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ALTERNATE PROOFS OF TWO CHARACTERIZATION THEOREMS OF MILLER AND JANKO ON 2-GROUPS, AND SOME RELATED RESULTS

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Abstract. We study the p-groups all of whose nonabelian maximal subgroups are decomposable in direct or central product of two groups with specific structures.

1. Introduction

Let Θ be a group theoretical property inherited by subgroups. There are a lot of papers where the finite non Θ-groups all of whose proper subgroups are Θ-groups are investigated (such groups we call Θ_1-groups). However, if the property Θ is not inherited by subgroups, Θ_1-groups, as a rule, do not exist. In that case, however, one can try to classify non Θ-groups G all of whose maximal subgroups are Θ-groups.

As Janko has reported [J1], he has classified the 2-groups all of whose minimal nonabelian subgroups (=$\mathcal{A}_1$-subgroups) are $\cong Q_8$; this coincides with Theorem 2.4 (in fact, in [J2] the 2-groups all of whose $\mathcal{A}_1$-subgroups have the same order 8 are classified). He also noticed that his result implies the classification of minimal non Dedekindian 2-groups (this coincides with Lemma 2.1). Theorem 2.4 follows from Lemma 2.3, below. Our proof of Lemma 2.3 uses Lemma 2.1.[1]

Recall that a group is said to be *Dedekindian* if all its subgroups are normal. If G is a nonabelian Dedekindian group, then $G = Q \times E \times A$,

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[1]It appears that Lemma 2.1 was proved by G.A. Miller [M1] in 1907 (I learned about this from Internet, after completing this paper). Janko's proof of Theorem 2.4 is independent of Lemma 2.1.

where $Q \cong Q_8$, E is elementary abelian 2-group and A is abelian of odd order (Dedekind). As follows from general definition, a p-group G is said to be *minimal nonabelian* ($=\mathcal{A}_1$-group), if it is nonabelian but all its proper subgroups are abelian. In this paper G is a p-group, where p is a prime.

A p-group $M \times E$ is said to be an $M^\times$-*group* if M is of maximal class and E elementary abelian (we consider the group $\{1\}$ as elementary abelian p-group for every prime p). The above group is said to be an $M_3^\times$-group if, in addition, $|M| = p^3$. All nonabelian epimorphic images of $M^\times$-groups are $M^\times$-groups. Every nonabelian subgroup of $M_3^\times$-group is also an $M_3^\times$-group. All nonabelian maximal subgroups of an $M^\times$-group G are $M^\times$-groups if and only if G has an abelian subgroup of index p.

It follows from Lemma J(i) that, if G is a 2-group of maximal class and order 2^m, then it is one of the following groups: dihedral D_{2^m}, generalized quaternion Q_{2^m} or semidihedral SD_{2^m} ($m > 3$). These three groups together with $M_{2^m} = \langle a, b \mid o(a) = 2^m, o(b) = 2, a^b = a^{1+2^{m-2}}, m > 3\rangle$ present the complete list of nonabelian 2-groups of order 2^m with cyclic subgroup of index 2. By Γ_1 we denote the set of maximal subgroups of G.

Remark 1.1. Let a p-group $G = M \times E$, where M is a nonabelian group with cyclic center and $E > \{1\}$ is elementary abelian and let $M_1 < G$ have no direct factor of order p and $|M_1| > p$. We claim that M_1 is isomorphic to a subgroup of M. It suffices to prove that $M_1 \cap E = \{1\}$. Assume that $X \le M_1 \cap E$ is of order p. Then $G = X \times G_0$ so, by the modular law, $M_1 = X \times (M_1 \cap G_0)$, a contradiction. In particular, if $M_1 < G$ is minimal nonabelian, then M_1 is isomorphic to a subgroup of $G/E \cong M$.

A nonabelian 2-group G is said to be *generalized dihedral* if it is nonabelian and contains a subgroup A such that all elements of the set $G - A$ are involutions. Then A is abelian of exponent > 2, $|G : A| = 2$, all subgroups of A are G-invariant, $\Omega_1(A) = Z(G)$ and G/G' is elementary abelian since $\Omega_1(G) = G$ (Burnside). Clearly, A is characteristic in G.

We use notation which is standard for finite p-group theory (see references [B1, B2, B3]). In Lemma J some elementary results which we use in what follows, are gathered.

Lemma J. *Let G be a nonabelian p-group.*

(a) [B2, Proposition 19(a)] *Let $B < G$ be nonabelian of order p^3. If $C_G(B) < B$, then G is of maximal class.*
(b) [B1, Lemma 5.3] *Suppose that $E < G$ is such that $|E'| = p$, $Z(E) = \Phi(E)$ and $[G, E] = E'$. Then $G = E * C_G(E)$. The last equality holds whenever $E < G$ is either minimal nonabelian or extraspecial and $[G, E] = E'$.*
(c) *(O. Schreier) If* $d(G) = 2$ *and* $|G : H| = 2$, *then* $d(H) \le 3$.

(d) *(L. Redei* [R]*; see also* [BJ2, Lemma 3.1.]*) If G is minimal nonabelian, then $|G'| = p$, $\mathrm{d}(G) = 2$, $|\Omega_1(G)| \le p^3$ so all proper subgroups of G are of rank ≤ 3. If $\Omega_1(G) = G$, then either $p > 2$ and G is of order p^3 and exponent p or $p = 2$ and $G \cong \mathrm{D}_8$. If $|\Omega_1(G)| \le p^2$, then G is metacyclic.*

(e) *(Z. Janko; see* [B5, Theorem 10.28, 10.32, 10.33] *and* [J3]*) All $\mathcal{A}_1$-subgroups of a 2-group G are generated by involutions if and only if G is generalized dihedral.*

(f) [B3, Theorem 7.4(c)] *If $|G| > p^3$ and G is not of maximal class, then the number of subgroups of maximal class and index p in G is a multiple of p^2.*

(g) *(Kazarin-Mann; see also* [BJ2, Lemma 3.2(d)]*) If $|H'| \le p$ for all $H \in \Gamma_1$, then $|G'| \le p^3$. If, in addition, G has an abelian subgroup of index p, then $|G'| \le p^2$.*

(h) *(Tuan; see* [I, Lemma 12.12]*) If G has an abelian maximal subgroup, then $|G'| = \frac{1}{p}|G : \mathrm{Z}(G)|$. If G has two distinct abelian maximal subgroups, then $|G'| = p$.*

(i) *(O. Taussky) If $p = 2$ and $|G : G'| = 4$, then G is of maximal class.*

(j) [B4, Remark 6.2] *If G is neither cyclic nor a 2-group of maximal class, then the number of cyclic subgroups of order $p^k > p$ in G is a multiple of p.*[2]

(k) [BJ2, Lemma 3.2(a)] *If $G' \le \mathrm{Z}(G)$ is of exponent p and $\mathrm{d}(G/G') = 2$, then G is an $\mathcal{A}_1$-group.*

(l) [B1, Theorem 6] *If $p > 2$ and $\Phi(G)$ is cyclic, then $\Phi(G) \le \mathrm{Z}(G)$.*

(m) *If $|G'| = |\mathrm{Z}(G)| = p$, then G is extraspecial.*

(n) *The number of abelian members in the set Γ_1 is $0, 1$ or $p + 1$. In particular, the number of nonabelian members in the set Γ_1 is $\ge p$, unless G is an $\mathcal{A}_1$-group.*

Remark 1.2. Let a p-group $G = M \times C$, where M is of maximal class and $C = \langle c \rangle \cong \mathrm{C}_{p^n}$, $n > 1$. We claim that G contains an $\mathcal{A}_1$-subgroup H of order p^{n+2} with $|H \cap M| = p^2$. Indeed, by Blackburn's Theorem (see [B5, Theorem 9.6]), G contains a nonabelian subgroup $D = \langle R, a \rangle$ of order p^3, where $|R| = p^2$ and $o(a) \le p^2$. Set $u = ac$; then $R \cap \langle u \rangle = \{1\}$, $o(u) = o(c) = p^n$. We claim that $L = \langle u, R \rangle$ is an $\mathcal{A}_1$-subgroup. Indeed, L is nonabelian so $|L'| = p$ since $L' < R$, and $\mathrm{d}(L/L') = 2$ so L is an $\mathcal{A}_1$-subgroup of order p^{n+2}, by Lemma J(k). We also have $|L \cap M| = R$ since $\langle u \rangle \cap M = \{1\}$. If M is not generalized quaternion, one can take from the start $R \cong \mathrm{E}_{p^2}$; in that case, L is not metacyclic since $\Omega_1(L) \cong \mathrm{E}_{p^3}$. If M is generalized quaternion, then $|\Omega_1(G)| = 4$, so all $\mathcal{A}_1$-subgroups of G are metacyclic (Lemma J(d)).

[2]As I knew from Internet, this result was proved by G.A. Miller many years ago; see also the part written by Miller, in [MBD]. However, in the existing literature I did not see references on this result.

Similarly, if $2 \le k < n$, then G contains an $\mathcal{A}_1$-subgroup of order p^{k+2} not contained in M.

REMARK 1.3. Suppose that a group G of order $2^m > 2^4$ is not of maximal class. Let $H \in \Gamma_1$ be of maximal class. Then the set Γ_1 has exactly four members of maximal class (Lemma J(f)). Suppose that all nonabelian members of the set Γ_1 are $M^\times$-groups. We claim that then G itself is an $M^\times$-group. Assume that our claim is false. Let $Z < H$ be cyclic of index 2; then, since $|H| \ge 16$, Z is characteristic in H so normal in G. Next, G contains a normal abelian subgroup R of type $(2,2)$ (Lemma J(j)); then $R \cap H = \Omega_1(Z)$. Since $A = RZ \in \Gamma_1$ is not an $M^\times$-group and $|A| > 8$, it must be abelian. Let F be a nonabelian maximal subgroup of H. Then $RF \in \Gamma_1$ since $|RF| = |H|$, and, by hypothesis, RF is an $M^\times$-group which is not of maximal class since $|RF| \ge 16$. It follows that $R = \mathrm{Z}(RF)$ (indeed, $R \not\le F$ since $R \not\le H$). Since $R < A$, we get $\mathrm{C}_G(R) \ge A(RF) = AF = G$ so $R = \mathrm{Z}(G)$. If $L < R$ is of order 2 and $L \not\le H$, then $G = HL = H \times L$ is an $M^\times$-group.

The following lemma is known.

LEMMA 1.4. *Suppose that a group G is of order p^{2m+1} and $|G'| = p$. Then the following assertions are equivalent:*

(a) *G is extraspecial.*
(b) *G has no abelian subgroup of index p^{m-1}.*

PROOF. Let G be extraspecial and let A be an abelian subgroup of G of maximal order; then $A \lhd G$ since $G' = \mathrm{Z}(G) < A$. It follows from decomposition of G in the central product of nonabelian subgroups of order p^3 that $|G : A| \le p^m$. We want to show that there we have equality. The class number of G equals $|G/G'| + p - 1 = p^{2m} + p - 1$ so that G has exactly $p - 1$ nonlinear irreducibles. Since the sum of squares of degrees of nonlinear irreducibles equals $|G| - |G/G'| = p^{2m}(p-1)$, it follows that the degrees of all irreducibles equal p^m. By Ito's theorem on degrees [BZ, Theorem 7.2.7], $|G : A| \ge \chi(1) = p^m$ so (a) $\Rightarrow$ (b).

Now assume that (b) is true. Let $\chi \in \mathrm{Irr}_1(G)$. Then $\chi = \lambda^G$, where λ is a linear character of some subgroup H of index $\chi(1)$ in G. We have $G' \not\le \ker(\chi) = \mathrm{core}_G(\ker(\lambda^G))$. Assuming that H is nonabelian, we get $G' = H' \le \ker(\lambda)$, a contradiction. Thus, H is abelian. Then, by (b), we get $\chi(1) = |G : H| \ge p^m$. We have

$$p^{2m+1} = |G| = |G : G'| + \sum_{\chi \in \mathrm{Irr}_1(G)} \chi(1)^2 \ge p^{2m} + |\mathrm{Irr_1(G)}|\mathrm{p^{2m}}$$

so $|\mathrm{Irr}_1(G)| \le p - 1$ and, by [BZ, Lemma 3.35], G is extraspecial so (b) $\Rightarrow$ (a). □

LEMMA 1.5. *Let G be an extraspecial group of order p^{2m+1}, $m > 1$, and let $M \in \Gamma_1$. Then $M = E\mathrm{Z}(M)$, where E is an extraspecial maximal subgroup* *of M and $|\mathrm{Z}(M)| = p^2$. If $L \triangleleft G$ is of order p^2, then $N = \mathrm{C}_G(L) = L * E$, where E is extraspecial.*

PROOF. By Lemma 1.4, M is nonabelian. Since $|M| = p^{2m}$, the subgroup M is not extraspecial. It follows from Lemma J(m) that $|\mathrm{Z}(M)| > p$. Let $R \leq \mathrm{Z}(M)$ be G-invariant of order p^2; then $\mathrm{C}_G(R) = M$ since $R \not\leq \mathrm{Z}(G)$. On the other hand, $R \not\leq \Phi(G) = \Phi(M)$ so there is a maximal subgroup E of M such that $M = ER$. But M is nonabelian so is E. We have $|E| = p^{2m-1} = p^{2(m-1)+1}$. Assume that E has an abelian subgroup, say A, of index p^{m-2}; then AR is an abelian subgroup of index p^{m-1} in G, contrary to Lemma 1.4. Thus, E has no abelian subgroup of index p^{m-2} so E is extraspecial (Lemma 1.4). It follows from $M = E\mathrm{Z}(M)$ that $|\mathrm{Z}(M)| = p^2$. □

DEFINITION 1.6. *A nonabelian p-group G is said to be*

1. *a $\mathcal{Z}$-group provided $|\mathrm{Z}(G)| = p$ and G' is cyclic.*
2. *a $\mathcal{Z}^\times$-group ($M^\times$-group) provided $G = U \times E$, where U is a $\mathcal{Z}$-group (group of maximal class) and E is elementary abelian.*
3. *$(\mathcal{Z} * \mathcal{C})$-group ($(\mathcal{M} * \mathcal{C})$-group) provided $G = A * Z$, a central product, where A is a $\mathcal{Z}$-group (group of maximal class), $Z = \mathrm{Z}(G)$ is cyclic.*

The center of $\mathcal{Z}^\times$-group ($M^\times$-group) is elementary abelian. The center of $\mathcal{Z} * \mathcal{C}$-group ($\mathcal{M} * \mathcal{C}$-group) is cyclic. Extraspecial p-groups and 2-groups of maximal class are $\mathcal{Z}$-groups. A $\mathcal{Z}$-group G with $|G'| = p$ is extraspecial (Lemma J(m)). If A is a cyclic p-group of order $p^n > p$ and G the Sylow p-subgroup of the holomorph of A, then G is a $\mathcal{Z}$-group (if, in addition, $p > 2$, then G is metacyclic). If p-group $G = E * L$, where E is extraspecial and L is a $\mathcal{Z}$-group, $E \cap L = \mathrm{Z}(E)$, then G is a $\mathcal{Z}$-group. If a $\mathcal{Z}$-group G is minimal nonabelian, then $|G| = p^3$. If a $\mathcal{Z}$-group G of order $> p^3$ is of maximal class, then $p = 2$. Clearly, the property $(\mathcal{M} * \mathcal{C})$ is not inherited by subgroups and epimorphic images.

REMARK 1.7. Suppose that all nonabelian maximal subgroups of a $\mathcal{Z}$-group G are $\mathcal{Z}^\times$-groups. We claim that if $|G'| > p$, then G is a 2-group of maximal class, and if $|G'| = p$, then G is extraspecial of exponent p, unless G is of order p^3 and exponent p^2. (i) Assume that $|G'| > p$; then $R = \Omega_2(G') \cong \mathrm{C}_{p^2}$ and $\mathrm{C}_G(R) \in \Gamma_1$ is abelian. Then, by Lemma J(h), $|G : G'| = p|\mathrm{Z}(G)| = p^2$. If $p = 2$, then G is of maximal class (Lemma J(i)). Now let $p > 2$. Then $G' = \Phi(G)$ is cyclic and so $\Phi(G) = \mathrm{Z}(G)$ (Lemma J(l)) hence G is an $\mathcal{A}_1$-group and $|G'| = p$, contrary to the assumption. (ii) Let $|G'| = p$; then $G' = \mathrm{Z}(G)$ so G is extraspecial (Lemma J(m)). However, if $\exp(G) > p$ and $|G| > p^3$, then the set Γ_1 contains a nonabelian member which is not a $\mathcal{Z}^\times$-group (Lemma 1.5). Thus, if $|G| > p^3$, then G is extraspecial of exponent p, and every such G satisfies the hypothesis, by the same lemma.

REMARK 1.8. Suppose that all nonabelian maximal subgroups of a $\mathcal{Z}$-group G are $(\mathcal{Z} * \mathcal{C})$-groups. Let G be not a 2-group of maximal class; then G contains a normal subgroup $R \cong \mathrm{E}_{p^2}$ (Lemma J(j)). Since the center of the $\mathcal{Z}$-group G is of order p, $\mathrm{C}_G(R) \in \Gamma_1$ must be abelian so $|G : G'| = p|\mathrm{Z}(G)| = p^2$ (Lemma J(h)), and we conclude that $G' = \Phi(G)$. If $p > 2$, then $\Phi(G) \leq \mathrm{Z}(G)$ (Lemma J(l)) so $|G| = p^3$. If $p = 2$, then G is a 2-group of maximal class (Lemma J(l)), contrary to the assumption.

LEMMA 1.9. *Let G be neither abelian nor an $\mathcal{A}_1$-group. Suppose that all nonabelian members of the set Γ_1 are $\mathcal{Z}^\times$-groups. Then one of the following holds:*

(a) *The set Γ_1 has an abelian member. Then all nonabelian members of the set Γ_1 are $M^\times$-groups for $p = 2$ and $M_3^\times$-groups for $p > 2$.*
(b) *The set Γ_1 has no abelian member. Then nonabelian members of the set Γ_1 are of the form $E_1 \times E_2$, where E_2 is elementary abelian and E_1 is extraspecial. If, in addition, G itself is a $\mathcal{Z}^\times$-group of the form $E_1 \times E_2$, where E_1 and E_2 are as above, then $p > 2$ and $\exp(E_1) = p$, $|E_1| \geq p^5$.*

PROOF. Take a nonabelian $H = M \times E \in \Gamma_1$, where M is a $\mathcal{Z}$-group and E is elementary abelian; then $M' = H' \triangleleft G$ is cyclic. Let $A \in \Gamma_1$ be abelian. In that case, $M \cap A$ is a maximal abelian subgroup of M. Then $|M : M'| = p|\mathrm{Z}(M)| = p^2$ (Lemma J(h)) so $M' = \Phi(M)$ is cyclic. If $p = 2$, then M is of maximal class (Lemma J(i)). If $p > 2$, then $M' = \mathrm{Z}(M)$ (Lemma J(l)) so $|M| = p^3$.

Suppose that $|M'| > p$; then $\mathrm{C}_G(\Omega_2(M')) = A \in \Gamma_1$ is abelian since its center has exponent $> p$. In that case, $M \cap A$ is a maximal abelian subgroup of M. Arguing, as in the previous paragraph, we conclude that $p = 2$ and M is of maximal class. This completes the proof of (a).

Now assume that the set Γ_1 has no abelian member. Then $|H'| = |M'| = p$ for all nonabelian $H \in \Gamma_1$ so M is extraspecial (Lemma J(m)).

Now, in addition, let G be a $\mathcal{Z}^\times$-group and the set Γ_1 has no abelian member. Then $G = M \times E$, where M is extraspecial of order $\geq p^5$ and E elementary abelian. Let $U < M$ be maximal; then $U \times E \in \Gamma_1$ so U is a $\mathcal{Z}^\times$-group. Then $\exp(M) = p > 2$ (Remark 1.7). □

LEMMA 1.10. *Suppose that a nonabelian p-group G has an abelian subgroup of index p. Then the following conditions are equivalent:*

(a) $|\mathrm{Z}(G)| = p$.
(b) $|G : G'| = p^2$.
(c) *G is of maximal class.*

PROOF. By Lemma J(h), (a) and (b) are equivalent and follow from (c). Now let (a) hold and prove (c) using induction on $|G|$. We have $\mathrm{Z}(G) \leq G'$ and $|G : G'| = p^2$ (Lemma J(h)). One may assume that $|G| > p^3$. Set

$\bar{G} = G/\mathrm{Z}(G)$. Then $|\bar{G} : \bar{G}'| = p^2$ hence $|\mathrm{Z}(\bar{G})| = p$ (Lemma J(h)) and $\bar{G}$ is of maximal class so is G since $|\mathrm{Z}(G)| = p$. (It is easy to show that if G is as in Lemma 1.10, then all nonabelian subgroups of G are of maximal class; in particular, all $\mathcal{A}_1$-subgroups of G are of order p^3.) □

Remark 1.11. Let G be a nonabelian p-group of order $> p^3$ and suppose that, whenever $H \le G$ is nonabelian, then $|H : H'| = p^2$. We claim that then G is of maximal class with abelian subgroup of index p. Indeed, let $N \triangleleft G$ be of index p^4. Then G/N has an abelian subgroup A/N, of index p so A is abelian, and we are done (Lemma 1.10).

Lemma 1.12. *Suppose that a p-group G, which is a $\mathcal{Z}$-group, contains an abelian subgroup of index p. Then one and only one of the following holds:*

(a) *If $p = 2$, then G is of maximal class.*
(b) *If $p > 2$, then $|G| = p^3$.*

Proof. By Lemma J(h), $|G : G'| = p|\mathrm{Z}(G)| = p^2$ so $\mathrm{d}(G) = 2$. Then G is of maximal class if $p = 2$ (Lemma J(i)). Let $p > 2$. Then $\Phi(G) = G'$ is cyclic so $\Phi(G) \le \mathrm{Z}(G)$ (Lemma J(l)), and we conclude that G is an $\mathcal{A}_1$-group since $\mathrm{d}(G) = 2$. Since $|\mathrm{Z}(G)| = p$, we get $|G| = p^3$. □

Lemma 1.13. *Let G be a p-group which is not of maximal class and $A, H \in \Gamma_1$, where A is abelian and H is of maximal class. Then $|\mathrm{Z}(G)| = p^2$ and $G = H\mathrm{Z}(G)$.*

Proof. By Lemma J(f), $G' = H'$ is of index p^3 in G. By Lemma J((h), $|\mathrm{Z}(G)| = \frac{1}{p}|G : G'| = p^2$ so $G = H\mathrm{Z}(G)$, by the product formula. □

Our main results are the following five theorems.

Theorem A. *Suppose that all maximal subgroups of a nonabelian 2-group G are $\mathcal{Z}^\times$-groups. Then one of the following holds:*

(a) *G is an $M^\times$-group.*
(b) *G is minimal nonabelian.*
(c) *$G = D * C$ is of order 16, where D is nonabelian of order 8 and C is cyclic of order 4.*
(d) *G is a generalized dihedral group of order 2^5 with abelian Hughes subgroup subgroup of type $(4, 4)$.*

Theorem B. *Suppose that all nonabelian maximal subgroups of a nonabelian p-group G, $p > 2$, are $\mathcal{Z}^\times$-groups. Then one of the following holds:*

(a) *G is an $M_3^\times$-group.*
(b) *G is minimal nonabelian.*
(c) *G is of maximal class and order p^4.*
(d) *$G = M * C$ is of order p^4, where M is nonabelian of order p^3 and C is cyclic of order p^2. We also have $G = M_1 * C$ where a nonabelian subgroup M_1 of order p^3 is not isomorphic with M.*

(e) *G is of order p^5 without abelian subgroup of index p, $|G'| = p^3$, $\mathrm{Z}(G) < G'$ is abelian of type (p,p). If $R < \mathrm{Z}(G)$ is of order p, then G/R is of maximal class.*
(f) *G is special of order p^5, $d(G) = 3$.*
(g) *G is special of order p^6 and exponent p, $d(G) = 3$.*
(h) *$G = E \times E_0$, where E_0 is elementary abelian and E is extraspecial; if $|E| \geq p^5$, then $\exp(E) = p$.*

Theorem C. *Suppose that all nonabelian maximal subgroups of a 2-group G are $(\mathcal{Z} * \mathcal{C})$-groups but G is not an $(\mathcal{Z} * \mathcal{C})$-group. Then one of the following holds:*
(a) *G is minimal nonabelian.*
(b) *$G = F \times D$, where F is nonabelian of order 8 and $|D| = 2$.*

Theorem D. *Suppose that $p > 2$ and all nonabelian maximal subgroups of a nonabelian p-group G are $(\mathcal{Z}*\mathcal{C})$-groups. Then one of the following holds:*
(a) *G is minimal nonabelian.*
(b) $|G| = p^4$.

Theorem E. *Let G be a nonabelian p-group of order $> p^4$, $p > 2$, which is not an $\mathcal{A}_1$-group. Suppose that all nonabelian maximal subgroups of G are $(\mathcal{M} * \mathcal{C})$-groups. Then G has an abelian subgroup of index p and one of the following holds:*
(a) *$G = M * C$ is an $(\mathcal{M} * \mathcal{C})$-group, where M of order $> p^3$ is of maximal class with abelian subgroup of index p and $C = \mathrm{Z}(G)$ is cyclic of order $\leq p^2$.*
(b) *$G = M \times L$, where M is nonabelian of order p^3 and $|L| = p$.*
(c) *$\mathrm{Z}(G)$ is cyclic of order $> p$, $\mathrm{Z}(G) < \Phi(G)$, $G/\mathrm{Z}(G)$ is either of maximal class or of order p^4 and class 2.*

2. Proof of Theorem A

We begin with the following partial case of Theorem 2.4.

Lemma 2.1 (Miller [M1]). *If G is a minimal non Dedekindian 2-group, then G is either minimal nonabelian or $\cong \mathrm{Q}_{16}$.*

Proof. Assume that G is not an $\mathcal{A}_1$-group so $|G| = 2^m > 2^3$. Let $H = Q \times E \in \Gamma_1$, where $Q \cong \mathrm{Q}_8$ and $\exp(E) \leq 2$. Suppose that $E = \{1\}$; then $m = 4$. If $\mathrm{C}_G(Q) \not\leq Q$, then $G = Q\mathrm{Z}(\mathrm{G})$ so $\mathrm{Z}(G)$ is cyclic of order 4 since G is not Dedekindian. Then $G = Q * \mathrm{Z}(G) = D * \mathrm{Z}(G)$, a contradiction since $D \cong \mathrm{D}_8$ is non Dedekindian. Thus, $\mathrm{C}_G(Q) < Q$ so G is of maximal class (Lemma J(a)); then $G \cong \mathrm{Q}_{16}$. Next assume that $|G| > 2^4$ so $E > \{1\}$ for arbitrary choice of nonabelian $H \in \Gamma_1$. We have $H' = Q' \triangleleft G$ and H/Q' is elementary abelian maximal subgroup of G/Q'. Assume that G/Q' has a nonabelian maximal subgroup $F/Q' = (Q_1/Q') \times (E_1/Q')$. where $Q_1/Q' \cong \mathrm{Q}_8$

and $\exp(E_1/Q') \le 2$. Then $(Q_1/Q') \cap (H/Q')$ is maximal in Q_1/Q' so cyclic of order 4 and elementary abelian as a subgroup of H/Q', a contradiction. Thus, $\bar{G} = G/Q'$ is either abelian or minimal nonabelian.

(i) Let $\bar{G}$ be minimal nonabelian; then $|G'| = 4$. Since $\exp(\bar{H}) = 2$, we get $\exp(\bar{G}) = 4$ and $|\bar{H}| \le 8$ (Lemma J(d)). Since $m > 4$, we get $\bar{H} \cong \mathrm{E}_8$. Since $\Omega_1(\bar{G}) = \bar{H}$ (Lemma J(d)), $\bar{G}$ is generated by elements of order 4 so it has two distinct maximal subgroups $\bar{A}$ and $\bar{B}$ of exponent 4. Then A and B are abelian (if, for example, $\bar{A}$ is nonabelian, then $A' = Q'$ and $\exp(A/A') = 2$, a contradiction). In that case, $A \cap B = \mathrm{Z}(G)$ so $|G'| = 2$ (Lemma J(h)), a contradiction.

(ii) Let $\bar{G}$ be abelian; then $G' = Q'$ is of order 2 so $G = Q * \mathrm{C}_G(Q)$ (Lemma J(b)). If $\mathrm{C}_G(Q)$ has a cyclic subgroup L of order 4, then $Q * L$ is not Dedekindian so $Q * L = G$. If $Q \cap L = \mathrm{Z}(Q)$, then G contains a proper subgroup $\cong \mathrm{D}_8$, a contradiction. If $Q \cap L = \{1\}$, then $G = Q \times L$ contains an $\mathcal{A}_1$-subgroup B of order 16 (Remark 1.2); since $B < G$ and B is not Dedekindian, we get a contradiction. Thus, $\exp(\mathrm{C}_G(Q)) = 2$ so $\mathrm{C}_G(Q) = \mathrm{Z}(G)$. If $\mathrm{Z}(G) = Q' \times E_1$, then $G = Q \times E_1$ is Dedekindian, a final contradiction. □

A 2-group G is said to be a $Q^\times$-group if $G = Q \times E$, where Q is generalized quaternion and E is elementary abelian. The center of every $Q^\times$-group is elementary abelian.

Remark 2.2. Let us show that if a 2-group $G = Q \times E$, where Q is generalized quaternion and $\exp(E) = 2$, and $A < G$ is nonabelian, then A is a $Q^\times$-group. We use induction on $|G|$. Obviously, $K \in \Gamma_1$ such that $G = K \times L$, where $L \le E$, is a $Q^\times$-group. One may assume that $A \cap E > \{1\}$. Let $X \le A \cap E$ be of order 2. Then $G = X \times G_0$ since $X \not\le \Phi(G)$. In that case, by the modular law, $A = X \times (A \cap G_0)$. Since G_0 is a $Q^\times$-group, it follows, by induction in G_0, that $A \cap G_0$ is also a $Q^\times$-group. Then $A = (A \cap G_0) \times X$ is a $Q^\times$-group, as desired. Similarly, if a 2-group G is an $M^\times$-group, then all its nonabelian subgroups are $M^\times$-groups. In particular, all $\mathcal{A}_1$-subgroups of G have the same order 8.

Lemma 2.3. *Suppose that all nonabelian maximal subgroups of a nonabelian 2-group G are $Q^\times$-groups. Then G is either a $Q^\times$- or $\mathcal{A}_1$-group.*

Proof. Assume that G is neither minimal nonabelian nor of maximal class (if G is of maximal class, it is generalized quaternion so a $Q^\times$-group). We also may assume, in view of Lemma 2.1, that $m > 4$. Then all proper nonabelian subgroups of G are $Q^\times$-groups, by Remark 2.2. There is a nonabelian $H = Q \times E \in \Gamma_1$, where Q is generalized quaternion and E elementary abelian. If $E = \{1\}$, then, by Remark 1.3, G is a $Q^\times$-group. Next we assume that $E > \{1\}$ for arbitrary choice of nonabelian $H \in \Gamma_1$.

In view of Lemma 2.1, one may assume that the subgroup H of the previous paragraph is chosen so that $|Q| > 2^3$. Then $H' = Q' \triangleleft G$ is cyclic of order > 2. In that case, $A = \mathrm{C}_G(\Omega_2(Q')) \in \Gamma_1$ is abelian since $\exp(\mathrm{Z}(A)) > 2$. Since $E < A$, we get $\mathrm{C}_G(E) \geq HA = G$ so that $E < \mathrm{Z}(\mathrm{G})$ ($<$, since $\mathrm{Z}(Q) < \mathrm{Z}(G)$ and $\mathrm{Z}(Q) \not\leq E$). It follows from $|G'| > 2$ that A is the unique abelian member of the set Γ_1 (Lemma J(h)). Take a nonabelian $F \in \Gamma_1 - \{H\}$ (F exists, by Lemma J(n)) and assume that $E \not\leq F$. Then there is $X \leq E$ of order 2 such that $X \not\leq F$. In that case, $G = F \times X$ is a $Q^\times$-group, and we are done. Therefore, one may assume that $E < \Phi(G)$. Write $\bar{G} = G/E$; then $\bar{G} = 2|\bar{H}| = 2|Q| > 2^4$. Therefore, if $L \in \Gamma_1$ is nonabelian, then $\bar{L}$ is an $M^\times$-group since, generally speaking, E is not a direct factor of L. By the above, $\bar{G}$ contains a maximal subgroup $\bar{H}$, which is generalized quaternion of order > 8. In view of Lemma 1.13, the following two possibilities for $\bar{G}$ must be considered.

(i) Let $\bar{G}$ be not of maximal class. Then $\bar{G} = \bar{H} \times \bar{C} = \bar{Q} \times \bar{C}$, where $|\bar{C}| = 2$ so that $\bar{G}$ is a $Q^\times$-group. Since $E < \mathrm{Z}(G)$ and $\bar{C} = C/E$ is of order 2, the subgroup $C \triangleleft G$ is abelian and $C \cap H \leq E \cap H = \{1\}$ so $G = Q \cdot C$ is a semidirect product with kernel C. If $F < Q$ is nonabelian maximal, then $F \cdot C \in \Gamma_1$ is a $Q^\times$-group so $FC = C \times F$ hence $\exp(C) = 2$. Since Q is generated by its nonabelian maximal subgroups, we get $G = Q \times C$ so that G is a $Q^\times$-group.

(ii) Now let $\bar{G}$ be of maximal class. Then $\mathrm{d}(G) = 2$ since $E < \Phi(G)$, and hence, by Lemma J(c), we get $\mathrm{d}(F) \leq 3$ for all $F \in \Gamma_1$. It follows that $|E| = 2$. Since $E \not\leq G'$ (otherwise, by Lemma J(i), G is of maximal class), we get $E \cap G' = \{1\}$; then G' is cyclic of index 8 in G and G/G' is abelian of type $(4, 2)$ since $\mathrm{d}(G) = 2$. Let A/G' and B/G' be two distinct cyclic subgroup of order 4 in G/G'. Since abelian epimorphic images of $Q^\times$-groups have exponent 2, it follows that A and B are abelian maximal subgroups of G so $A \cap B = \mathrm{Z}(G)$. In that case, $|G'| = 2 < |H'|$, a final contradiction. □

THEOREM 2.4 (Janko [J2]). *Suppose that every $\mathcal{A}_1$-subgroup of a nonabelian 2-group G is $\cong \mathrm{Q}_8$. Then G is a $Q^\times$-group.*

PROOF. We use induction on $|G|$. By induction, every proper nonabelian subgroup of G is a $Q^\times$-group. Then, by Lemma 2.3, G is either an $\mathcal{A}_1$- or $Q^\times$-group. In the first case, however, $G \cong \mathrm{Q}_8$. □

A 2-group G is said to be a $D^\times$-group if $G = D \times E$, where D is dihedral and $\exp(E) \leq 2$.

PROPOSITION 2.5 (Compare with [M2]). *Suppose that all nonabelian maximal subgroups of a nonabelian 2-group G are $D^\times$-groups. Then one of the following holds:*

(a) *G is minimal nonabelian.*
(b) *G is a $D^\times$-group.*

(c) *G is a generalized dihedral group of order 2^5 with abelian subgroup of type $(4,4)$. The group G is special,* $\mathrm{d}(G) = 3$.

PROOF. Suppose that G is neither an $\mathcal{A}_1$- nor a $D^\times$-group. All $\mathcal{A}_1$-subgroups of G are $\cong \mathrm{D}_8$ (Remark 1.1) so, by Lemma J(e), $G = C \cdot A$ is a generalized dihedral group; here $|C| = 2$ and A is abelian of exponent > 2 and all elements of the set $G - A$ are involutions inverting A. Since G is not dihedral, $\mathrm{d}(A) > 1$. Let $A_2 \leq A$ be of type $(4,4)$; then the nonabelian subgroup $B = C \cdot A_2 \leq G$ is not a $D^\times$-group so $B = G$, $A_2 = A$, and G is as stated in (c). Thus, A has no proper subgroup of type $(4,4)$. Thus, assuming that all invariants of A are > 2, we conclude that A is abelian of type $(4,4)$. Assume that A is not of type $(4,4)$. Then $A = L \times A_0$, where $|L| = 2$, $|A_0| > 2$. In that case, $G = L \times G_0$, where $G_0 = C \cdot A_0 \in \Gamma_1$; then G_0 is a $D^\times$-group, by the above and hypothesis, so G is also $D^\times$-group. We have $\mathrm{Z}(G) = \Omega_1(A) \leq G'$ (indeed, if $K < A$ is of order 2, then $K < C \cdot U$, where $\mathrm{C}_4 \cong U < A$ and $C \cdot U \cong \mathrm{D}_8$ so $K = (C \cdot U)' < G'$). By Lemma J(i), $|G : G'| > 4$ so $\mathrm{Z}(G) = G'$ (compare orders!). It follows from $\Omega_1(G) = G$ that $G' = \Phi(G)$, so G is special and $\mathrm{d}(G) = 3$. □

LEMMA 2.6. *Suppose that all nonabelian maximal subgroups of a nonabelian 2-group G of order 2^m are $M_3^\times$-groups. Then one of the following holds:*

(a) *G is minimal nonabelian.*
(b) *G is of maximal class and order 16.*
(c) *$G = M * C$ is the central product, where M is nonabelian of order 8 and C is cyclic of order 4, $m = 4$.*
(d) *G is generalized dihedral, $m = 5$, with abelian subgroup A of type $(4,4)$ (as in Proposition 2.5(c)).*
(e) *G is an $M_3^\times$-group.*

PROOF. Groups (a-e) satisfy the hypothesis. Since the lemma is true for $m \leq 4$, we assume that $m > 4$ and G is neither minimal nonabelian nor of maximal class.

Let $M < G$ be an $\mathcal{A}_1$-subgroup; then $|M| = 8$ (Remark 1.1). In that case, $M < H \in \Gamma_1$, where $H = M \times E$ and $\exp(E) = 2$ since $m > 4$. Set $D = \langle H' \mid H \in \Gamma_1 \rangle$. Then $D \leq G' \cap \Omega_1(\mathrm{Z}(G)) (\leq \Phi(G))$ so all maximal subgroups of G/D are abelian. Set $\bar{G} = G/D$. By Lemma J(n), $\Omega_1(\bar{G}) = \bar{G}$. Thus, either $\exp(\bar{G}) = 2$ or G is an $\mathcal{A}_1$-group so $\cong \mathrm{D}_8$ (Lemma J(d)).

Assume that $|D| = 2$; then $\exp(\bar{G}) = 2$ since $m > 4$, so $G' = D$ and all $\mathcal{A}_1$-subgroups of G are normal. Let $M < G$ be an $\mathcal{A}_1$-subgroup. Then $G = M * \mathrm{C}_G(M)$ (Lemma J(b)). If $C \leq \mathrm{C}_G(M)$ is cyclic of order 4, then $M * C$ is not an $M^\times$-group so $G = M * C$. Since $m > 4$, we get $M \cap C = \{1\}$ so $G = M \times C$. Then, by Remark 1.2, G has an $\mathcal{A}_1$-subgroup K of order 2^4 and $K \in \Gamma_1$ is not an $M^\times$-group, a contradiction. Thus, $\exp(\mathrm{C}_G(M)) = 2$ so

$C_G(M) = Z(G)$. If $Z(G) = Z(M) \times E$, then $G = M \times E$ is an $M^\times$-group. In what follows we assume that $|D| > 2$.

By the above, if $U < G$ is nonabelian of order 2^n, then $d(U) = n - 1$.

Suppose that $\exp(\bar{G}) = 2$. Let $M < G$ be minimal nonabelian; then there is $H = M \times E \in \Gamma_1$, where $\exp(E) = 2$. Since $|D| > 2$, there is an $\mathcal{A}_1$-subgroup $M_1 < G$ such that $M_1' \neq M'$. In view of Theorem 2.4 and Proposition 2.5, one may assume from the start that $M \cong Q_8$. Then $M \cap M_1 = \{1\}$ so $|\langle M, M_1\rangle| \geq |MM_1| = 2^6$. Set $U = \langle M, M_1\rangle$; then $d(U) \leq d(M) + d(M_1) = 4 < 6 - 1$ so $U = G$. We have $[M, M_1] > \{1\}$ (otherwise, $U = M \times M_1$ contains an $\mathcal{A}_1$-subgroup of order 2^4, by Remark 1.2). Therefore, one of subgroups M, M_1 is not normal in U. Let M is not normal in U. Then some cyclic subgroup $C_1 < M_1$ does not normalize some cyclic subgroup $C < M$ (of order 4). Since $U_1 = \langle C, C_1\rangle$ of order $\geq 2^4$ is generated by two elements and $2 < 4 - 1$, we get $U_1 = G$. It follows that G is minimal nonabelian (Lemma J(k)), a contradiction. Now let M_1 is not normal in U. Then some subgroup $Z < M$ of order 4 does not normalize some cyclic subgroup $Z_1 < M_1$. Since $V = \langle Z, Z_1\rangle$ of order ≥ 16 is two-generator, we get $V = G$ so G is an $\mathcal{A}_1$-subgroup, a contradiction.

Now we let $\bar{G} \cong D_8$. Since $D < G'$, we get $|G : G'| = |\bar{G} : \bar{G}'| = 4$ so G is of maximal class (Lemma J(i), a contradiction since $|Z(G)| \geq |D| > 2$. ∎

Remark 2.7. Suppose that a nonabelian p-group G is neither minimal nonabelian nor of maximal class and all nonabelian members of the set Γ_1 are of maximal class. Since G has a subgroup A with center of order $> p$, A is abelian. By Lemma J(f), the set Γ_1 has exactly $p + 1$ abelian members. In that case, $|G'| = p$ (Lemma J(h)) so $cl(G) = 2$ and $G = MZ(G)$ is of order p^4, where M is nonabelian of order p^3.

For $p = 2$, we get the following stronger result.

Lemma 2.8. *Suppose that all nonabelian maximal subgroups of a nonabelian 2-group G are $M^\times$-groups. Then one of the following holds:*

(a) *G is minimal nonabelian.*
(b) *The central product $G = M * C$ is of order 16, M is nonabelian of order 8 and C is cyclic of order 4.*
(c) *G is generalized dihedral of order 2^5 with abelian subgroup A of type $(4, 4)$.*
(d) *G is an $M^\times$-group.*

Proof. Groups (a-d) satisfy the hypothesis. All nonabelian members of the set Γ_1 are $\mathcal{Z}^\times$-groups. By Lemma 1.9, either the set Γ_1 has an abelian member or else all its members are $M_3^\times$-groups. In the second case, however, the set Γ_1 also has an abelian member, by Lemma 2.6. Thus, in any case, there is abelian $A \in \Gamma_1$. Assume that G is not an $\mathcal{A}_1$-group. Take a nonabelian $H = M \times E \in \Gamma_1$, where M is of maximal class and $\exp(E) \leq 2$. Set $|G| = 2^m$.

Suppose that $E = \{1\}$ and G is not of maximal class. Then, by Lemma 1.13, $G = H\mathrm{Z}(G)$, where $\mathrm{Z}(G)$ is of order 4. If $m = 4$, then G is as in (b) or (d). Let $m > 4$. If $F < H$ is nonabelian maximal, then $F\mathrm{Z}(G)$ is an $M^{\times}$-group so $\mathrm{Z}(G)$ is noncyclic, and we conclude that H is a direct factor of G so G is an $M^{\times}$-group. In what follows we assume that $E > \{1\}$ for every choice of nonabelian $H \in \Gamma_1$; then $m > 4$.

In view of Lemma 2.6, one may assume that $H(= M \times E)$ is chosen so that $|M| \geq 16$. Obviously, H has only one abelian maximal subgroup, say A_1, and $E < \mathrm{Z}(H) < A_1$. It follows that $A \cap H = A_1$ so $\mathrm{C}_G(E) \geq HA = G$, and we get $E < \mathrm{Z}(G)$ ($<$ since $\mathrm{Z}(M) < \mathrm{Z}(G)$ and $\mathrm{Z}(M) \not\leq E$). If $E \not\leq \Phi(G)$, then $G = X \times G_0$, where $X \leq E$ is of order 2 and a nonabelian $G_0 \in \Gamma_1$. However, G_0 is an $M^{\times}$-group so is G. Next we assume that $E < \Phi(G)$.

Suppose that $\bar{G} = G/E$ is not of maximal class. Since $M \cong \bar{M} = \bar{H} < \bar{G}$, we get $\exp(\bar{G}) = \exp(\bar{M}) = \exp(M) \geq 8$. By Remark 1.3, we get $\bar{G} = \bar{H} \times \bar{C} = \bar{M} \times \bar{C}$, where $|\bar{C}| = 2$. Also, $C \lhd G$ is abelian and $C \cap H = E \cap H = \{1\}$ so $G = M \cdot C$, a semidirect product with kernel C. As in part (i) of the proof of Lemma 2.3, we prove that $G = M \times C$ so G is an $M^{\times}$-group.

Next we assume that $\bar{G}$ is of maximal class. Then $\mathrm{d}(G) = 2$ since $E < \Phi(G)$, and hence, by Lemma J(c), we get $\mathrm{d}(F) \leq 3$ for all $F \in \Gamma_1$ so $|E| = 2$. Since $E \not\leq G'$ (otherwise, by Lemma J(i), G is of maximal class), we get $E \cap G' = \{1\}$ and so G/G' is abelian of type $(4, 2)$ since $\mathrm{d}(G) = 2$ and $4 < |G/G'| \leq 8$. Let $U/G', V/G' < G/G'$ be distinct cyclic of order 4. Then U, V are abelian since $\exp(X/X') = 2$ for every $M^{\times}$-group X. We have $U \cap V = \mathrm{Z}(G)$ so $|G'| = 2$ (Lemma J(h)) so G is an $\mathcal{A}_1$-group (Lemma J(k)), a final contradiction. ∎

Proof of Theorem A. Set $|G| = 2^m$. As above, we may assume that $m > 4$ and G is not an $\mathcal{A}_1$-group.

(A) Suppose that the set Γ_1 has no abelian member. Take $H = M \times E \in \Gamma_1$, where M is a $\mathcal{Z}$-group and $\exp(E) \leq 2$. Then, by Lemma 1.9, M is extraspecial. Write $D = \langle F' \mid F \in \Gamma_1 \rangle$; then $D \leq G' \cap \Omega_1(\mathrm{Z}(G))$ and all maximal subgroups of $\bar{G} = G/D$ are elementary abelian so $\exp(\bar{G}) = 2$.

(i) Suppose that $|D| = 2$ so $D = G' = \Phi(G)$. Then, by Lemma 1.5, G is not extraspecial so that $|\mathrm{Z}(G)| > 2$. If $\mathrm{Z}(G)$ is noncyclic, then $G = G_0 \times L$, where $L < \mathrm{Z}(G)$ is of order 2 and $L \not\leq D$. However, $G_0 \in \Gamma_1$ is a $\mathcal{Z}^{\times}$-group so is G. Now assume that $\mathrm{Z}(G)$ is cyclic; then $\mathrm{Z}(G) \cong \mathrm{C}_4$. In that case, all members of the set Γ_1, containing $\mathrm{Z}(G)$, must be abelian, contrary to the assumption.

(ii) Now suppose that $|D| > 2$. Then there are nonabelian $F, H \in \Gamma_1$ such that $F' \neq H'$. In that case, $\exp(F/F') = 2 = \exp(H/H')$. Let $H = M \times E$ be as above; then $F' \not\leq M$ so $MF'/F' \cong M$. The intersection $(MF'/F') \cap (F/F')$ is an abelian maximal subgroup of the extraspecial group MF'/F' so $|M| = |MF'/F'| = 8$ (Lemma 1.4). Since a nonabelian $H \in \Gamma_1$ is arbitrary, G

satisfies the hypothesis of Lemma 2.6 so there is an abelian $A \in \Gamma_1$, contrary to the assumption.

(B) Now let $A \in \Gamma_1$ be abelian. Let a nonabelian $H = M \times E$ be as above. Then $M \cap A$ is an abelian maximal subgroup of M so, by Lemma 1.12(a), M is of maximal class, and the result follows from Lemma 2.8. □

3. Proof of Theorem B

In this section $p > 2$. We begin with the following

Lemma 3.1. *Suppose that $p > 2$ and all nonabelian maximal subgroups of a nonabelian p-group G are $M_3^\times$-groups. Then either G is an $M_3^\times$-group or one of the following holds:*

(a) *G is minimal nonabelian.*
(b) *G is of maximal class and order p^4.*
(c) *$G = M * C = N * C$ is of order p^4, where M is nonabelian of order p^3 and exponent p, $N \cong \mathrm{M}_{p^3}$ and C is cyclic of order p^2.*
(d) *G is extraspecial of order p^5 and exponent p.*
(e) *G is special of order p^5, $d(G) = 3$.*
(f) *G is special of order p^6 and exponent p, $d(G) = 3$.*
(g) *G is of order p^5 without abelian subgroup of index p, $|G'| = p^3$, $\mathrm{Z}(G) < G'$ is abelian of type (p, p). If $R < \mathrm{Z}(G)$ is of order p, then G/R is of maximal class.*

Proof. Groups (a-d), (f) and also groups of exponent p from parts (e) and (g) satisfy the hypothesis (if the group of (e) is of exponent p^2, it may be an $\mathcal{A}_2$-group [BJ2, §5] and so does not satisfy the hypothesis). Set $|G| = p^m$. One may assume that G is not an $\mathcal{A}_1$-group so $m > 3$. In view of Lemma J(a), one may also assume that $m > 4$. All proper nonabelian subgroups of G are $M_3^\times$-groups (Remark 1.1).

Let $M < G$ be an $\mathcal{A}_1$-subgroup and let $M < H \in \Gamma_1$. Then $H \leq M * C$, where $C = \mathrm{C}_G(M)$. Suppose that $M * C = G$. If $U \leq C$ is cyclic of order p^2, then $M * U$ is not an $M^\times$-group. By Remark 1.2, $M \cap C = \Omega_1(C)$ so $G = M * C$, a contradiction since $m > 4$. Now let $\exp(C) = p$. Since $m > 4$, then $C \not\leq M$ (Lemma J(a)).

Suppose that $G = M * C$. By modular law and Remark 1.1, all maximal subgroups of C are elementary abelian so C is either elementary abelian or nonabelian of order p^3 and exponent p. If C is elementary abelian, then $\mathrm{Z}(G) = C = \mathrm{Z}(M) \times E$, and then $G = M \times E$ is an $M_3^\times$-group. If C is nonabelian, then $G = M * C$ is extraspecial of order p^5 and exponent p (Lemma 1.5). Next we assume that $M * C < G$; then $M * C \in \Gamma_1$ is an $M_3^\times$-group.

Set $D = \langle H' \mid H \in \Gamma_1 \rangle$; then $D \leq G' \cap \mathrm{Z}(G) \leq \Phi(G)$. If $M < G$ is minimal nonabelian and $M < H \in \Gamma_1$, then $M' = H' \triangleleft G$ and H/H' is elementary abelian. It follows that all maximal subgroups of $\bar{G} = G/D$ are abelian and $\Omega_1(\bar{G}) = \bar{G}$ (Lemma J(n)) so $\bar{G}$ is either elementary abelian or minimal nonabelian of order p^3 and exponent p since $p > 2$ (Lemma J(d)). By Lemma J(g), $|D| \le |G'| \le p^3$.

(i) Suppose that $|D| = p$; then $\bar{G}$ is elementary abelian since $m > 4$ so $D = G'$ and all $\mathcal{A}_1$-subgroups are normal in G. Let $M < G$ be minimal nonabelian. Then, by Lemma J(b), $G = M\mathrm{C}_G(M)$ and $\exp(\mathrm{C}_G(M)) = p$ (Remark 1.2). In that case, as we have proved, G is either $M_3^{\times}$-group or extraspecial of order p^5 and exponent p.

(ii) Now let $|D| > p$. Then there are two distinct $F, H \in \Gamma_1$ such that $H' \ne F'$. The set Γ_1 has at most one abelian member since $|G'| \ge |D| > p$ (Lemma J(h)). In that case, H/H' and F/F' are distinct elementary abelian so $\Omega_1(\bar{G}) = \Omega_1(\bar{F}\bar{H}) = \bar{F}\bar{H} = \bar{G}$. Since $p > 2$ and $\mathrm{cl}(\bar{G}) \le 2$, we get $\exp(\bar{G}) = p$. It follows that if $\bar{G}$ is minimal nonabelian, then $|\bar{G}| = p^3$ ((Lemma J(d)).

(ii1) Assume that $\bar{G}$ is an $\mathcal{A}_1$-group of order p^3 and exponent p; then $\mathrm{d}(G) = \mathrm{d}(\bar{G}) = 2$. Since $|G' : D| = p$, we get $|D| = p^2$ and $|G'| = p^3$ so $|G| = |D||\bar{G}| = p^5$. Let F and H be such as in the previous paragraph. Then $F = M \times H' = M \times M_1'$ and $H = M_1 \times F' = M_1 \times M'$, where M and M_1 are nonabelian of order p^3 (note that $F'H' \le \Phi(G) \le F \cap H$). Since $F/H' < G/H'$ is nonabelian of order p^3 and $\mathrm{d}(G/H') = 2$, it follows from Lemma J(a) that G/H' is of maximal class. Similarly, G/F' is of maximal class. If G has an abelian subgroup of index p, then $p^5 = |G| = p|G'||\mathrm{Z}(G)| = p^6$ (Lemma J(h)), a contradiction. Thus, all members of the set Γ_1 are nonabelian and G is from part (g). It is easy to check that if, in addition, $\exp(G) = p$, then indeed G satisfies the hypothesis, by Lemma J(d,a)).

(ii2) Now let $\bar{G}$ be elementary abelian; then $G' = D = \Phi(G)$ and $\mathrm{cl}(G) = 2$.

Assume that exp(Z(G)) > p and let $C \le \mathrm{Z}(G)$ by cyclic of order p^2. Then all members of the set Γ_1 containing C, are abelian so $|G'| = p < |D|$ (Lemma J(h)), a contradiction.

Thus, $\exp(\mathrm{Z}(G)) = p$. As above, $\mathrm{Z}(G) \le \Phi(G)$ (otherwise, G is an $M_3^{\times}$-group). In that case, $D \le \mathrm{Z}(G) \le \Phi(G) \le D$ so G is special. If $M < G$ is minimal nonabelian, then $M\Phi(G)/\Phi(G) = MD/D \cong M/(M \cap D) \cong \mathrm{E}_{p^2}$ so $\mathrm{d}(G) > 2$.

Suppose that $\mathrm{d}(G) > 3$. Then there exist distinct $\bar{F}, \bar{H} > \bar{M}$, where $F, H \in \Gamma_1$. Since M is a direct factor in F and H (Remark 1.1), we get $\mathrm{N}_G(M) \ge FH = G$ so $M \lhd G$ whence all $\mathcal{A}_1$-subgroups are normal in G. We have $G = M\mathrm{C}_G(M)$ since $M\mathrm{C}_G(M) \ge FH = G$. Assume that $\mathrm{C}_G(M)$ has an $\mathcal{A}_1$-subgroup N and let $M \cap N = \{1\}$. It follows from Remark 1.2 that $\exp(M) = p = \exp(N)$ so $M \cong N$. Let $T < M \times N$ be the diagonal subgroup; then $T \cong M$ is an $\mathcal{A}_1$-subgroup so $T \lhd G$. Since $T \cap M = \{1\} = T \cap N$, we get $\mathrm{C}_{MN}(T) \ge MN$, a contradiction since T is nonabelian. Now let $M \cap N > \{1\}$; then $M \cap N = \mathrm{Z}(M) = \mathrm{Z}(N)$. In that case, $M * N$ is extraspecial so it is not a subgroup of any $M_3^\times$-group, and we conclude that $G = M * N$. Then $|G'| = p < p^2 \leq |D|$, a contradiction. Thus, N does not exist so $\mathrm{C}_G(M)$ is elementary abelian whence coincides with $\mathrm{Z}(G)$. Since $G = M\mathrm{C}_G(M)$, we get $|G'| = p < |D|$, a contradiction.

Thus, $\mathrm{d}(G) = 3$. In that case, $|G| = |G'||G/G'| \leq p^6$. Suppose that $|G'| = p^3$. Then $|G| = p^6$ and $G' = D = F' \times H' \times L'$, where F, H, L are $\mathcal{A}_1$-subgroups of G. Then $\exp(G/F'H') = \exp(G/H'L') = \exp(G/L'F') = p$ so, since $F'H' \cap H'L' \cap L'F' = \{1\}$, we conclude that $\exp(G) = p$.

Now let G be (special) of order p^5 or p^6, $\exp(G) = p$, $|G'| = p^2$ or p^3, respectively, and $\mathrm{d}(G) = 3$. If $M < G$ is an $\mathcal{A}_1$-subgroup (of order p^3), then the $M_3^\times$-group $MG' = M \times E$ (here $G' = M' \times E$) is the unique member of the set Γ_1 containing M. It follows that G satisfies the hypothesis. □

Proof of Theorem B. Set $|G| = p^m$. As above, assume that G is not an $\mathcal{A}_1$-group and $m > 4$. By Lemma 1.5, if G is extraspecial, then $\exp(G) = p$ and all such G satisfy the hypothesis. Next we assume that G is not extraspecial. Since $m > 4$ and $p > 2$, G is not of maximal class.

(A) Let the set Γ_1 have no abelian member. Then, by Lemma 1.9, each nonabelian member $H \in \Gamma_1$ is of the form $E_1 \times E_2$, where E_1 is extraspecial and E_2 is elementary abelian so $|K'| \leq p$ for all $K \in \Gamma_1$, and we get $|G'| \leq p^3$ (Lemma J(g)). Put

$$D = \langle H' \mid H \in \Gamma_1 \rangle (\leq G' \cap \Omega_1(\mathrm{Z}(G))).$$

As above in similar situation, $\bar{G} = G/D$ is either elementary abelian or nonabelian of order p^3 and exponent p.

(i) Suppose that $|D| = p$; then $\bar{G}$ is elementary abelian since $m > 4$, and we conclude that $D = G'$. If $\mathrm{Z}(G) = G'$, then G is extraspecial (Lemma J(m), and so $\exp(G) = p$. Now assume that $\mathrm{Z}(G) > G'$. If $\mathrm{Z}(G)$ contains a cyclic subgroup of order p^2, then all members of the set Γ_1, containing $\mathrm{Z}(G)$, are abelian, contrary to assumption. Thus, $\exp(\mathrm{Z}(G)) = p$. If $L < \mathrm{Z}(G)$ is of order p and $L \neq G'(= \Phi(G))$, then $G = L \times G_0$; then G is an $\mathcal{Z}^\times$-group since G_0 is.

(ii) Suppose that $|D| > p$. Then there are nonabelian $F, H \in \Gamma_1$ such that $F' \neq H'$ and F/F' is elementary abelian maximal subgroup of G/F'. Let $H = M \times E$, where M is extraspecial and E is elementary abelian; then $F' \not\leq M$ so $MF'/F' \cong M$. The intersection $(MF'/F') \cap (F/F')$ is an abelian maximal subgroup of the extraspecial group MF'/F' so $|M| = |MF'/F'| = p^3$ (Lemma 1.13). Since a nonabelian $H \in \Gamma_1$ is arbitrary, G satisfies the hypothesis of Lemma 3.1, and we are done.

(B) Now suppose that there is abelian $F \in \Gamma_1$. Let a nonabelian $H = M \times E \in \Gamma_1$ be as above. Then $M \cap F$ is an abelian maximal subgroup of M so, by Lemma 1.12, $|M| = p^3$. Thus, all nonabelian members of the set Γ_1 are $M_3^\times$-groups so result follows from Lemma 3.1. □

4. Proof of Theorem C

In this section we classify the nonabelian 2-groups, all of whose nonabelian maximal subgroups are $(\mathcal{Z} * \mathcal{C})$-groups.

The proof of the following lemma is straightforward (see also [BJ1, Appendix 16]).

Lemma 4.1. *Suppose that $m > 1$ and $G = Q * C$, where*

$$Q = \langle a, b \mid a^4 = 1, a^2 = b^2, a^b = a^{-1}\rangle \cong \mathrm{Q}_8$$

and $C = \langle c_0 \rangle \cong \mathrm{C}_{2^m}$, $Q \cap C = \mathrm{Z}(Q) = \Omega_1(C)$. Write $d = ab$, $c = c_0^{2^{m-2}}$. Then

(a) *$\Omega_1(G) = Q\Omega_2(C)$, G has exactly seven involutions ($ac, ac^3, bc, bc^3, dc, dc^3, a^2$) so exactly four cyclic subgroups of order 4.*

(b) *G has exactly four proper nonabelian subgroups of order 8, namely Q, $D_1 = \langle a, bc\rangle \cong \mathrm{D}_8$, $D_2 = \langle d, bc\rangle \cong \mathrm{D}_8$, $D_3 = \langle b, dc\rangle \cong \mathrm{D}_8$. It follows that Q is characteristic in G and $G = \mathrm{D}_i * C$ ($i = 1, 2, 3$).*

Lemma 4.2 ([BJ1, Appendix 16]). *Suppose that $n > 3$ and $G = Q * C$, where*

$$Q = \langle a, b \mid a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, a^b = a^{-1}\rangle \cong \mathrm{Q}_{2^n},\ C = \langle c\rangle \cong \mathrm{C}_4.\ |G| = 2^{n+1}$$

Then $\Omega_1(G) = G$ and the set Γ_1 contains exactly four members of maximal class, namely Q, $D = \langle a, ba\rangle \cong \mathrm{D}_{2^n}$, $S_1 = \langle ac, abc\rangle \cong \mathrm{SD}_{2^n}$, $S_2 = \langle ac, bc\rangle \cong \mathrm{SD}_{2^n}$.

Proof. Since $(bc)^2 = b^2c^2 = b^2b^2 = 1$, we get $o(bc) = 2$. It follows from $a^{bc} = a^b = a^{-1}$ that $D = \langle a, bc\rangle \cong \mathrm{D}_{2^n}$. Next,

$$(abc)^2 = ababc^2 = ab^2a^{-1}b^2 = 1,\ o(ac) = 2^{n-1},$$

$$(ac)^{abc} = a^bc = a^{-1}c^2c^{-1} = a^{-1+2^{n-2}}c^{-1+2^{n-2}} = (ac)^{-1+2^{n-2}},$$

so that $S_1 = \langle ac, abc\rangle \cong \mathrm{SD}_{2^n}$. It follows from $o(bc) = 2$ and

$$(ac)^{bc} = (ac)^{abc} = (ac)^{-1+2^{n-2}}$$

that $S_2 = \langle ac, bc\rangle \cong \mathrm{SD}_{2^n}$. We have $Q, D, S_1, S_2 \in \Gamma_1$ and these subgroups are all members of maximal class in the set Γ_1 (Lemma J(f)). Since, by Lemma J(j), the set $G - D$ contains an involution x, we get $\Omega_1(G) \geq \langle x, D\rangle = G$. □

Lemma 4.3 ([BJ1, Appendix 16]). *Suppose that $n > 3$, $m > 2$ and $G = Q * C$, where $|G| = 2^{m+n-1}$ and*

$$Q = \langle a, b \mid a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, a^b = a^{-1}\rangle \cong \mathrm{Q}_{2^n},\ C = \langle c\rangle \cong \mathrm{C}_{2^m}.$$

Then

(a) *$\Omega_1(G) = Q * \Omega_2(C)$ is of order 2^{n+1} and contains all subgroups of G of maximal class.*

(b) *G contains exactly one subgroup, namely Q, that is* $\cong \mathrm{Q}_{2^n}$*, exactly one subgroup* $D \cong \mathrm{D}_{2^n}$*, and exactly two subgroups, say* S_1 *and* S_2*, that are isomorphic to* SD_{2^n}*. If* $M < G$ *is of maximal class and order* 2^n*, then* $G = M * C$*. The intersections* $D \cap Q$ *and* $S_1 \cap S_2$ *are cyclic,* $S_1 \cap D \neq S_2 \cap D$ *are isomorphic to* $\mathrm{D}_{2^{n-1}}$*,* $S_1 \cap Q \neq S_2 \cap Q$ *are isomorphic to* $\mathrm{Q}_{2^{n-1}}$*. Next, G has no subgroup of maximal class and order* 2^{n+1}*.*

Proof. Since G/Q is cyclic, we get $\Omega_1(G) \leq Q * \Omega_2(C) \leq \Omega_1(G)$ (Lemma 4.2(a)). Let $T < Q$ be nonabelian of order 8. Then $T' = \Omega_1(Q') = \Omega_1(C)$. Since $\Omega_1(T * \Omega_2(C)) = T * \Omega_2(C)$ and every 2-group of maximal class, say U, is generated by its nonabelian subgroups of order 8, we get $U \leq \Omega_1(G)$. Next, by Lemma 4.2(b), $\Omega_1(G)$ contains exactly one subgroup $\cong \mathrm{D}_{2^n}$, exactly one subgroup $Q \cong \mathrm{Q}_{2^n}$, and exactly two subgroups $\cong \mathrm{SD}_{2^n}$. The last assertion is true since $\mathrm{cl}(G) = n - 1$. The rest of (b) follows from Lemma 4.2 applied to $\Omega_1(G)$. ∎

Lemma 4.4. *Suppose that a 2-group* $G = U * Z$*, where U is of maximal class,* $Z = \mathrm{Z}(G) = \langle c \rangle$ *is cyclic of order* $2^n > 2$*. Then*

(a) *All* $\mathcal{A}_1$*-subgroups of G are metacyclic and have orders* $\leq 2^{n+1}$*.*
(b) *The group G contains an* $\mathcal{A}_1$*-subgroup* $\cong \mathrm{M}_{2^{n+1}}$*.*
(c) *If* $M < G$ *is minimal nonabelian and* $M \not\leq U$*, then* $M \cap U \cong \mathrm{C}_4$ *and* $M/(M \cap U)$ *is cyclic.*
(d) *G has no subgroup* $\cong \mathrm{E}_8$*.*

Proof. To prove that G contains an $\mathcal{A}_1$-subgroup of order 2^{n+1}, one may assume that $|U| = 8$ and $n > 2$. Let $U = \langle a, R \rangle$, where $R < U$ is of order 4, $a \in U - R$, $b = ac$, $H = \langle b, R \rangle$. Then $R \cap \langle b \rangle = \Omega_1(Z)$ is of order 2, $o(b) = o(c) = 2^n$ so $|H| = 2^{n+1}$ and $H \cong \mathrm{M}_{2^{n+1}}$ since $\mathrm{cl}(H) \leq \mathrm{cl}(G) = 2$, $n > 2$ and H is nonabelian.

Let $H < G$ be an $\mathcal{A}_1$-subgroup such that $H \not\leq U$. To describe the structure of H, one may assume, in view of Lemma 4.3(b), that U is generalized quaternion. Then HU/U is cyclic as a subgroup of $G/U \cong Z/(Z \cap U)$ so $|H \cap U| > 2$ since H is nonabelian. Since $H \cap U$ is abelian, it is cyclic so H is metacyclic. Assume that $|H \cap U| > 4$. Then $\mho_1(H \cap U) = \Phi(H \cap U) \leq \Phi(H) = \mathrm{Z}(H)$ so $\mathrm{C}_G(\mho_1(H \cap U)) \geq H$ is nonabelian, a contradiction. Thus, $|H \cap U| = 4$. Since $|H/(H \cap U)| = |HU/U| \leq |G/U| = 2^{n-1}$ we get $|H| = |H \cap U||HU/U| \leq 4 \cdot 2^{n-1} = 2^{n+1}$.

Assume that G has a subgroup $E \cong \mathrm{E}_8$. As above, let U be a generalized quaternion group. Then $E < \Omega_1(G) = U * \Omega_2(Z)$ so one may assume that $|Z| = 4$. In that case, $E \cap U$ is of exponent 2 and order 4, a contradiction since U has no abelian subgroup of type $(2, 2)$. ∎

Thus, the property $(\mathcal{M} * \mathcal{C})$ is not inherited by nonabelian subgroups.

LEMMA 4.5. *Suppose that every nonabelian maximal subgroup of a* 2*-group* G*,* $|G| = 2^m > 2^3$*, is an* $(\mathcal{M} * \mathcal{C})$*-group. If* G *is neither* $\mathcal{A}_1$*- nor* $(\mathcal{M} * \mathcal{C})$*-group, then* $G = M \times D$*, where* M *is nonabelian of order* 8 *and* $|D| = 2$*.*

PROOF. In view of Remark 2.7, one may choose a nonabelian $M * Z = H \in \Gamma_1$ so that M is of maximal class and $\mathrm{Z}(H) = Z$ is cyclic of order > 2. Assume that $\mathrm{Z}(G)$ is noncyclic. Then $\mathrm{Z}(G)$ contains a subgroup L of order 2 such that $L \not\leq H$ so $G = H \times L$. Since $(M * \mho_1(Z)) \times L \in \Gamma_1$ is neither abelian nor $(\mathcal{M} * \mathcal{C})$-group, we get a contradiction. Thus, $\mathrm{Z}(G)$ is cyclic.

We claim that $\mathrm{Z}(G) = Z$. Indeed, by Lemma J(j), H contains a G-invariant abelian subgroup R of type $(2, 2)$. Then $A = \mathrm{C}_G(R) \in \Gamma_1$ is abelian since $\mathrm{Z}(A)$ is noncyclic. In that case, $\mathrm{C}_G(Z) \geq AH = G$ so $Z \leq \mathrm{Z}(G)$. If $Z < \mathrm{Z}(G)$, then $G = M\mathrm{Z}(G)$ is an $(\mathcal{M} * \mathcal{C})$-group, contrary to the hypothesis. Assume that $F \in \Gamma_1$ is of maximal class. Then $G = F * \mathrm{Z}(G)$ is an $(\mathcal{M} * \mathcal{C})$-group, a contradiction. Thus, $\mathrm{Z}(G) = \mathrm{Z}(H)$ for all nonabelian $H \in \Gamma_1$. As above, we write $\mathrm{Z}(G) = Z$. We have also proved that $Z \leq \Phi(G)$.

If $F = B * Z, K = L * Z \in \Gamma_1$ are nonabelian, then $|B| = |L|$. Write $\bar{G} = G/Z$. Then, for nonabelian $F, H \in \Gamma_1$, $\bar{F} \cong \bar{H}$ is either $\cong \mathrm{E}_4$ or dihedral. Thus, either $\bar{G}$ has at least two maximal subgroups $\cong \mathrm{E}_4$ (Lemma J(n)) or all nonabelian maximal subgroups of G are dihedral. In that case, $\Omega_1(\bar{G}) = \bar{G}$ (of order ≥ 8) is one of the following groups: (i) D_8, (ii) E_8, (iii) $\mathrm{D}_8 \times \mathrm{C}_2$, (iv) $\mathrm{D}_{2^n}, n > 3$ (Proposition 2.5).

(i) Suppose that $\bar{G} = \mathrm{D}_8$. We have $\mathrm{d}(G) = 2$ since $Z < \Phi(G)$ and, if $\bar{U} < \bar{G}$ is cyclic of order 4, then U is abelian. Two other members of the set Γ_1, say F and H, are nonabelian. Let $F = B * Z$ be as above. By Lemma 4.1, F contains exactly one subgroup $\cong \mathrm{Q}_8$ and exactly three subgroups $\cong \mathrm{D}_8$ so one may assume from the start that $B \cong \mathrm{Q}_8$; then $B \triangleleft G$. If G/B is noncyclic, then $B \leq \Phi(G)$ since $\mathrm{d}(G) = 2$ so $F = B * Z \leq \Phi(G)$, a contradiction. Thus, G/B is cyclic so $G = BZ_1$, where $Z_1 < G$ is cyclic. We get $G' < B$. Since G is not an $\mathcal{A}_1$-group, we get $G' \cong \mathrm{C}_4$ (Lemma J(k)). Thus, G/G' is abelian of type $(2^n, 2)$, where $n > 1$ since $m > 4$. In that case, G/G' contains two distinct cyclic subgroups Z_1/G' and Z_2/G' of index 2. Then the metacyclic subgroups $Z_1, Z_2 \in \Gamma_1$ must be abelian since all nonabelian members of the set Γ_1 are not metacyclic, a contradiction since the set Γ_1 has only one abelian member in view of $|G'| = 4 > 2$ (Lemma J(h)).

(ii) Suppose that $\bar{G} \cong \mathrm{E}_8$. Then $G' \leq Z = \mathrm{Z}(G)$ is cyclic and $\mathrm{cl}(G) = 2$. If $x, y \in G$, then $[x, y]^2 = [x, y^2] = 1$ so $|G'| = 2$ since G' is cyclic. If $F \in \Gamma_1$ is nonabelian, then $F = B * Z$, where B is nonabelian of order 8. Then $B' = G'$. By Lemma J(b), $G = B * \mathrm{C}_G(B)$. We have $|\mathrm{C}_G(B) : Z| = 2$ so $\mathrm{C}_G(B)$ is abelian. Then $\mathrm{C}_G(B) = \mathrm{Z}(G) = Z$, a contradiction.

(iii) Suppose that $\bar{G} \cong \bar{D} \times \bar{L}$, where $\bar{D} \cong \mathrm{D}_8$ and $|\bar{L}| = 2$. In that case, $\bar{G}$ has exactly three abelian maximal subgroups: $\bar{T}_1$ of type $(4, 2)$ and $\bar{T}_2$, $\bar{T}_3$

of type $(2,2,2)$. Then T_i, $i = 1,2,3$, are abelian since they are not $(\mathcal{M} * \mathcal{C})$-groups (indeed, if X is an $(\mathcal{M} * \mathcal{C})$-group, then $X/Z \ncong \bar{T}_i$, $i = 1,2,3$). In that case, $Z = \mathrm{Z}(G) = T_1 \cap T_2$ has index 4 in G, a contradiction since $|G : Z| = |\bar{G}| = 16$.

(iv) Suppose that $\bar{G} = G/Z \cong \mathrm{D}_{2^n}, n > 3$, and let $|Z| = 2^m$, $m > 1$. Then $\mathrm{d}(G) = 2$ since $Z < \Phi(G)$. If $T/Z < G/Z$ is cyclic of index 2, then $T \in \Gamma_1$ is abelian. Therefore, by Lemma J(h), $|G'| = \frac{1}{2}|G/Z| = 2^{n-1} \geq 8$ so T is the unique abelian member of the set Γ_1 (Lemma J(h)). If $F = A * Z \in \Gamma_1$ is nonabelian, then one may assume that $A \triangleleft G$ (Lemma 4.3). Since the set Γ_1 has exactly three members and one of them is abelian, the quotient group G/A must be cyclic, and we conclude that $G/A \cong \mathrm{C}_{2^m}$ since $F/A \cong \mathrm{C}_{2^{m-1}}$ is maximal in G/A. But $G' < A$ so G' is cyclic, by Burnside (recall that $|G'| \geq 8$). Since G is not of maximal class, we get $|G : G'| \geq 8$ (Lemma J(i)). We have $|G| = |Z||G/Z| = 2^{m+n}$ so $|G/G'| = 2^{m+1}$ since $|G'| = 2^{n-1}$. Since $G/A \cong \mathrm{C}_{2^m}$, it follows that G/G' has a cyclic subgroup of index 2. Let $U/G', V/G' < G/G'$ be distinct cyclic subgroups of index 2. Since U, V being metacyclic, are not $(\mathcal{M} * \mathcal{C})$-groups, a contradiction: G has only one abelian maximal subgroup. □

Proof of Theorem C. Assume that G is not minimal nonabelian.

Let a nonabelian $H \in \Gamma_1$ be not of maximal class (if such H does not exist, we are done, by Remark 2.7). Then H has a G-invariant four-subgroup R. In that case, $A = \mathrm{C}_G(R) \in \Gamma_1$ since $R \not\leq \mathrm{Z}(H)$, and A is abelian since $\mathrm{Z}(A)$ is noncyclic. Let $F = B * Z \in \Gamma_1$ be a $(\mathcal{Z} * \mathcal{C})$-subgroup. Then $B \cap A$ is an abelian maximal subgroup of B so $|B : B'| = 2|\mathrm{Z}(B)| = 4$ (Lemma J(h)) whence B is of maximal class, by Lemma J(i). Thus, all nonabelian members of the set Γ_1 are $(\mathcal{M} * \mathcal{C})$-groups, and the theorem follows from Lemma 4.5. □

Let a 2-group $G = M * C$ be an $\mathcal{M}_3 * \mathcal{C}$-group, where M is nonabelian of order 8 and C is cyclic of order $2^n > 2^2$; then $|G| = 2^{n+2}$. By Lemma 4.4(b), there is in G an $\mathcal{A}_1$-subgroup $H \cong \mathrm{M}_{2^{n+1}}$. Then $H \in \Gamma_1$ is not an $(\mathcal{M}_3 * \mathcal{C})$-group.

5. Proof of Theorem D

In this section we classify the nonabelian p-groups, $p > 2$, all of whose nonabelian maximal subgroups are $(\mathcal{Z} * \mathcal{C})$-groups.

A p-group $G = A * Z$, where A is nonabelian of order p^3 and $Z = \mathrm{Z}(G)$ is cyclic, is said to be $(\mathcal{M}_3 * \mathcal{C})$-group.

Lemma 5.1. *If $p > 2$ and G is an $(\mathcal{M}_3 * \mathcal{C})$-group and $|\mathrm{Z}(G)| > p$, then $G = \Omega_1(G) * \mathrm{Z}(G)$, where $\Omega_1(G)$ is nonabelian of order p^3 and exponent p.*

Proof. Since $\mathrm{cl}(G) = 2$, G is regular so we get

$$|\Omega_1(G)| = |G/\mho_1(G)| = |G/\mho_1(\mathrm{Z}(G))| = p^3,\ \exp(\Omega_1(G)| = p.$$

By the product formula, $G = \Omega_1(G)\mathrm{Z}(G)$ so $\Omega_1(G)$ is nonabelian. □

Lemma 5.2. *Suppose that $p > 2$ and all nonabelian maximal subgroups of a nonabelian p-group G, $p > 2$, are $(\mathcal{M}_3 * \mathcal{C})$-groups. Then G is either minimal nonabelian or of order p^4.*

Proof. Set $|G| = p^m$. As above, assume that G is not an $\mathcal{A}_1$-group and $m > 4$.

Assume that $G = U * Z$ is an $(\mathcal{M}_3 * \mathcal{C})$-group, where $U = \Omega_1(G)$ is nonabelian of order p^3 and exponent p (Lemma 5.1) and $Z = \mathrm{Z}(G)$ is cyclic of order $> p^2$. Let $F \in \Gamma_1$. If $U \not\leq F$, then $|\Omega_1(F)| = p^2$ so F is metacyclic so it is not an $(\mathcal{M}_3 * \mathcal{C})$-group; then F is abelian. If $U \leq F$, then F is an $(\mathcal{M}_3 * \mathcal{C})$-group, by the modular law. Since $\mathrm{d}(G) = 3$, the set Γ_1 contains $|\Gamma_1| - 1 = p^2 + p$ abelian members, which is impossible. Thus, G is not an $(\mathcal{M}_3 * \mathcal{C})$-group.

Assume that G is of maximal class. In that case, there is $H \in \Gamma_1$ of maximal class [Bla]. Then H is not an $(\mathcal{M}_3 * \mathcal{C})$-group since $|H| > p^3$, a contradiction.

Let $H = M * Z \in \Gamma_1$, where M is nonabelian of order p^3 and exponent p and Z is cyclic of order $> p$ (Lemma 5.1). Then H has a G-invariant subgroup R of type (p, p) (Lemma J(j)). Since $R \not\leq \mathrm{Z}(H)$, we get $A = \mathrm{C}_G(R) \in \Gamma_1$ so A is abelian since $\mathrm{Z}(A)$ is noncyclic. Then $\mathrm{C}_G(Z) \geq AH = G$ so $Z \leq \mathrm{Z}(G)$.

Suppose that $Z < \mathrm{Z}(G)$; then $|\mathrm{Z}(G) : Z| = p$, by the product formula. If $\mathrm{Z}(G)$ is cyclic, then $G = M * \mathrm{Z}(G)$ is an $(\mathcal{M}_3 * \mathcal{C})$-group, a contradiction. Now assume that $\mathrm{Z}(G)$ is noncyclic. Then $\mathrm{Z}(G) = Z \times L$, where $|L| = p$. In that case, $G = H \times L = (M * Z) \times L$, and $(M * \mho_1(Z)) \times L \in \Gamma_1$ is not an $(\mathcal{M}_3 * \mathcal{C})$-group, a contradiction. Thus, $\mathrm{Z}(H) = Z$ for every choice of H. Since, in addition, $Z < A$ for every abelian $A \in \Gamma_1$, it follows that $\mathrm{Z}(G) = Z \leq \Phi(G)$.

Let distinct nonabelian $F, H \in \Gamma_1$ (Lemma J(n)), where H is as above and $F = M_1 * Z$, where $M_1 = \Omega_1(F)$ is nonabelian of order p^3 and exponent p (Lemma 5.1); then $M, M_1 \triangleleft G$. Since $Z \leq \Phi(G) < H$ and $M_1 Z = F \neq H$, it follows that $M_1 \neq M$. Since $M_1 \cap M = M_1 \cap H$, we get $M_1 \cap M \cong \mathrm{E}_{p^2}$ so MM_1 is of order p^4, by the product formula. Let $MM_1 \leq W \in \Gamma_1$; then $|\Omega_1(W)| \geq p^4$ so W is not an $(\mathcal{M}_3 * \mathcal{C})$-group, a contradiction. □

Proof of Theorem D. In view of Lemma 5.2, one may assume that $|G| = p^m > p^4$; we also assume that G is not an $\mathcal{A}_1$-group. Assume that there exist $H = B * Z$, where B is a $\mathcal{Z}$-group of order $> p^3$ and $Z = \mathrm{Z}(H)$ is cyclic. In that case, there is in H a G-invariant subgroup $R \cong \mathrm{E}_{p^2}$ (Lemma J(j)); then $R \not\leq Z = \mathrm{Z}(H)$ so $A = \mathrm{C}_G(R) \in \Gamma_1$ is abelian. In that case, $B \cap A$ is an abelian maximal subgroup of B; then $|B| = p^3$ (Lemma 1.12(b)), contrary to the assumption. Thus, all nonabelian members of the set Γ_1 are $(\mathcal{M}_3 * \mathcal{C})$-groups, and the result now follows from Lemma 5.2. □

6. Proof of Theorem E

If $p = 2$, then an $(\mathcal{M} * \mathcal{C})$-group $G = M * C$ is a $(\mathcal{Z} * \mathcal{C})$-group but this is not the case for $p > 2$ and $|M| > p^3$. In this section we consider the nonabelian p-groups, $p > 2$, all of whose nonabelian maximal subgroups are $(\mathcal{M} * \mathcal{C})$-groups.

Proof of Theorem E. In view of Lemma 5.2, one may assume that $\mathrm{cl}(X) > 2$ for some $X \in \Gamma_1$; then $|G| > p^4$.

Suppose that G is of maximal class. Let $\mathrm{E}_{p^2} \cong R \triangleleft G$; then $\mathrm{C}_G(R) \in \Gamma_1$ is abelian. Conversely, every p-group of maximal class with abelian subgroup of index p satisfies the hypothesis (this follows immediately from Fitting's Lemma). In what follows we assume that G is neither an $\mathcal{A}_1$-group nor of maximal class.

Now let $G = M * \mathrm{Z}(G)$ be an $(\mathcal{M} * \mathcal{C})$-group. Then, as in the previous paragraph, M has an abelian subgroup of index p. Assume that $|\mathrm{Z}(G)| = p^n, n > 2$, and $|M| > p^3$. Let S be a G-invariant subgroup of index p in $M'(= G')$. Then $G/S \cong (M/S) \times (\mathrm{Z}(G)/\Omega_1(\mathrm{Z}(S))$ so G/S contains a maximal subgroup U/S of order p^{n+1} which is an $\mathcal{A}_1$-group (Remark 1.2). Then $U \in \Gamma_1$ is not an $(\mathcal{M} * \mathcal{C})$-group, a contradiction. Thus, if $|M| > p^3$, then $|\mathrm{Z}(G)| \leq p^2$. Let $\mathrm{Z}(G) \cong \mathrm{C}_{p^2}$. Then every member of the set Γ_1, not containing $\mathrm{Z}(G)$, is of the same class as G so of maximal class. If $\mathrm{Z}(G) < H \in \Gamma_1$ and H is nonabelian, then $H = \mathrm{Z}(G) * (H \cap M)$ is an $(\mathcal{M} * \mathcal{C})$-group. If $|M| = p^3$ (then $|\mathrm{Z}(G)| > p^2$), then G does not satisfy the hypothesis (see the second paragraph of the proof of Lemma 5.2). In what follows we assume that G is not an $(\mathcal{M} * \mathcal{C})$-group.

Assume that $H \in \Gamma_1$ is of maximal class. Let $\mathrm{E}_{p^2} \cong R < H$ be G-invariant (R exists, by Lemma J(j)). Then $A = \mathrm{C}_G(R) \in \Gamma_1$ is abelian since the center of $(\mathcal{M} * \mathcal{C})$-group must be cyclic. In that case, either G is of maximal class or $|\mathrm{Z}(G)| = p^2$ (Lemma 1.13). In the last case, as easily seen, $\mathrm{Z}(G)$ is cyclic and $G = H\mathrm{Z}(G)$ is an $(\mathcal{M} * \mathcal{C})$-group, contrary to the assumption. Thus, the set Γ_1 has no member of maximal class.

Let $X = K * Z \in \Gamma_1$, where K is of maximal class and order $> p^3$ and $Z = \mathrm{Z}(X)$ is cyclic of order $> p$ (in view of Lemma 5.2 and the previous paragraph, such X exists); then $X' = K' \triangleleft G$ is noncyclic of order $\geq p^2$ so it contains a G-invariant subgroup $R \cong \mathrm{E}_{p^2}$ (Lemma J(j)). In that case, $A = \mathrm{C}_G(R) \in \Gamma_1$ is abelian. Since $Z < A$, we get $\mathrm{C}_G(Z) \geq AX = G$ so $Z \leq \mathrm{Z}(G)$. As in the proof of Lemma 5.2, $\mathrm{Z}(G) = Z$ is cyclic and $|Z| \geq p^2$.

Take a nonabelian $Y \in \Gamma_1$. By the previous paragraph, $\mathrm{Z}(Y) = Z$. Thus, $\mathrm{Z}(G) < \Phi(G)$. Since the set Γ_1 has an abelian member, we get $|G'| \leq p|K'|$ (Lemma J(h)).

Write $\bar{G} = G/Z$; then $|\bar{G}| \geq p^4$ and $\bar{G}$ is neither abelian nor $\mathcal{A}_1$-group (indeed, X/Z is nonabelian). In that case, all nonabelian maximal subgroups of $\bar{G}$ are of maximal class so, by Remark 2.7, $\bar{G}$ is either of maximal class or $\bar{G} = \bar{K}Z(\bar{G})$ is of order p^4 with $|Z(G)| = p^2$ (Remark 2.7). ∎

7. Problems

1. Classify the p-groups G, $p > 2$, all of whose $\mathcal{A}_1$-subgroups have the same order p^3. (For the case where $\exp(G) > p > 2$ and all $\mathcal{A}_1$-subgroups of G are of order p^3 and exponent p, Mann showed that then the Hughes subgroup of G is abelian and maximal in G; see item 115 in [B5, Research Problems and Themes I].)

2. Find the types of $\mathcal{A}_1$-subgroups in a group $G = M_1 \times \cdots \times M_n$ ($G = M_1 * \cdots * M_n$), where all M_i are 2-groups of maximal class.

3. Classify the 2-groups G, all of whose nonabelian maximal subgroups are either generalized dihedral or $M^{\times}$-groups or $(\mathcal{M} * \mathcal{C})$-groups.

4. Classify the nonabelian p-groups, $p > 2$, all of whose maximal subgroups are $M^{\times}$-groups.

5. Describe all $\mathcal{A}_1$-subgroups of a p-group $G = M \times C$ ($G = M * C$ with $M \cap C = \Omega_1(C)$), where M is minimal nonabelian and C is cyclic.

6. Does there exist a p-group all of whose maximal subgroups are of the form $A \times B$, where A and B are (i) of maximal class, (ii) extraspecial?

7. Classify the p-groups G such that, whenever $H \in \Gamma_1$, then $H \in \{M \times C, M * C\}$, where M is minimal nonabelian and C is cyclic.

8. Study the nonabelian p-groups all of whose nonabelian maximal subgroups have cyclic centers.

9. Classify the p-groups all of whose maximal subgroups (nonabelian maximal subgroups) are special.

10. Classify the p-groups all of whose maximal subgroups are nontrivial direct (central) products.

11. Classify the 2-groups with odd number of dihedral subgroups of order 8.

12. Classify the nonabelian 2-groups G such that, whenever $H \in \Gamma_1$ is nonabelian, then $H = MZ(H)$, where M is of maximal class.

13. Classify the 2-groups G containing an $\mathcal{A}_1$-subgroup M of order 16 such that $C_G(M) < M$.

14. Classify the p-groups G containing a nonabelian subgroup M of order p^3 such that (i) $|C_G(M)| = p^2$, (ii) $C_G(M)$ is cyclic.

15. Study the p-groups all of whose $\mathcal{A}_1$-subgroups are isomorphic.

16. Classify the 2-groups all of whose nonabelian subgroups have a section $\cong Q_8$ (compare with Lemma 2.1).

17. Study the p-groups all of whose $\mathcal{A}_1$-subgroups of minimal order are conjugate.

18. Study the p-groups G such that $|G : H^G| = p$ for all $\mathcal{A}_1$-subgroups $H < G$.

19. Study the p-groups all of whose $\mathcal{A}_1$-subgroups are metacyclic. (See [J2]. See also [BJ3] where the 2-groups all of whose $\mathcal{A}_1$-subgroups are isomorphic with M_{16}, are classified.)

20. Classify the 2-groups all of whose subgroups of index 4 are (i) $M^{\times}$-groups, (ii) Dedekindian.

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