$ has a TCLF in $\mathrm{Pro}\,\mathcal{C}$,
- there is an equivalence

$$Sh(\mathrm{Pro}\,\mathcal{C}, \mathrm{Pro}\,\mathcal{K}) \simeq \mathrm{Pro}\,\mathcal{K}.$$

and an isomorphism

$$Sh(\mathrm{Pro}\,\mathcal{C}, \mathrm{Pro}\,\mathcal{K}) \cong \mathrm{Pro}\,\mathcal{C}[\tilde{\Sigma}^{-1}].$$

LEMMA 3.1. *Let $\mathcal{K} \subset \mathcal{C} \subset \mathcal{E}$ be categories. Then*

(i) $Sh(\mathrm{Pro}\,\mathcal{C}, \mathrm{Pro}\,\mathcal{K}) \cong Sh(\mathrm{Pro}\,\mathcal{C}, \mathcal{K})$,
(ii) $Sh(\mathcal{C}, \mathcal{K}) \subset Sh(\mathcal{E}, \mathcal{K})$.

As a consequence of the above the shape category of $(\mathcal{C},\mathcal{K})$ can be viewed as the full subcategory of $\mathrm{Pro}\,\mathcal{C}[\tilde{\Sigma}^{-1}]$ with objects those of $\mathcal{C}$, which we write

$$Sh(\mathcal{C},\mathcal{K}) \cong \mathcal{C}[\tilde{\Sigma}^{-1}].$$

It follows that a shape morphism $\varphi : X \to Y$ is a left fraction

$$\varphi = P_{\tilde{\Sigma}}(\underline{s})^{-1} \circ P_{\tilde{\Sigma}}(\underline{f}),$$

for $\underline{f} : X \to \underline{Z}$, $\underline{s} : Y \to \underline{Z}$, with $\underline{s} \in \tilde{\Sigma}$ and $\underline{Z} \in \mathrm{Ob}(\mathrm{Pro}\,\mathcal{C})$ .

Passing from the reflective to the proreflective case both $\Sigma$ and $\mathcal{K}$ inherit good properties:

THEOREM 3.2. *Let $\mathcal{K}$ be proreflective in $\mathcal{C}$ and let $\Sigma$ be the class of shape equivalences for $(\mathcal{C},\mathcal{K})$. Then*

(i) $\Sigma$ *has a* CLF *and is internally saturated w.r.t.* $\mathcal{C}$,
(ii) $\mathcal{K}$ *is internally saturated w.r.t.* $\mathcal{C}$.

PROOF. $\tilde{\Sigma}$ is internally saturated and has a CLF in $\mathrm{Pro}\,\mathcal{C}$. By Proposition 2.2, $\Sigma = \tilde{\Sigma} \cap \mathrm{Mor}(\mathcal{C})$ has a CLF in $\mathcal{C}$. Moreover, from Proposition 2.6, one has

$$\Sigma = \tilde{\Sigma} \cap \mathrm{Mor}(\mathcal{C}) = \tilde{\Sigma}^{\top_{\mathrm{Pro}\,\mathcal{C}} \perp_{\mathrm{Pro}\,\mathcal{C}}} \cap \mathrm{Mor}(\mathcal{C}) = \tilde{\Sigma}^{\top_{\mathcal{C}} \perp_{\mathcal{C}}} \supset \Sigma^{\top_{\mathcal{C}} \perp_{\mathcal{C}}}.$$

The proof of (ii) follows again from Proposition 2.6, since $\mathrm{Pro}\,\mathcal{K}$ is is internally saturated w.r.t. $\mathrm{Pro}\,\mathcal{C}$. ∎

Assume now that $\mathcal{K}$ is an internally saturated class of objects of $\mathcal{C}$. Recall that under such assumptions

- $\Sigma = \mathcal{K}^{\perp}$ is also internally saturated,
- $\Sigma$ has a CLF and, for every $X \in \mathrm{Ob}(\mathcal{C})$, the comma category $X \downarrow \Sigma$ is filtered ([10], 19.2.3)

Theorem 3.3. *Let $\mathcal{K} \subset \mathcal{C}$ be internally saturated and let $\Sigma = \mathcal{K}^{\perp_{\mathcal{C}}}$. $\mathcal{K}$ is proreflective in $\mathcal{C}$ whenever the comma category $X \downarrow \Sigma$ is essentially small, for all $X \in \mathrm{Ob}(\mathcal{C})$.*

Proof. First of all note that, by Proposition 2.8, $\mathrm{Pro}\,\mathcal{K}$ (resp. $\mathrm{Ind}\,\mathcal{K}$) is internally saturated in $\mathrm{Pro}\,\mathcal{C}$ (resp. $\mathrm{Ind}\,\mathcal{C}$). For the same reason $\mathrm{Ind}\,\Sigma$ is also internally saturated in $\mathrm{Ind}\,\mathcal{C}$, so that it has a CLF in $\mathrm{Ind}\,\mathcal{C}$. For each $X \in \mathrm{Ob}(\mathcal{C})$, let $\mathcal{I}_X$ denote the small final subcategory of the filtered category $X \downarrow \Sigma$. The range functor $\mathcal{I}_X \to X \downarrow \Sigma \to \mathcal{C}$ determines a direct system $\overline{X} = (X_i)_{i \in \mathcal{I}_X}$ and a structural morphism $\overline{x} : X \to \overline{X}$, $\overline{x} \in \mathrm{Ind}\,\Sigma$, which actually is a final object for $X \downarrow \mathrm{Ind}\,\Sigma$. From [9, 15.3, p.110] it follows that $\overline{X} \in \Sigma^{\top_{\mathrm{Ind}\,\mathcal{C}}}$, hence $\overline{X} \in \mathrm{Ind}\,\mathcal{K}$ and $X_i \in \mathcal{K}$, for all $i \in \mathcal{I}_X$. Dualizing, the morphism $\underline{x} : X \to \underline{X} = (X_i)_{i \in \mathcal{I}_X}$ is an initial object in $X \downarrow \mathrm{Pro}\,\Sigma$: for every $s : X \to S$, $s \in \Sigma$, there is an $i \in \mathcal{I}_X$ and an $s_i : X_i \to S$ such that $s_i \circ x_i = s$, in particular $s_i \in \Sigma$. Let now $f : X \to K$, $K \in \mathcal{K}$. Since $\mathcal{K} = \Sigma^{\top_{\mathcal{C}}}$, then there is a bijection

$$x^*_{i,K} : \mathcal{C}(X_i, K) \to \mathcal{C}(X, K),$$

for all $i \in \mathcal{I}_X$. Passing to the colimit one obtains a bijection

$$\underline{x}^*_K : \mathrm{Pro}\,\mathcal{C}(\underline{X}, K) \to \mathcal{C}(X, K)$$

which gives the proreflectiveness of $\mathcal{K}$ in $\mathcal{C}$. ∎

Corollary 3.4. *Let $\mathcal{K} \subset \mathcal{C}$ be internally saturated class and let $\Sigma = \mathcal{K}^{\perp}$. $\mathcal{K}$ is reflective in $\mathcal{C}$, whenever the comma category $X \downarrow \Sigma$ has an in initial object, for all $X \in \mathrm{Ob}(\mathcal{C})$.*

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L. Stramaccia
Dipartimento di Matematica e Informatica
Università di Perugia
via Pascoli
06123 Perugia
Italia
*E-mail*: stra@dipmat.unipg.it

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