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CYCLIC SUBGROUPS OF ORDER 4 IN FINITE 2-GROUPS

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Abstract. We determine completely the structure of finite 2-groups which possess exactly six cyclic subgroups of order 4. This is an exceptional case because in a finite 2-group is the number of cyclic subgroups of a given order 2^n ($n \geq 2$ fixed) divisible by 4 in most cases and this solves a part of a problem stated by Berkovich. In addition, we show that if in a finite 2-group G all cyclic subgroups of order 4 are conjugate, then G is cyclic or dihedral. This solves a problem stated by Berkovich.

1. Introduction and known results

For a finite 2-group G and a fixed integer $n \geq 1$ we denote with $c_n(G)$ the number of cyclic subgroups of order 2^n. The starting point are the following results of Y. Berkovich. Suppose that a finite 2-group G is neither cyclic nor of maximal class. Then $c_1(G) \equiv 3 \pmod 4$ and if $n \geq 2$, then $c_n(G)$ is even (Berkovich [2, Theorem 1.17]). If in addition G is nonabelian and $n \geq 3$, then $c_n(G) \equiv 0 \pmod 4$ unless G is an L_2-group or a U_2-group (Berkovich [1] and [2, Corollary 18.7]).

We shall use freely the above two results and we consider here only finite 2-groups with a standard notation. In addition, a 2-group G is called an L_2-group if $\Omega_1(G) \cong E_4$ is a four-subgroup and and $G/\Omega_1(G)$ is cyclic of order ≥ 4. We note that an L_2-group G is either abelian of type $(2, 2^m)$, $m \geq 3$, or

$$G \cong M_{2^{m+1}} = \langle a, b \mid a^{2^m} = b^2 = 1,\ m \geq 3,\ a^b = a^{1+2^{m-1}} \rangle.$$

A 2-group G is called a U_2-group (with respect to R) if G possesses a normal four-subgroup R such that G/R is a group of maximal class (i.e., G/R is dihedral D_{2^n}, generalized quaternion Q_{2^n} or semi-dihedral SD_{2^n}) and whenever

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T/R is a cyclic subgroup of index 2 in G/R, then $\Omega_1(T) = R$. It is easy to see that the four-subgroup R is uniquely determined. All U_2-groups are completely classified in Janko [3, section 6]. Finally, for a 2-group G, we define $\Omega_n^*(G) = \langle x \in G \mid o(x) = 2^n \rangle$.

Here we shall consider the exceptional cases, where in a 2-group G we have $c_2(G) \equiv 2 \pmod 4$. If $c_2(G) = 2$, then such 2-groups G are already known (see Janko [4, Proposition 1.4, Theorems 5.1 and 5.2]). If $c_2(G) = 6$, then such 2-groups G are determined only in the special case where $|\Omega_2^*(G)| = 2^4$. Such 2-groups G with $|G| > 2^4$ are determined by Janko [4, Theorem 2.1] when $|\Omega_2(G)| = 2^4$ (since in that case $\Omega_2(G) \cong Q_8 \times C_2$ or $\Omega_2(G) \cong C_4 \times C_4$) and by Janko [3, Theorem 4.1] when $|\Omega_2(G)| > 2^4$. In this paper we shall classify 2-groups G with $c_2(G) = 6$ and $|\Omega_2^*(G)| > 2^4$. First we show that we must have $|\Omega_2^*(G)| = 2^5$ and we get three possibilities for the structure of $\Omega_2^*(G)$ (Theorem 2.6). The corresponding 2-groups G are determined up to isomorphism in Theorem 2.7. The general case, where $c_2(G) \equiv 2 \pmod 4$ and $c_2(G) \geq 10$ is very difficult and is still open.

At the end we consider 2-groups G which possess only one conjugate class of cyclic subgroups of order 4 and we show that in that case G has only one cyclic subgroup of order 4 and therefore G is either cyclic or dihedral (Theorem 3.1).

For convenience we state another known result which is of special importance in the proof of Theorem 2.6.

Proposition 1.1. *(see [3, Proposition 1.2]) Let K be a 2-group of order $> 2^3$ possessing exactly two cyclic subgroups U_1, U_2 of order 4 and assume that neither of them is a characteristic subgroup of K. Then one of the following holds:*

(i) *$K \cong D_8 \times C_2$ (of order 2^4) with $\Phi(K) = U_1 \cap U_2 \cong C_2$;*
(ii) *K is a uniquely determined group of order 2^5 with $\Phi(K) = \langle U_1, U_2 \rangle \cong C_4 \times C_2$.*

2. New results for $c_2(G) = 6$

In what follows G will denote a 2-group with $c_2(G) = 6$ and $H = \Omega_2^*(G)$ is of order $> 2^4$. Since H has exactly six cyclic subgroups of order 4, H is neither cyclic nor a group of maximal class. It follows that H possesses a G-invariant four-subgroup W (see [6, Proposition 2.19]).

Lemma 2.1. *If a cyclic subgroup V of order 4 in G normalizes another cyclic subgroup U of order 4, then U normalizes V and $UV \cong C_4 \times C_2$ or $UV \cong Q_8$.*

Proof. First suppose $U \cap V = \{1\}$. Then $|UV| = 2^4$ and $(UV)' < U$ and we have either $UV = U \times V \cong C_4 \times C_4$ or $(UV)' \cong C_2$ in which case UV is a metacyclic minimal nonabelian group of order 2^4 and exponent 4. In any case, $c_2(UV) = 6$ and so $UV = H = \Omega_2^*(G)$, contrary to our assumption that $|H| > 2^4$. Thus, $U \cap V \cong C_2$ and so $|UV| = 2^3$. In this case, U also normalizes V and the only possibilities are $UV \cong C_4 \times C_2$ or $UV \cong Q_8$. □

LEMMA 2.2. *Suppose that $H = \Omega_2^*(G)$ contains a quaternion subgroup $Q \cong Q_8$. Then $|H| = 2^5$ and we have the following two possibilities:*

(a) $H \cong Q_8 * Q_8$;
(b) $H \cong Q_{16} * C_4$.

PROOF. First we determine the structure of $S = WQ$, where W is a normal four-subgroup in H and $|W \cap Q| \leq 2$. Let z be an involution in $W \cap Z(S)$. If $z \notin Q$, then $c_2(Q \times \langle z \rangle) = 6$ and therefore $Q \times \langle z \rangle = \Omega_2^*(G) = H$, a contradiction. Hence $Q \cap W = \langle z \rangle \cong C_2$, $|S| = 2^4$, and $[W, Q] = \langle z \rangle$. This gives $S = Q * \langle v \rangle$, where $\langle v \rangle \cong C_4$ and $\langle v \rangle \cap Q = \langle z \rangle$. We have $c_2(S) = 4$, $\langle v \rangle = Z(S)$ and since all elements in $S - (Q \cup \langle v \rangle)$ are involutions, Q is a unique quaternion subgroup of S and therefore Q is characteristic in S.

Assume that S is not normal in H and set $K = N_H(S)$ so that $|K : S| \geq 2$ and $|H : K| \geq 2$. Let M be a subgroup of H containing K so that $|M : K| = 2$ and take an element $m \in M - K$. Since m normalizes W, we have $Q^m \neq Q$, $Q^m \not\leq S$ and $Q^m \leq K$. We have $|Q^m \cap S| \leq 4$ and so $Q^m - S$ contains at least four elements of order 4. It follows that $c_2(K) = 6$. But $\Omega_2^*(G) = H$ and so there are elements of order 4 in $H - K$, a contradiction. We have proved that S is normal in H and so Q and $Z(S) = \langle v \rangle$ are normal in H. Since $c_2(S) = 4$, we have exactly four elements of order 4 in $H - S$. Set $C = C_H(Q)$ so that C is normal in H and $|H : (QC)| \leq 2$ (since $Aut(Q) \cong S_4$).

If there is an involution u in $C - \langle v \rangle$, then $c_2(Q \times \langle u \rangle) = 6$ and $Q \times \langle u \rangle = H$, a contradiction. Hence C is either generalized quaternion or cyclic. In the first case (since $C - \langle v \rangle$ can contain at most four elements of order 4), $C \cong Q_8$, $QC \cong Q_8 * Q_8$, $c_2(QC) = 6$ and therefore $QC = H$ is an extraspecial group of order 2^5 (case (a) of our lemma).

We may assume that C is cyclic so that $|H : (QC)| = 2$ because $H - S$ must contain exactly four elements of order 4. Since $H/C \cong D_8$, there is $x \in H - (QC)$ such that $x^2 \in C$ and x induces an involutory outer automorphism on Q. There are elements $a, b \in Q$ such that $\langle a, b \rangle = Q$, $a^x = a^{-1}$ and $b^x = ab$.

Suppose that $\langle x, C \rangle$ is cyclic so that $\langle x, C \rangle = \langle x \rangle$. If $o(x) \geq 16$, then there are no elements of order 4 in $H - (QC)$, a contradiction. Hence $o(x) = 8$ so that we may assume that $x^2 = v$. In this case all eight elements lx ($l \in S - \langle a, v \rangle$) in $H - S$ are of order 4, a contradiction. Hence $\langle x, C \rangle$ is noncyclic.

Assume that $\langle x, C \rangle$ is abelian or $\langle x, C \rangle \cong M_{2^m}$, $m \geq 4$, so that in both cases we may assume that x is an involution centralizing $\langle v \rangle$. We have $o(xv) = o(xva) = 4$ and we see that $c_2(S\langle x \rangle) = 6$ and so we get $H = S\langle x \rangle = \langle Q, xv \rangle * \langle v \rangle$, where $\langle Q, xv \rangle \cong Q_{16}$ and $H \cong Q_{16} * C_4$ (case (b) of our lemma).

Assume that $\langle x, C\rangle \cong Q_{2^n}$ or $\langle x, C\rangle \cong SD_{2^n}$. In both cases we may assume that $x^2 = z$ and $\langle v, x\rangle \cong Q_8$ since Q_8 is a subgroup of Q_{2^n} and SD_{2^n} and Q_8 contains the subgroup $\langle v\rangle$ of C . But then x inverts $\langle v\rangle$ and $\langle a\rangle$ (see above) and so all eight elements in $\langle a, v\rangle x$ from $H - (QC)$ are of order 4, a contradiction. Indeed, we compute for any integers i, j :

$$(a^i v^j x)^2 = a^i v^j x a^i v^j x = a^i v^j x^2 (a^i v^j)^x = a^i v^j z a^{-i} v^{-j} = z.$$

Finally, suppose that $\langle x, C\rangle \cong D_{2^n}$, $n \geq 3$, where x is an involution. But then all elements in $\langle a, C\rangle x$ from $H - (QC)$ are involutions since x inverts $\langle a\rangle$ and C and all other elements in $(QC - \langle a, C\rangle)x$ from $H - (QC)$ are elements of order 8, a contradiction (since $H - S$ does not contain any elements of order 4). Indeed, we set $C = \langle c\rangle$ and we know that $b^x = ab$ so that for any integers i, j we compute (noting that in Q we have $ba^i = a^i b z^i$ and $bab = a$):

$$(ba^i c^j x)^2 = ba^i c^j (ba^i c^j)^x = ba^i c^j \cdot aba^{-i} c^{-j} = a^i b z^i c^j aba^{-i} c^{-j} =$$
$$a^i z^i (bab) c^j a^{-i} c^{-j} = a^i z^i \cdot a \cdot c^j a^{-i} c^{-j} = z^i a,$$

which is an element of order 4. Hence, all elements $ba^i c^j x$ are of order 8, as claimed. Our lemma is proved. □

In the next three lemmas we assume in addition that Q_8 is not a subgroup of H.

Lemma 2.3. *Assuming that Q_8 is not a subgroup of H, we have $|H : N_H(X)| \leq 2$ for each cyclic subgroup X of order 4 in H.*

Proof. Suppose that the lemma is false. Then there is a cyclic subgroup U_1 of order 4 in H such that $K = N_H(U_1)$ is of index 4 in H. Let M be a maximal subgroup of H containing K so that $|H : M| = |M : K| = 2$ and let $m \in M - K$. Then $U_2 = U_1^m \neq U_1$, $N_H(U_2) = K$ and so $A = U_1U_2 \cong C_4 \times C_2$ and A is normal in M (Lemma 2.1). Let $x \in H - M$ so that $A^x \neq A$ since $c_2(A) = 2$ (and so A cannot be normal in H) and $A^x \leq M$. We have $c_2(M) \in \{3, 4, 5\}$ because there must exist elements of order 4 in $H - M$. If $c_2(M)$ is odd, then M (of order $\geq 2^4$) is of maximal class, a contradiction (since M possesses an abelian subgroup of type $(4, 2)$). Hence $c_2(M) = 4$.

Suppose that $|M| > 2^4$. If $|\Omega_2(M)| > 2^4$, then U_1 is normal in M (see [3, Introduction]), contrary to the fact that $N_H(U_1) = K < M$. Hence we must have $|\Omega_2(M)| = 2^4$. In that case we may use Janko [4, Theorems 3.1, 3.3, and 3.4] since Q_8 is not a subgroup of M and $c_2(\Omega_2(M)) = 4$. This implies that $\Omega_2(M)$ is abelian of type $(4, 2, 2)$ and there is a cyclic subgroup of order 4 which is normal in M. This is a contradiction since $\Omega_2(M) = AA^x$ and so all four cyclic subgroups of order 4 in M are conjugate in H and so no one of them could be normal in M.

We have proved that $|M| = 2^4$ so that $K = A = N_H(U_1)$, $AA^x = M$ is of order 2^4 and $|H| = 2^5$. In this case A and A^x are two distinct abelian maximal subgroups of M which implies $|Z(M)| = 4$, $|M'| = 2$, the class of

M is 2 and M is of exponent 4. Suppose that M is not minimal nonabelian. Then M possesses a subgroup $D \cong D_8$ and since M is not of maximal class, we have $C_M(D) \not\leq D$ (see [3, Proposition 1.9]). Since $c_2(M) = 4$, we get $M = D * C$ with $C \cong C_4$ and $D \cap C = Z(D)$. But $D_8 * C_4 \cong Q_8 * C_4$, contrary to our assumption. Hence M is minimal nonabelian. If M is metacyclic, then M possesses a cyclic normal subgroup of order 4 which contradicts the fact that all four cyclic subgroups of order 4 in M are conjugate in H. Hence M is a uniquely determined nonmetacyclic minimal nonabelian group of order 2^4 and exponent 4 (see [4, Proposition 1.3]).

Since $N_H(X) < M$ for each cyclic subgroup X of order 4 in M, there are no elements of order 8 in $H - M$. It follows that $H - M$ consists of four elements of order 4 and 12 involutions. Set $E = \Omega_1(M)$ so that E is elementary abelian of order 8, $Z(M) = \Phi(M) \cong E_4$ and $Z(H) \leq Z(M)$. Let v be an element of order 4 in $H - M$ so that $v^2 \in E$ and $C_E(v) \cong E_4$ since $H - M$ contains exactly four elements of order 4 (and they are all contained in $(E\langle v\rangle) - E$). All eight elements in $H - (M \cup E\langle v\rangle)$ are involutions and so if $u \in H - (M \cup E\langle v\rangle)$, then u centralizes E and $F = E \times \langle u\rangle \cong E_{16}$. In particular, $Z(M) < E < F$ and so $Z(M) = Z(H)$. Let $y \in M - E$ and we know that all cyclic subgroups of order 4 in M are conjugate in H to $\langle y\rangle$. But $y^2 \in \Phi(M) = Z(H)$ and so $\mho_1(M) = \langle y^2\rangle$, contrary to $\Phi(M) \cong E_4$. Our lemma is proved. □

Lemma 2.4. *Assuming that Q_8 is not a subgroup of H, we have $|H : N_H(X)| = 2$ for each cyclic subgroup X of order 4 in H.*

Proof. Suppose that the lemma is false. Then there are at least two distinct cyclic subgroups U_1 and U_2 which are normal in H. Let $\{U_1, U_2, ..., U_6\}$ be the set of six cyclic subgroups of H. Since each U_i, $i = 1, 2, ..., 6$, normalizes U_1 and U_2, it follows (Lemma 2.1) that $A = \langle U_1, U_2\rangle \cong C_4 \times C_2$ and $A \leq Z(H)$. For each U_j, $j = 3, ..., 6$, we have $\langle U_1, U_j\rangle \cong \langle U_2, U_j\rangle \cong C_4 \times C_2$ and so $U_1 \cap U_j = U_2 \cap U_j = U_1 \cap U_2 = \langle z\rangle = \mho_1(A)$. It follows that $B = AU_3 = \langle U_1, U_2, U_3\rangle$ is abelian of order 2^4 and exponent 4 with $\mho_1(B) = \langle z\rangle$ and so B is abelian of type $(4, 2, 2)$. Since $c_2(B) = 4$, we nay assume that $\{U_1, U_2, U_3, U_4\}$ is the set of cyclic subgroups of order 4 in B. Similarly, $C = AU_5$ is abelian of type $(4, 2, 2)$ with $\mho_1(C) = \langle z\rangle$ so that $\{U_1, U_2, U_5, U_6\}$ is the set of cyclic subgroups of order 4 in C. We have $B \cap C = A$ and $H = \langle B, C\rangle$. Thus, H/A is generated with two distinct involutions $(B/A)^\sharp$ and $(C/A)^\sharp$ and so $H/A \cong E_4$ or $H/A \cong D_{2^n}$, $n \geq 3$. In particular, B and C are not conjugate in H. Let t be an involution in $H - (B \cup C)$ and let v be an element of order 4 in $A \leq Z(H)$. Then tv is an element of order 4 in $H - (B \cup C)$, a contradiction. Hence, all elements in $H - (B \cup C)$ are of order ≥ 8. This implies that B and C are normal in H and so $H = \langle B, C\rangle = BC$ is of order 2^5 with two distinct abelian maximal subgroups B and C. It follows that $|H'| \leq 2$ and so H is of class ≤ 2.

But H is generated by its elements of order 4 and so H is of exponent 4, a contradiction. □

LEMMA 2.5. *Assuming that Q_8 is not a subgroup of H, we have $c_2(N_H(X)) = 2$ for each cyclic subgroup X of order 4 in H.*

PROOF. Let U_1 be a cyclic subgroup of order 4 in H so that $|H : N_H(U_1)| = 2$ (Lemma 2.4). Set $M = N_H(U_1)$ and taking an element $h \in H - M$ we get $U_2 = U_1^h \neq U_1$, $N_H(U_2) = M$, $A = \langle U_1, U_2\rangle \cong C_4 \times C_2$ (Lemma 2.1), and A is normal in H. Assume that M possesses a further cyclic subgroup $U_3 \not\leq A$ of order 4 so that $\langle U_1, U_3\rangle \cong \langle U_2, U_3\rangle \cong C_4 \times C_2$ and therefore $B = \langle U_1, U_2, U_3\rangle$ is abelian of type $(4, 2, 2)$. Since $c_2(B) = 4$, we may assume that $\{U_1, U_2, U_3, U_4\}$ is the set of all cyclic subgroups of order 4 in B. There is an element g of order 4 in $H - M$ and since $|H : N_H(\langle g\rangle)| = 2$, $U_5 = \langle g\rangle$ and $U_6 = \langle g^x\rangle$ (with an $x \in H - N_H(\langle g\rangle))$ give two last cyclic subgroups of order 4 in H which give exactly four elements of order 4 in $H - M$. This implies that $B = \Omega_2^*(M)$ is normal in H and U_3 and U_4 are conjugate in H.

Set $H_0 = BU_5$. If $c_2(H_0) = 5$, then H_0 is of maximal class, a contradiction. Hence $c_2(H_0) = 6$ and so $H_0 = BU_5 = H$. Set $B_0 = \Omega_1(B) \cong E_8$. Suppose $B \cap U_5 = \{1\}$ so that $|H : B| = 4$. Since $|H : N_H(U_5)| = 2$, U_5 centralizes a four-subgroup S in B_0. But then all eight elements of order 4 in $S \times U_5$ lie in $H - B$, a contradiction. Hence $B \cap U_5 \cong C_2$ and so $|H : B| = 2$, $|H| = 2^5$, $B = M$, and $N_H(U_3) = N_H(U_4) = B$. This implies that there are no elements of order 8 in $H - B$ and so $H - B$ consists of four elements of order 4 and twelve involutions.

We have $|B : N_B(\langle g\rangle)| = 2$, where $\langle g\rangle = U_5$ and $N_B(\langle g\rangle)$ cannot contain an element x of order 4 (otherwise, that element x would centralize U_5, contrary to the fact that $C_H(x) = B$). Hence $N_B(\langle g\rangle) = B_0$. If g centralizes B_0, then there are eight elements of order 4 in $H - B$, a contradiction. Hence $C_B(g) = C_{B_0}(g) = Z \cong E_4$ and so $Z(H) = Z$. The set B_0g consists of four elements of order 4 and four involutions. Hence all eight elements in $H - (B \cup B_0\langle g\rangle)$ are involutions and if t is one of them, then $H - B = B_0g \cup B_0t$ and $B_0g \cap B_0t = \emptyset$ so that t must centralize B_0 and therefore $B_0 \leq Z(H)$, contrary to the fact that $Z(H) = Z \cong E_4$. Our lemma is proved. □

THEOREM 2.6. *Let G be a 2-group with exactly six cyclic subgroups of order 4 and let $H = \Omega_2^*(G) = \langle x \in G \mid o(x) = 4\rangle$ be of order $> 2^4$. Then H is of order 2^5 and we have the following three possibilities:*

(a) *$H \cong Q_8 * Q_8$ is extraspecial (of type "+");*
(b) *$H \cong Q_{16} * C_4$ with $Q_{16} \cap C_4 = Z(Q_{16})$;*
(c) *H is a special group possessing a unique elementary abelian subgroup E of order 2^4 and there is an involution $t \in H - E$ such that $H = \langle E, t\rangle$ and $C_E(t) = Z(H) \cong E_4$.*

PROOF. In view of Lemma 2.2, we may assume that Q_8 is not a subgroup of H and so we may use Lemmas 2.1, 2.4, and 2.5. Let U_1 be a cyclic subgroup of order 4 in H. Set $K = N_H(U_1)$ so that $|H : K| = 2$ and if $h \in H - K$, then $U_2 = U_1^h \neq U_1$, $A = \langle U_1, U_2 \rangle = \Omega_2^*(K) \cong C_4 \times C_2$ is normal in H, $N_H(U_2) = K$ and so no one of U_1, U_2 is characteristic in K. Note that $|H| > 2^4$ and so $|K| > 2^3$.

We are in a position to use Proposition 1.1 which gives that K is a uniquely determined group of order 2^5 or 2^4. We may assume that we have the following conjugacy classes of our six cyclic subgroups of order 4 in H: $U_1 \sim U_2$, $U_3 \sim U_4$, and $U_5 \sim U_6$. Assume that $|K| = 2^5$ in which case $|H| = 2^6$. It follows that $\Phi(N_H(U_1)) = \langle U_1, U_2 \rangle$ and similarly (since $|H : N_H(U_3)| = |H : N_H(U_5)| = 2$), $N_H(U_3) \cong N_H(U_5) \cong K$. But then $\Phi(N_H(U_3)) = \langle U_3, U_4 \rangle$, $\Phi(N_H(U_5)) = \langle U_5, U_6 \rangle$ and therefore $\Phi(H) \geq \langle U_1, U_2, U_3, U_4 \rangle = H$, a contradiction.

We have proved that $K = N_H(U_1) = N_H(U_2)$ is of order 2^4 and then $K \cong D_8 \times C_2$ (Proposition 1.1(i)) and $|H| = 2^5$. The subgroup K has exactly three abelian maximal subgroups: $F_1 \cong E_8$, $F_2 \cong E_8$, and $A = \langle U_1, U_2 \rangle \cong C_4 \times C_2$, where $F_1 \cap F_2 = F_0 = Z(K) \cong E_4$. There are no elements of order 8 in $H - K$ since $N_H(U_1) = N_H(U_2) = K$ and so $H - K$ consists of eight elements of order 4 and eight involutions. Let r be an involution in $H - K$. Then rk ($k \in K$) is an involution if and only if $k^r = k^{-1}$. But $U_1^r = U_2$ and so the fact that $H - K$ contains exactly eight involutions gives at once that $C_K(r)$ is elementary abelian of order 8 and therefore we may assume that $C_K(r) = F_1$. Indeed, the fact that $U_1^r = U_2$ implies that r neither centralizes nor inverts any of the four elements of order 4 in $K - (F_1 \cup F_2)$. Since $H - K$ contains exactly eight involutions, it follows that r inverts (and therefore centralizes) the element 1 and seven further involutions in K and so in $F_1 \cup F_2$. Note that no involution in $F_1 - F_0$ commutes with any involution in $F_2 - F_0$ (otherwise, K would be abelian!). If all these seven involutions are not contained in F_1 or in F_2, then r centralizes an involution $t_1 \in F_1 - F_0$ and an involution $t_2 \in F_2 - F_0$ and so r centralizes t_1t_2. But t_1 and t_2 do not commute and so r centralizes the element t_1t_2 of order 4, a contradiction. The subgroup $E = F_1 \times \langle r \rangle$ is a unique elementary abelian subgroup of order 16 in H and for each involution $x \in E - F_1$, $C_K(x) = F_1$ and so if t is an involution in $F_2 - F_0$, then $C_E(t) = C_{F_1}(t) = F_0 = Z(K) = Z(H) \cong E_4$. We have obtained the group (c) stated in our theorem. □

THEOREM 2.7. *Let G be a 2-group with exactly six cyclic subgroups of order 4 and let $H = \Omega_2^*(G)$ be of order $> 2^4$. Then H is of order 2^5 and we have three possibilities for the structure of H (Theorem 2.6). However, if $G > H$, then $H \cong Q_{16} * C_4$, $|G : H| = 2$, $|G| = 2^6$, and we have the following two possibilities:*

(i) *G has a dihedral subgroup $D = \langle f, \xi \mid f^{16} = \xi^2 = 1,\ f^{\xi} = f^{-1}\rangle \cong D_{16}$ of index 2 and an involution $u \in G - D$ so that $[u, \xi] = 1$ and $f^u = fz$, $z = f^8$.*

(ii)

$$G = \langle a, t \mid a^{16} = t^2 = 1,\ a^8 = z,\ a^4 = v,\ a^t = a^{-1}vu,\ u^2 = 1,\ [u, a] = 1,\ u^t = uz\rangle,$$

where G is a U_2-group with respect to $U = \langle u, z\rangle \cong E_4$, $G/U \cong SD_{16}$ and $Z(G) = \langle uv\rangle \cong C_4$.

PROOF. For the structure of $H = \Omega_2^*(G)$ we use Theorem 2.6. We assume in addition that $G > H$. If $H \cong Q_8 * Q_8$ (Theorem 2.6(a)), then we have a contradiction by [5, Theorem 2].

Suppose that H is a special group given in Theorem 2.6(c). Let L be a subgroup of G containing H so that $|L : H| = 2$. Let E be a unique elementary abelian subgroup of order 16 in H so that H is normal in L. Let j be an involution in $H - E$ so that $C_E(j) = E_0 = Z(H) \cong E_4$ is normal in L and $F = E_0 \times \langle j\rangle = C_H(j) \cong E_8$ is normal in L since 12 elements in $H - (E \cup F)$ are of order 4. Four involutions in $F - E_0$ form a single conjugate class in H and so $I = C_L(j)$ covers L/H and $I \cap H = F$. Since there are no elements of order 4 in $L - H$, all elements in $I - F$ must be involutions and therefore $I \cong E_{16}$ and $E_0 \le Z(L)$. Let i be an involution in $I - F$ and consider the subgroup $J = E\langle i\rangle$ of order 2^5, where $J \cap H = E$. Again, all elements in $J - E$ must be involutions and so $J \cong E_{32}$. We get $C_H(i) \ge \langle E, F\rangle = H$. If v is an element of order 4 in H, then vi is of order 4 and $vi \in L - H$, a contradiction.

We have proved that $H \cong Q_{16} * C_4$ must be a group given in Theorem 2.6(b). We may set $H = Q * C$, where

$$Q = \langle b, t \mid b^8 = 1,\ t^2 = b^4 = z,\ b^t = b^{-1}\rangle \cong Q_{16}, \quad C = \langle v\rangle \cong C_4, \quad v^2 = z,$$

and $Q \cap C = \langle z\rangle$. The subgroup Q is generated by all (five) noncentral cyclic subgroups of order 4 in H and so Q is normal in G. Set $D = C_G(Q)$ so that $D \ge C$ and $D \cap H = C$. If there is an involution $i \in D - C$, then $o(b^2 i) = 4$ and $b^2 i \notin H$, a contradiction. Hence z is a unique involution in D. Since $c_2(D) = 1$, D cannot be generalized quaternion and so D is cyclic. Let $d \in D - C$ be an element of order 8. Then $b^4 = d^4 = z$ and so $o(bd) = 4$ with $bd \notin H$, a contradiction. We have proved that $D = C = C_G(Q)$.

The automorphism group $Aut(Q)$ is generated by $Inn(Q) \cong D_8$ and two involutory outer automorphisms α and β induced by $t^{\alpha} = tb$, $b^{\alpha} = b^{-1}$, $t^{\beta} = t$, $b^{\beta} = bz$, where $[\alpha, \beta] = i_{b^2}$ (the inner automorphism of Q induced by conjugation with the element b^2) and so $Aut(Q)/Inn(Q) \cong E_4$ (and in fact $\langle \alpha, \beta\rangle \cong D_8$). The subgroup Q contains exactly two quaternion subgroups Q_1 and Q_2 and we have $Q_1^{\beta} = Q_1$, $Q_2^{\beta} = Q_2$, and $Q_1^{\alpha} = Q_2$. It follows that $G/H \ne \{1\}$ is elementary abelian of order ≤ 4.

Assume that $L = N_G(Q_1) > H$ so that $|L : H| = 2$. Since $Q/\langle z\rangle \cong D_8$ is isomorphic to an S_2-subgroup of $Aut(Q_1) \cong S_4$, it follows that $C_0 = C_L(Q_1) > C$ and $|C_0 : C| = 2$. If $y \in C_0 - C$ is an involution, then $o(b^2y) = 4$ (because $b^2 \in Q_1$) and $b^2y \notin H$, a contradiction. Since $c_2(C_0) = 1$, we get that $C_0 = \langle c\rangle \cong C_8$ is cyclic with $c^4 = z$. Now, c normalizes $\langle b\rangle$ (since Q is normal in G and $\langle b\rangle$ is a unique cyclic subgroup of index 2 in Q) and centralizes $\langle b^2\rangle = \langle b\rangle \cap Q_1$, but c does not centralize $\langle b\rangle$ (otherwise, c would centralize $\langle b, Q_1\rangle = Q$, a contradiction) and so we get $b^c = bz$, $\langle b, c\rangle' = \langle z\rangle$, class of $\langle b, c\rangle$ is 2, $(bc)^4 = b^4c^4[c, b]^6 = zzz^6 = 1$, $o(bc) = 4$, and $bc \notin H$, a contradiction.

We have proved that $|G/H| = 2$, $|G| = 2^6$, and if $g \in G - H$, then $Q_1^g = Q_2$. In particular, $C_G(t) = \langle t, v\rangle \cong C_4 \times C_2$ and so eight elements of order 4 in $Q - \langle b\rangle$ form a single conjugate class in G. Set $T = \langle b\rangle\langle v\rangle \cong C_8 \times C_2$ which is normal in G and eight elements in $H - (Q \cup T)$ are involutions which form a single conjugate class in G and so if tv is one of them, then $C_G(tv) = \langle t, v\rangle$. In particular, if $x \in G - H$, then $x^2 \in T$. We have $U = \Omega_1(T) = \langle z, b^2v\rangle \cong E_4$ is normal in G, $\Omega_2(T) = \langle b^2, v\rangle \cong C_4 \times C_2$, and $\langle b^2\rangle$ and $\langle v\rangle$ are normal in G.

Suppose that there is an involution $\xi \in G - H$. Then ξ inverts $\langle v\rangle$ and $\langle b^2\rangle$ (otherwise, ξ centralizes v or b^2 and then ξv or ξb^2 would be an element of order 4 in $G - H$, a contradiction). If $b^\xi = b^{-1}z$, then $(\xi b)^2 = b^\xi b = z$ and $o(\xi b) = 4$ with $\xi b \notin H$, a contradiction. Hence ξ inverts each element in T and so, in particular, ξ centralizes U. Since $Q_1^\xi = Q_2$, we have $t^\xi = tb^i$, where i is odd. Set $b^2v = u$ and $\xi t = f$ so that ξ centralizes the involution u,

$$f^2 = \xi t\xi t = t^\xi t = (b^i)^t = b^{-i},\ o(f) = 16,\ f^8 = (b^{-i})^4 = z,$$

$$f^\xi = (\xi t)^\xi = \xi t^\xi = \xi tb^i = fb^i = f^{-1}(f^2b^i) = f^{-1}b^{-i}b^i = f^{-1},\ \langle f, \xi\rangle \cong D_{16}$$

and

$$f^u = (\xi t)^{b^2v} = v^{-1}b^{-2}\xi tb^2v = \xi(v^{-1}b^{-2})^\xi tb^2v = \xi vb^2tb^2v =$$
$$(\xi t)(vb^2)^tb^2v = (\xi t)vb^{-2}b^2v = \xi tv^2 = fz.$$

We have obtained the group given in part (i) of our theorem.

It remains to investigate the case, where there are no involutions in $G-H$. Then 32 elements in $G - H$ are of order 8 or 16. If all 32 elements in $G - H$ are of order 8, then $c_3(G) = 10$ and therefore G is a U_2-group (see section 1). But then G must also have elements of order 16 which is not the case. If all 32 elements in $G - H$ are of order 16, then $c_4(G) = 4$ and $c_3(G) = 2$. Again, G is a U_2-group. But a U_2-group of order 2^6 has exactly two cyclic subgroups of order 16, a contradiction. Hence $G - H$ contains elements of order 8 and 16. Since the number of cyclic subgroups of order 16 must be even (otherwise, G would be of maximal class), it follows that $G - H$ has exactly 16 elements of order 16 (and so $c_4(G) = 2$) and exactly 16 elements of order 8. Hence $c_3(G) = 6$ and so G is a U_2-group with respect to U since in a U_2-group a normal four-subgroup is unique. If R/U is a cyclic subgroup of index 2 in G/U, then $G-R$ contains exactly eight involutions, eight elements of order 4, and 16 elements of order 8. Hence $G/U \cong SD_{16}$ and $\Phi(R) \cong C_8$. Since H is nonmetacyclic, G is also nonmetacyclic. We have $\Phi(G) \leq T$ and so there are exactly three maximal subgroups of G containing T. They are H, R and a certain subgroup V with the property that all 16 elements in $V-T$ are of order 8. Since $\Omega_2(V) = \Omega_2(T) = \langle b^2, v\rangle \cong C_4 \times C_2$, $|V| = 2^5$, and V has no elements of order 16, V must be isomorphic to a group (d) given in [4, Proposition 1.4] and so $\Phi(V) = \Omega_2(V)$ and $Z(V) \cong C_4$. We get $\Phi(G) \geq \langle \Phi(R), \Phi(V)\rangle = T$ and so G is 2-generated, i.e., $d(G) = 2$. Also, $Z(V) \cong C_4$ implies that $U \not\leq Z(V)$ and so $C_G(U) = R$ (because $C_G(U)$ must be a maximal subgroup of G containing T and also $U \not\leq Z(H)$). Since $\Phi(T) = \langle b^2\rangle$ and $\Phi(V) = \langle b^2, v\rangle$ (and no involution in $\Phi(V) - \langle z\rangle$ could be a square of an element in $V-T$ because $U \not\in Z(V)$), there is an element $s \in V-T$ such that $s^2 = v$. Hence, $C_G(v) \geq \langle H, s\rangle = G$ and so $Z(G) \cong C_4$. We have obtained a nonmetacyclic U_2-group G of order 2^6 with respect to $U \cong E_4$ such that $G/U \cong SD_{16}$, $d(G) = 2$, and $Z(G) \cong C_4$. It follows that G must be isomorphic to a U_2-group given in [3, Theorem 6.3(c)]. We have obtained the group given in part (ii) of our theorem. □

3. 2-groups with one conjugacy class of cyclic subgroups of order 4

THEOREM 3.1. *Let G be a 2-group of exponent > 2 all of whose cyclic subgroups of order 4 are conjugate. Then G has exactly one cyclic subgroup of order 4 and G is either cyclic or dihedral.*

PROOF. First suppose that G has more than one cyclic subgroup of order 4. Let U be one of them and set $K = N_G(U)$ so that $|G : K| \geq 2$ and let M be a maximal subgroup of G containing K. Then each cyclic subgroup of order 4 is contained in M and if X is one of them, then $N_G(X) \leq M$ (since X is conjugate in G to U). Let x be any element in $G-M$. We know that x is not of order 4 and suppose that $o(x) \geq 8$. But then $x^2 \in M$ and $o(x^2) \geq 4$ and so x centralizes a cyclic subgroup of order 4 in M, a contradiction. Hence each element x in $G-M$ is an involution and so M must be abelian and x acts invertingly on M. But then U is normal in G, a contradiction.

We have proved that G has a unique cyclic subgroup $V = \langle v\rangle$ of order 4 so that V is normal in G. Then our result follows by [2, Theorem 1.17]. Here we give also a direct proof. Set $C = C_G(V)$ and we have $|G : C| \leq 2$. If C possesses an involution $t \neq v^2$, then $\langle tv\rangle$ is a cyclic subgroup of order 4 distinct from V, a contradiction. It follows that C has the unique involution v^2 and so C is cyclic. If $|G : C| = 2$, then G is dihedral and we are done. □

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