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ON A CHARACTERIZATION OF QUASICYCLIC GROUPS

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Abstract. Let G be an infinite solvable group (resp. an infinite group properly containing its commutator subgroup G'). We prove that G is isomorphic to a quasicyclic group if and only if all proper normal subgroups of G are finitely generated (resp. all proper normal subgroups of G are cyclic or finite).

In this paper, the symbols $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$ denote the rational numbers, the integers, the nonnegative integers, respectively. A quasicyclic group (or Prüfer group) is the p-primary component of $\mathbb{Q}/\mathbb{Z}$, that is, the unique maximal p-subgroup of $\mathbb{Q}/\mathbb{Z}$, for some prime number p. Any group isomorphic to it will also be called a quasicyclic group and denoted by $\mathbb{Z}_{p^\infty}$. Quasicyclic groups play an important roles in the infinite abelian group theory. They may also be defined in a number of equivalent ways (again, up to isomorphism):

- A quasicyclic group is the group of all p^n-th complex roots of 1, for all $n \in \mathbb{N}$.
- A quasicyclic group is the injective hull of $\mathbb{Z}/p\mathbb{Z}$ (viewing abelian groups as $\mathbb{Z}$-modules).
- A quasicyclic group is the direct limit of the groups $\mathbb{Z}/p^n\mathbb{Z}$.

The subgroup structure of $\mathbb{Z}_{p^\infty}$ is particularly simple: all proper subgroups are finite and cyclic, and there is exactly one of order p^n for each non-negative integer n.

One may naturally ask the inverse problem: is G a quasicyclic group if its all proper subgroups are finite or cyclic? The literature [1] gives an affirmative answer if G is an infinite solvable group or G is an infinite group with $G > G'$. In this paper, we will prove that G is a quasicyclic group if its

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all proper normal subgroups are finite or cyclic under the condition that G is an infinite solvable group or G is an infinite group properly containing its commutator group G'.

A characterization of a group from its subgroups is one of the key technical tools in the infinite group theory. Let G be a group and Σ be an absolute property about group G, i.e. whether G has a property Σ is only dependent on the group G itself. G is called a Σ *group* if G has a property Σ. G is called an *inner* Σ *group* if each proper subgroup of G has the property Σ but G itself doesn't. For more information about inner Σ group, we refer to [2]. In the same vein, we give the following definition for convenience of a later description.

DEFINITION 1. *A group G is called a hypo-inner Σ group if each proper normal subgroup of G has the property Σ but G itself doesn't.*

For example, $\mathbb{Z}_{p^\infty}$ is a hypo-inner finitely generated group, a hypo-inner cyclic group, a hypo-inner finite group, etc. But G itself is not finitely generated, not cyclic, not finite, etc. Clearly, inner Σ group is a hypo-inner Σ group. But the converse is not valid. A. J. Ol'sanskii in [4] gives a counter example. If G is an infinite abelian group, then a hypo-inner Σ group is also an inner Σ group. From [1], following results hold clearly.

1. G is hypo-inner finitely generated if and only if $G \cong \mathbb{Z}_{p^\infty}$.
2. G is a hypo-inner supper solvable group if and only if $G \cong \mathbb{Z}_{p^\infty}$.
3. G is not a direct product of cyclic groups. Then each proper normal subgroup of G is a direct product of cyclic groups if and only if $G \cong \mathbb{Z}_{p^\infty}$.
4. G is a hypo-inner cyclic group if and only if $G \cong \mathbb{Z}_{p^\infty}$.

In the sequel, let G be an infinite group. We prove

THEOREM 2. *Let G be a solvable group, then G is a hypo-inner finitely generated group if and only if $G \cong \mathbb{Z}_{p^\infty}$.*

PROOF. Sufficiency is clear. We prove the necessity by induction on the length of the derived series of G. Suppose

$$G = G^{(0)} \geq G' \geq G^{(2)} \geq \ldots \geq G^{(s)} = 1.$$

When G is an abelian group, The fact 1 implies that $G \cong \mathbb{Z}_{p^\infty}$.

The first step, we prove that the theorem holds if G' is abelian.

From inductional assumption, G' and G/G' are abelian groups. Since G is a hypo-inner finitely generated group then G' is a finitely generated group and G/G' is a hypo-inner finitely generated group. By the fact 1, $G/G' \cong \mathbb{Z}_{p^\infty}$.

Now, first we show that G' is in the center of G, i.e. $G' \leq Z(G)$. Since G' is an abelian group, $G' \leq C_G(G')$. If $C_G(G')$ is a proper normal subgroup of G, then $G/C_G(G')$ is a hypo-inner finitely generated abelian group, and $G/C_G(G') \cong \mathbb{Z}_{p^\infty}$. On the other hand, $G/C_G(G')$ is isomorphic to a subgroup of the automorphic group $Aut(G')$. Hence G' is an infinite group.

The structure theorem of finitely generated abelian groups ([5, 4.2.10]) asserts that G' is a direct sum of finitely many cyclic groups of infinite or prime-power orders. We can assume that G' is torsion-free. In fact, any torsion subgroup F of G' is a characteristic subgroup, so, $G/C_G(G')$ can be viewed as a subgroup of the automorphic group $Aut(G'/F)$, and G'/F is torsion-free. Hence an automorphism of G' can be represent by a matrix A with entries in $\mathbb{Z}$, which satisfies $\det(A) = \pm 1$.

Suppose the order of A is p^k and minimal polynomial of A is $m(x)$. We have

$$m(x)|(x^{p^k} - 1) \quad \text{and} \quad x^{p^k} - 1 = \Psi_1(x)\Psi_p(x)\cdots\Psi_{p^{k-1}}(x)\Psi_{p^k}(x),$$

where Ψ_i is ith cyclotomic polynomial.

It is easily shown that $m(x)$ must contain the irreducible polynomial $\Psi_{p^k}(x)$. Otherwise

$$m(x)|\Psi_1(x)\Psi_p(x)\cdots\Psi_{p^{k-1}}(x).$$

Then

$$x^{p^{k-1}} - 1 = m(x)q(x) \quad \text{and} \quad A^{p^{k-1}} - 1 = m(A)q(A) = 0.$$

This is a contradiction. Hence, the degree of the $m(x)$ is not less than the degree of $\Psi_{p^k}(x)$, which is equal to $p^{k-1}(p-1)$. Assume that the size of A is n, then the degree of the minimal polynomial $m(x)$ of A is less than n. We have

$$p^{k-1}(p-1) \leq n.$$

So, we obtain

$$|A| = p^k \leq pn \text{ (constant)},$$

where $|A|$ denote the order of A. This inequality shows that, for any $A \in Aut(G')$, the order of A is not greater than a constant. Hence, $G/C_G(G')$ can not be isomorphic to a subgroup of $Aut(G')$. Therefore, $G = C_G(G')$, and $G' \leq Z(G)$.

Second, we show that G is an abelian group. Since $G/G' \cong \mathbb{Z}_{p^\infty}$ and $G' \leq Z(G)$, for any elements $a, b \in G$, there exists $c \in G$ such that

$$a = cz_1, \quad b = c^r z_2, \quad \text{for} \quad z_1, z_2 \in G' \subset Z(G),$$

then

$$ab = cz_1 \cdot c^r z_2 = c^r z_2 \cdot cz_1 = ba.$$

So, G is an abelian group, and the fact 1 implies $G \cong \mathbb{Z}_{p^\infty}$.

The second step, clearly, $G/G^{(s-1)}$ is a hypo-inner finitely generated group. It has a derived series

$$G/G^{(s-1)} \geq G'/G^{(s-1)} \geq \ldots \geq G^{(s-2)}/G^{(s-1)} \geq 1.$$

By the induction hypothesis, we have $G/G^{(s-1)} \cong \mathbb{Z}_{p^\infty}$. Then $G^{(s-1)} \geq G'$ and $G' = G^{(s-1)}$. Thus, $(G')' = 1$ implies that G' is an abelian group. From the first step, we get $G \cong \mathbb{Z}_{p^\infty}$. $\square$

THEOREM 3. *Suppose that G properly contains its commutator subgroup G'. Then G is hypo-inner finite group if and only if $G \cong \mathbb{Z}_{p^\infty}$.*

PROOF. We need only to prove the necessity. G/G' is a torsional abelian group since it is a hypo-inner finite abelian group. By [3, Theorem 1], G is isomorphic to a direct sum of primary groups. By the assumption, G has only one primary component, i.e. $G/G' \cong \mathbb{Z}_{p^\infty}$. Since G' is finite, its automorphic group $Aut(G')$ is also finite. If $C_G(G')$ is a proper normal subgroup of G, then $C_G(G')$ is finite, and $G/C_G(G')$ is infinite. But $G/C_G(G')$ is isomorphic to a subgroup of $Aut(G')$. This is a contradiction. So $G = C_G(G')$. Using the similar proof of Theorem 2, we obtain that G is an abelian group. Therefore, G is a torsional abelian group. By assumption and [3, Theorem 1], we have that $G \cong \mathbb{Z}_{p^\infty}$. $\square$

LEMMA 4. *Suppose that G properly contains its commutator subgroup G'. If all proper normal subgroups of G are abelian groups then G is an abelian group.*

PROOF. Suppose that G is not an abelian group. Then $1 \neq G' \lhd G$. Let Ω be the set of all proper normal subgroups of G which contain G'. Assume that

$$A_1 \leq A_2 \leq \ldots \leq A_n \leq \ldots$$

is an ascending series of Ω. Set $B = \cup_i A_i$. It easily shows that B is an abelian subgroup containing G' since every A_i is an abelian group containing G'. From the hypothesis, B is a proper normal subgroup of G, then $B \in \Omega$. By Zorn Lemma, there exists a maximal normal subgroup M of G containing G'. So, G/M is an abelian simple group, i.e. G/M is a cyclic group of order p for some prime number p. Suppose that aM is a generator of G/M. Since G is not an abelian group, there exists $b \in M$ such that

$$a^{-1}ba = b_1 \in M, \quad b_1 \neq b.$$

Assume that $b_1, b_2, \ldots, b_r$ are all distinct elements in G which conjugate with b. Note that $1 \leq r \leq p$. Accordingly, $M_1 =< b_1, b_2, \ldots, b_r >$ is a normal subgroup of G, and $M_1 \leqslant M$.

Set $N =< a, M_1 >=< a, b_1, \ldots, b_r >$. It is clear that N is a nonabelian subgroup of G. We show that N is normal in G. For $x \in G$, $y \in N$, we have $x = a^k m$ with $m \in M$ and $k \in \mathbb{Z}$, and $y = a^l m_1$ with $m_1 \in M_1$ and $l \in \mathbb{Z}$.

$$x^{-1}yx = (a^k m)^{-1}(a^l m_1)(a^k m) = (m^{-1}a^l m)\tilde{m_1} = a^s \tilde{m_1},$$

where $\tilde{m_1} \in M_1$ and $s \in \mathbb{Z}$. It follows that $N \trianglelefteq G$, and $G = N$. Hence, we may choose $M =< b_1, \ldots, b_r >$. So, M is a finitely generated infinite abelian group. Without loss of generality, assume that the order of b_1 is infinite. If the order of $b_i\,(2 \le i \le r)$ is finite, then denote their order by q_i. Let $n > 1$ be a positive integer and be coprime with all q_i. We claim that $M^n \neq M$. Otherwise, b_1 can be written as

$$b_1 = x_1 n b_1 + \ldots + x_k n b_r \text{ for } x_i \in \mathbb{Z}.$$

Since $b_1, \ldots, b_r$ is a base of the abelian group M, $nx_1 = 1$. This is a contradiction. M^n is a characteristic subgroup of M, then $M^n \trianglelefteq G$. Using the same technique as in the previous proof of this lemma, we have that $< a, M^n >$ is a proper normal subgroup of G. Thus $< a, M^n >$ is an abelian group. We have

$$nb_i = a^{-1}(nb_i)a = n(a^{-1}b_i a) \text{ for } 1 \le i \le r.$$

Since $a^{-1}b_i a \in M$, we have

$$a^{-1}b_i a = t_1 b_1 + \ldots + t_i b_i + \ldots + t_r b_r \text{ for } 1 \le i \le r \tag{1}$$

where $t_i \in \mathbb{Z}$. When the order of b_i is a finite number q_i then $1 \le t_i \le q_i$. It follows that

$$nb_i = nt_1 b_1 + \ldots + nt_i b_i + \ldots + nt_r b_r,$$

where $b_1, \ldots, b_r$ is a basis of M. The uniqueness of representation implies that

$$nt_j \equiv 0 \pmod{q_j} \quad \text{or} \quad nt_j = 0, \ \ j \neq i,$$

and

$$n(t_i - 1) \equiv 0 \pmod{q_i} \quad \text{or} \quad n(t_i - 1) = 0.$$

Since n and q_i are coprime, anyway we obtain

$$t_i = 1 \quad \text{and} \quad t_j = 0, \ j \neq i.$$

Hence, the equality (1) becomes $a^{-1}b_i a = b_i$ for $1 \le i \le r$. That is to say, $G =< a, b_1, \ldots, b_r >$ is an abelian group. This is a contradiction to the hypothesis. ☐

THEOREM 5. *Suppose that G properly contains G'. Then G is a hypo-inner cyclic group if and only if $G \cong \mathbb{Z}_{p^\infty}$.*

PROOF. Cyclic groups are of course abelian groups, then all proper normal subgroups of G are abelian groups. Hence, from Lemma 4 we have that G is an abelian group. By the fact ④, $G \cong \mathbb{Z}_{p^\infty}$. ☐

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