

QUASILINEAR DIRICHLET PROBLEM IN A PERIODICALLY PERFORATED DOMAIN

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ABSTRACT. Consider the following quasilinear Dirichlet problem

$$\begin{cases} -\Delta_p u_\varepsilon = f_\varepsilon & \text{in } \Omega_\varepsilon = \Omega \setminus T_\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \end{cases} \quad (P_\varepsilon)$$

where $1 < p \leq 2$, (f_ε) a sequence of functions in $L^{p'}(\Omega)$ such that $f_\varepsilon \rightarrow f$ in $L^{p'}(\Omega)$ where p' is the conjugate of p , Ω is a bounded domain in \mathbb{R}^N ($N \geq 2$) and T_ε the union of inclusions contained in Ω which are ε -periodically distributed. In this paper we give the limit problem and errors estimates for the problem (P_ε) .

1. INTRODUCTION

We consider here the homogenization of the following quasilinear Dirichlet problem

$$\begin{cases} -\Delta_p u_\varepsilon = f_\varepsilon & \text{in } \Omega_\varepsilon = \Omega \setminus T_\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \end{cases} \quad (P_\varepsilon)$$

where,

- $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$
- $1 < p \leq 2$
- (f_ε) is a convergent sequence of functions to f in $L^{p'}(\Omega)$, p' is the conjugate of p , $(\frac{1}{p} + \frac{1}{p'} = 1)$
- Ω is a bounded domain in \mathbb{R}^N ($N \geq 2$) which contains small inclusions $T_{\varepsilon,j}$ ($j \in \mathbb{N}$) periodically distributed in Ω with period ε , $T_{\varepsilon,j}$ are defined by translating εT , where T is a closed connected regular domain inside

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the unit cube $Y =]-\frac{1}{2}, \frac{1}{2}[^N$ of \mathbb{R}^N . T_ε is the union of all the periodic inclusions $T_{\varepsilon,j}$.

The analysis of (P_ε) begins by considering the associated minimization problem

$$\min \left\{ \int_{\Omega} |\nabla v|^p dx - p \int_{\Omega} f_\varepsilon v dx, v \in W_0^{1,p}(\Omega), v = 0 \text{ on } T_\varepsilon \right\} \quad (D_\varepsilon)$$

by standard argument, the problem (D_ε) admits a unique solution u_ε , which is the unique weak solution of (P_ε) .

In section 2, we give some estimates for the L^p -norms of u_ε and ∇u_ε in terms of ε (Proposition 2.1) and then present another equivalent version of the problem (D_ε)

$$\min \left\{ \varepsilon^p \int_{\Omega} |\nabla v|^p dx - p \int_{\Omega} f_\varepsilon v dx, v \in W_0^{1,p}(\Omega), v = 0 \text{ on } T_\varepsilon \right\} \quad (D_\varepsilon^*)$$

The goal of this section is to prove the epi-convergence of the problem (D_ε^*) to the following problem

$$\min \left\{ \frac{1}{m(w)^{p-1}} \int_{\Omega} |v|^p dx - p \int_{\Omega} f v dx, v \in L^p(\Omega) \right\} \quad (D^*)$$

where w is the solution of the problem

$$\begin{cases} -\Delta_p w = 1 & \text{in } Y \setminus T \\ w = 0 & \text{in } T \\ w & Y\text{-periodic} \end{cases}$$

and $m(w) = \int_Y w dy$, by showing the epi-convergence of the functional

$$F_\varepsilon(v) = \int_{\Omega} \varepsilon^p |\nabla v|^p dx - p \int_{\Omega} f_\varepsilon v dx \quad v \in W_0^{1,p}(\Omega), v = 0 \text{ on } T_\varepsilon$$

as ε goes to zero, to the functional

$$F(v) = \frac{1}{m(w)^{p-1}} \int_{\Omega} |v|^p dx - p \int_{\Omega} f v dx \quad v \in L^p(\Omega)$$

that means

- i) $\forall v \in L^p(\Omega), \exists (v_\varepsilon) \in W_0^{1,p}(\Omega), v_\varepsilon = 0 \text{ on } T_\varepsilon \text{ and } v_\varepsilon \rightharpoonup v \in L^p(\Omega)$
such that

$$\limsup_{\varepsilon} F_\varepsilon(v_\varepsilon) \leq F(v),$$

and

ii) For all $(v_\varepsilon) \subseteq W_0^{1,p}(\Omega)$, $v_\varepsilon = 0$ on T_ε , $(F_\varepsilon(v_\varepsilon))$ bounded and $v_\varepsilon \rightharpoonup v \in L^p(\Omega)$ then,

$$\liminf_\varepsilon F_\varepsilon(v_\varepsilon) \geq F(v);$$

the proof of this property is based on Tartar’s method using test functions.

For a detailed study of the concept of epi-convergence, we refer to [1].

In section 3, under some hypothesis on the sequences (f_ε) (Theorem 3.1), we give error estimates for convergences obtained in section 2.

The new aspects of these results are concentrated in Lemma 2.7 and Theorem 3.1. The proofs of these results are based on several vectorial inequalities given for example in [8] and [12].

For $p = 2$ and $f_\varepsilon = f$ (for all ε), the convergence of (D_ε) as $\varepsilon \rightarrow 0$ has been studied (see, [3], [11], [13]).

For $p \geq 2$ and $f_\varepsilon = f$, the limit problem of (D_ε) as $\varepsilon \rightarrow 0$ is giving in [9].

For $f_\varepsilon = f$ and $p \geq 2$, the error estimates for the homogenization of (D_ε) are given in [10].

Case $2 \leq p < \infty$ [2] (in preparation).

For a presentation of the homogenization theory see [4].

2. CONVERGENCE OF (D_ε)

2.1. *Estimates for u_ε .* The weak lower semicontinuity, coercivity and strict convexity of the functional

$$F_\varepsilon(v) = \int_\Omega |\nabla v|^p dx - p \int_\Omega f_\varepsilon v dx$$

defined on the closed convex set $\{v \in W_0^{1,p}(\Omega), v = 0 \text{ on } T_\varepsilon\}$ of $W_0^{1,p}(\Omega)$ implies the existence and uniqueness of a solution u_ε of the problem (D_ε) , that is a weak solution of (P_ε) .

PROPOSITION 2.1. *There exists a constant c independent of ε such that*

$$\|u_\varepsilon\|_{L^p(\Omega)} \leq c \varepsilon^{p/(p-1)} \quad \text{and} \quad \|\nabla u_\varepsilon\|_{(L^p(\Omega))^N} \leq c \varepsilon^{1/(p-1)}.$$

PROOF. From “nonstandard” Poincaré’s inequality (where v in $W^{1,p}(Y \setminus T)$ and v vanishes on a part of $\partial(Y \setminus T)$), there exists a constant $c > 0$ such that, for $v \in W^{1,p}(Y \setminus T), v = 0$ on T we have

$$\int_{Y \setminus T} |v|^p dy \leq c \int_{Y \setminus T} |\nabla v|^p dy.$$

Then, for $v \in W^{1,p}(Y), v = 0$ on T we have

$$\int_Y |v|^p dy \leq c \int_Y |\nabla v|^p dy.$$

By the change of variable $y = \frac{x}{\varepsilon}$ we deduce that, for every v_ε in $W^{1,p}(\varepsilon Y)$ such that $v_\varepsilon = 0$ on εT , we have

$$\int_{\varepsilon Y} |v_\varepsilon|^p dx \leq c \varepsilon^p \int_{\varepsilon Y} |\nabla v_\varepsilon|^p dx.$$

Then, since $u_\varepsilon = 0$ on T_ε we have

$$\int_{\Omega} |u_\varepsilon|^p dx \leq c \varepsilon^p \int_{\Omega} |\nabla u_\varepsilon|^p dx$$

which is equivalent to

$$\|u_\varepsilon\|_{L^p(\Omega)} \leq c \varepsilon \|\nabla u_\varepsilon\|_{(L^p(\Omega))^N}.$$

But u_ε is the solution of (D_ε) , so

$$\int_{\Omega} |\nabla u_\varepsilon|^p dx = \int_{\Omega} f_\varepsilon \cdot u_\varepsilon dx,$$

and we deduce that

$$\|\nabla u_\varepsilon\|_{(L^p(\Omega))^N}^p \leq \|f_\varepsilon\|_{L^{p'}(\Omega)} \|u_\varepsilon\|_{L^p(\Omega)} \leq c \|u_\varepsilon\|_{L^p(\Omega)}$$

(because the sequence (f_ε) is bounded in $L^{p'}(\Omega)$). It follows that

$$\|u_\varepsilon\|_{L^p(\Omega)}^p \leq c \varepsilon^p \|u_\varepsilon\|_{L^p(\Omega)},$$

so

$$\|u_\varepsilon\|_{L^p(\Omega)} \leq c \varepsilon^{p/(p-1)},$$

and

$$\|\nabla u_\varepsilon\|_{(L^p(\Omega))^N} \leq c \varepsilon^{1/(p-1)}$$

where c is a constant. □

REMARK 2.2. We deduce from Proposition 2.1 that (u_ε) is bounded in $W_0^{1,p}(\Omega)$ and then admits a converging subsequence, still denoted by (u_ε) .

REMARK 2.3. Let $u_\varepsilon^* = \varepsilon^{-p/(p-1)} u_\varepsilon$. Since u_ε is the solution of (D_ε) , u_ε^* is the solution of the following minimization problem

$$(D_\varepsilon^*) \quad \min \left\{ \varepsilon^p \int_{\Omega} |\nabla v|^p dx - p \int_{\Omega} f_\varepsilon \cdot v dx \quad v \in W_0^{1,p}(\Omega), v = 0 \text{ on } T_\varepsilon \right\}.$$

2.2. Convergence of (D_ε^*) .

THEOREM 2.4. *The problem (D_ε^*) converge as $\varepsilon \rightarrow 0$ to the following problem*

$$(D^*) \quad \min \left\{ \frac{1}{m(w)^{p-1}} \int_{\Omega} |v|^p dx - p \int_{\Omega} f v dx, v \in L^p(\Omega) \right\}$$

where w is the solution of the problem

$$\begin{cases} -\Delta_p w = 1 & \text{in } Y \setminus T \\ w = 0 & \text{in } T \\ w & Y - \text{periodic} \end{cases}$$

and $m(w) = \int_Y w dy$. Hence,

- i) $u_\varepsilon^* \rightharpoonup u^* = m(w)|f|^{p'-2}f$ in $L^p(\Omega)$,
- ii) $\varepsilon^{\frac{-1}{(p-1)}}u_\varepsilon \rightharpoonup 0$ in $W^{1,p}(\Omega)$,
- iii) $\int_{\Omega} \varepsilon^{-p/(p-1)}|\nabla u_\varepsilon|^p dx \rightarrow m(w) \int_{\Omega} |f|^{p'} dx$.

For proof of this theorem we use the following lemmas giving in [10, 2]. From properties of the epi-convergence notion (see [1, 5, 6, 7]), we get the limit problem of (D_ε^*) by using a direct proof making use of suitable test functions.

LEMMA 2.5. *Let w_ε be such that*

$$\begin{cases} w_\varepsilon(x) = w\left(\frac{x}{\varepsilon}\right), & \text{for } x \in \varepsilon Y \setminus \varepsilon T \\ w_\varepsilon = 0, & \text{on } \varepsilon T \\ w_\varepsilon, & \varepsilon Y - \text{periodic} \end{cases}.$$

Then we have

$$\begin{cases} w_\varepsilon \in W^{1,p}(\Omega), w_\varepsilon = 0, & \text{on } T_\varepsilon, \\ w_\varepsilon \rightharpoonup m(w), & \text{in } L^p(\Omega), \\ \varepsilon^p |\nabla w_\varepsilon|^p dx \rightarrow m(w) dx, & \text{in } \sigma\left(M_1(\Omega), C(\bar{\Omega})\right). \end{cases}$$

LEMMA 2.6. For all v in $D(\Omega)$ the sequence $(v_\varepsilon) = \left(\frac{1}{m(w)} w_\varepsilon v\right)$ satisfies the following

$$\begin{cases} v_\varepsilon \in W_0^{1,p}(\Omega), \quad v_\varepsilon = 0 \text{ on } T_\varepsilon, \\ v_\varepsilon \rightharpoonup v \text{ in } L^p(\Omega), \\ \int_\Omega \varepsilon^p |\nabla v_\varepsilon|^p dx \rightarrow \frac{1}{m(w)^{p-1}} \int_\Omega |v|^p dx. \end{cases}$$

LEMMA 2.7. Let $v \in L^p(\Omega)$ and $v_\varepsilon \in W_0^{1,p}(\Omega)$ such that

$$\begin{cases} v_\varepsilon = 0 & \text{on } T_\varepsilon \\ v_\varepsilon \rightharpoonup v & \text{in } L^p(\Omega) \\ (\varepsilon|\nabla v_\varepsilon|) & \text{is bounded in } L^p(\Omega) \end{cases}$$

then we have

$$\liminf_\varepsilon \int_\Omega \varepsilon^p |\nabla v_\varepsilon|^p dx \geq \frac{1}{m(w)^{p-1}} \int_\Omega |v|^p dx.$$

PROOF. Let $\varphi \in D(\Omega)$ and $\varphi_\varepsilon = \frac{1}{m(w)} \varphi w_\varepsilon$. By using inequality for convex functions we have

$$\int_\Omega \varepsilon^p |\nabla v_\varepsilon|^p dx \geq \int_\Omega \varepsilon^p |\nabla \varphi_\varepsilon|^p dx + p \int_\Omega |\varepsilon \nabla \varphi_\varepsilon|^{p-2} \varepsilon \nabla \varphi_\varepsilon \varepsilon \nabla (v_\varepsilon - \varphi_\varepsilon) dx.$$

But,

$$\begin{aligned} & \int_\Omega |\varepsilon \nabla \varphi_\varepsilon|^{p-2} \varepsilon \nabla \varphi_\varepsilon \varepsilon \nabla (v_\varepsilon - \varphi_\varepsilon) dx \\ &= \frac{1}{m(w)^{p-1}} \int_\Omega |\varepsilon \nabla (\varphi w_\varepsilon)|^{p-2} \varepsilon \nabla (\varphi w_\varepsilon) \varepsilon \nabla (v_\varepsilon - \varphi_\varepsilon) dx \end{aligned}$$

by using (3.5) of Lemma 3.1 given in [12] we obtain known inequality for $1 < p \leq 2$

$$(I) \quad \exists \alpha > 0 \quad \forall X, Y \in \mathbb{R}^N \quad |X|^{p-2}X - |Y|^{p-2}Y \mid \leq \alpha |X - Y|^{p-1}$$

and then,

$$\begin{aligned} & \left| \int_{\Omega} |\varepsilon \nabla(\varphi w_{\varepsilon})|^{p-2} \varepsilon \nabla(\varphi w_{\varepsilon}) \varepsilon \nabla(v_{\varepsilon} - \varphi_{\varepsilon}) \, dx \right. \\ & \left. - \int_{\Omega} |\varepsilon \varphi \nabla w_{\varepsilon}|^{p-2} \varepsilon \varphi \nabla w_{\varepsilon} \varepsilon \nabla(v_{\varepsilon} - \varphi_{\varepsilon}) \, dx \right| \\ & \leq \alpha \int_{\Omega} |\varepsilon w_{\varepsilon} \nabla \varphi|^{p-1} |\varepsilon \nabla(v_{\varepsilon} - \varphi_{\varepsilon})| \, dx. \end{aligned}$$

Since $\varepsilon w_{\varepsilon} \rightarrow 0$ (by Lemma 2.5) and $(\varepsilon \nabla \varphi_{\varepsilon}), (\varepsilon \nabla v_{\varepsilon})$ bounded in $L^p(\Omega)$ (by Lemma 2.6), we have

$$\begin{aligned} & \liminf_{\varepsilon} \int_{\Omega} |\varepsilon \nabla \varphi_{\varepsilon}|^{p-2} \varepsilon \nabla \varphi_{\varepsilon} \varepsilon \nabla(v_{\varepsilon} - \varphi_{\varepsilon}) \, dx \\ & = \frac{1}{m(w)^{p-1}} \liminf_{\varepsilon} \int_{\Omega} |\varepsilon \varphi \nabla w_{\varepsilon}|^{p-2} \varepsilon \varphi \nabla w_{\varepsilon} \varepsilon \nabla(v_{\varepsilon} - \varphi_{\varepsilon}) \, dx. \end{aligned}$$

Introducing the following function

$$\mathcal{T}_{\varepsilon}(s) = \begin{cases} |s|^{p-2} s, & \text{if } |s| > \varepsilon \\ |\varepsilon|^{p-2} s, & \text{if } |s| \leq \varepsilon \end{cases}$$

we have

$$\begin{aligned} & \int_{\Omega} |\varepsilon \varphi \nabla w_{\varepsilon}|^{p-2} \varepsilon \varphi \nabla w_{\varepsilon} \varepsilon \nabla(v_{\varepsilon} - \varphi_{\varepsilon}) \, dx \\ & = \int_{\Omega} \varepsilon^p |\nabla w_{\varepsilon}|^{p-2} \mathcal{T}_{\varepsilon}(\varphi) \nabla w_{\varepsilon} \varepsilon \nabla(v_{\varepsilon} - \varphi_{\varepsilon}) \, dx \\ & \quad + \int_{\{x \in \Omega: |\varphi| \leq \varepsilon\}} \varepsilon^p |\nabla w_{\varepsilon}|^{p-2} (|\varphi|^{p-2} \varphi - \mathcal{T}_{\varepsilon}(\varphi)) \nabla w_{\varepsilon} \varepsilon \nabla(v_{\varepsilon} - \varphi_{\varepsilon}) \, dx. \end{aligned}$$

But,

$$\begin{aligned} & \left| \int_{\{x \in \Omega: |\varphi| \leq \varepsilon\}} \varepsilon^p |\nabla w_{\varepsilon}|^{p-2} (|\varphi|^{p-2} \varphi - \mathcal{T}_{\varepsilon}(\varphi)) \nabla w_{\varepsilon} \varepsilon \nabla(v_{\varepsilon} - \varphi_{\varepsilon}) \, dx \right| \\ & \leq 2\varepsilon^{p-1} \left| \int_{\Omega} |\varepsilon \nabla w_{\varepsilon}|^{p-1} |\varepsilon \nabla(v_{\varepsilon} - \varphi_{\varepsilon})| \, dx \right| \\ & \leq 2\varepsilon^{p-1} \|\varepsilon \nabla w_{\varepsilon}\|_{(L^p(\Omega))^N}^{p-1} \|\varepsilon \nabla(v_{\varepsilon} - \varphi_{\varepsilon})\|_{(L^p(\Omega))^N} \end{aligned}$$

(from Hölder's inequality). Since the sequences $(|\varepsilon \nabla w_\varepsilon|)$ and $(|\varepsilon \nabla(v_\varepsilon - \varphi_\varepsilon)|)$ are bounded in $L^p(\Omega)$ (due to Lemmas 2.5 and 2.6) we deduce,

$$\begin{aligned} & \liminf_{\varepsilon} \int_{\Omega} \varepsilon^p |\varphi \nabla w_\varepsilon|^{p-2} \varphi \nabla w_\varepsilon \nabla(v_\varepsilon - \varphi_\varepsilon) dx \\ &= \liminf_{\varepsilon} \int_{\Omega} \varepsilon^p |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon \mathcal{T}_\varepsilon(\varphi) \nabla(v_\varepsilon - \varphi_\varepsilon) dx. \end{aligned}$$

Now,

$$\begin{aligned} & \int_{\Omega} \varepsilon^p |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon \mathcal{T}_\varepsilon(\varphi) \nabla(v_\varepsilon - \varphi_\varepsilon) dx \\ &= \int_{\Omega} \varepsilon^p |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon \nabla(\mathcal{T}_\varepsilon(\varphi)(v_\varepsilon - \varphi_\varepsilon)) dx \\ &\quad - \int_{\Omega} \varepsilon^p |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon \nabla(\mathcal{T}_\varepsilon(\varphi)) \cdot (v_\varepsilon - \varphi_\varepsilon) dx \end{aligned}$$

and

$$\begin{aligned} & \left| \int_{\Omega} \varepsilon^p |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon \nabla(\mathcal{T}_\varepsilon(\varphi)) \cdot (v_\varepsilon - \varphi_\varepsilon) dx \right| \\ &\leq c \varepsilon^{p-2} \int_{\Omega} \varepsilon^p |\nabla w_\varepsilon|^{p-1} |v_\varepsilon - \varphi_\varepsilon| dx \\ &\leq c \varepsilon^{p-1} \|\varepsilon \nabla w_\varepsilon\|_{(L^p(\Omega))^N}^{p-1} \|v_\varepsilon - \varphi_\varepsilon\|_{L^p(\Omega)} \end{aligned}$$

(by using again Hölder's inequality). Since the sequences $(|\varepsilon \nabla w_\varepsilon|)$ and $(|v_\varepsilon - \varphi_\varepsilon|)$ are bounded in $L^p(\Omega)$, we deduce again that

$$\begin{aligned} & \liminf_{\varepsilon} \int_{\Omega} \varepsilon^p |\varphi \nabla w_\varepsilon|^{p-2} \varphi \nabla w_\varepsilon \nabla(v_\varepsilon - \varphi_\varepsilon) dx \\ &= \liminf_{\varepsilon} \int_{\Omega} \varepsilon^p |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon \nabla(\mathcal{T}_\varepsilon(\varphi)(v_\varepsilon - \varphi_\varepsilon)) dx \\ &= \liminf_{\varepsilon} \int_{\Omega} \mathcal{T}_\varepsilon(\varphi)(v_\varepsilon - \varphi_\varepsilon) dx \end{aligned}$$

(by the definition of w_ε). Since

$$\begin{aligned} \int_{\Omega} \mathcal{T}_\varepsilon(\varphi)(v_\varepsilon - \varphi_\varepsilon) \, dx &= \int_{\Omega} |\varphi|^{p-2} \varphi (v_\varepsilon - \varphi_\varepsilon) \, dx \\ &\quad + \int_{\{x \in \Omega: |\varphi| \leq \varepsilon\}} (|\varepsilon|^{p-2} \varphi - |\varphi|^{p-2} \varphi) (v_\varepsilon - \varphi_\varepsilon) \, dx \end{aligned}$$

and (from Hölder's inequality)

$$\left| \int_{\{x \in \Omega: |\varphi| \leq \varepsilon\}} (|\varepsilon|^{p-2} \varphi - |\varphi|^{p-2} \varphi) (v_\varepsilon - \varphi_\varepsilon) \, dx \right| \leq c \varepsilon^{p-1} \|v_\varepsilon - \varphi_\varepsilon\|_{L^p(\Omega)},$$

it follows that

$$\begin{aligned} \liminf_{\varepsilon} \int_{\Omega} \varepsilon^p |\varphi \nabla w_\varepsilon|^{p-2} \varphi \nabla w_\varepsilon \nabla (v_\varepsilon - \varphi_\varepsilon) \, dx &= \liminf_{\varepsilon} \int_{\Omega} |\varphi|^{p-2} \varphi (v_\varepsilon - \varphi_\varepsilon) \, dx \\ &= \int_{\Omega} |\varphi|^{p-2} \varphi (v - \varphi) \, dx \end{aligned}$$

(because $v_\varepsilon \rightharpoonup v$ and $\varphi_\varepsilon \rightarrow \varphi$ in $L^p(\Omega)$). Finally,

$$\liminf_{\varepsilon} \int_{\Omega} \varepsilon^p |\nabla v_\varepsilon|^p \, dx \geq \liminf_{\varepsilon} \int_{\Omega} \varepsilon^p |\nabla \varphi_\varepsilon|^p \, dx + \frac{p}{m(w)^{p-1}} \int_{\Omega} |\varphi|^{p-2} \varphi (v - \varphi) \, dx$$

and (from Lemma 2.6)

$$\liminf_{\varepsilon} \int_{\Omega} \varepsilon^p |\nabla v_\varepsilon|^p \, dx \geq \frac{1}{m(w)^{p-1}} \int_{\Omega} |\varphi|^p \, dx + \frac{p}{m(w)^{p-1}} \int_{\Omega} |\varphi|^{p-2} \varphi (v - \varphi) \, dx$$

by choosing φ converges to v in $s - L^p(\Omega)$, we obtain

$$\liminf_{\varepsilon} \int_{\Omega} \varepsilon^p |\nabla v_\varepsilon|^p \, dx \geq \frac{1}{m(w)^{p-1}} \int_{\Omega} |v|^p \, dx.$$

□

PROOF OF THEOREM 2.4. Let $v \in D(\Omega)$. By Lemma 2.6 there exist v_ε in $W_0^{1,p}(\Omega)$ such that

$$\begin{cases} v_\varepsilon = 0 \text{ on } T_\varepsilon, \\ v_\varepsilon \rightharpoonup v \text{ in } L^p(\Omega), \\ \int_{\Omega} \varepsilon^p |\nabla v_\varepsilon|^p \, dx \rightarrow \frac{1}{m(w)^{p-1}} \int_{\Omega} |v|^p \, dx. \end{cases}$$

Now, by Proposition 2.1, the sequence (u_ε^*) is bounded in $L^p(\Omega)$ and admits a weakly converging subsequence in $L^p(\Omega)$ still denoted by (u_ε^*) .

Let $u^* = \lim_{\varepsilon \rightarrow 0} u_\varepsilon^*$ in $w - L^p(\Omega)$. Since

$$\liminf_{\varepsilon} \int_{\Omega} \varepsilon^p |\nabla u_\varepsilon^*|^p dx \geq \frac{1}{m(w)^{p-1}} \int_{\Omega} |u^*|^p dx \quad (\text{from Lemma 2.7})$$

we have

$$\begin{aligned} & \frac{1}{m(w)^{p-1}} \int_{\Omega} |u^*|^p dx - p \int_{\Omega} f u^* dx \\ & \leq \liminf_{\varepsilon} \left(\int_{\Omega} \varepsilon^p |\nabla u_\varepsilon^*|^p dx - p \int_{\Omega} f_\varepsilon u_\varepsilon^* dx \right) \\ & \leq \liminf_{\varepsilon} \left(\int_{\Omega} \varepsilon^p |\nabla v_\varepsilon^*|^p dx - p \int_{\Omega} f_\varepsilon v_\varepsilon^* dx \right) \quad (\text{by the definition of } u_\varepsilon^*) \\ & \leq \frac{1}{m(w)^{p-1}} \int_{\Omega} |v|^p dx - p \int_{\Omega} f v dx \end{aligned}$$

(by Lemma 2.6 and this for all $v \in D(\Omega)$). Then by the density of $D(\Omega)$ in $L^p(\Omega)$ for all v in $L^p(\Omega)$ we have

$$\frac{1}{m(w)^{p-1}} \int_{\Omega} |u^*|^p dx - p \int_{\Omega} f u^* dx \leq \frac{1}{m(w)^{p-1}} \int_{\Omega} |v|^p dx - p \int_{\Omega} f v dx$$

which is equivalent to: u^* is the solution of the minimization problem (D^*) . But the unique solution of the problem (D^*) is $m(w) |f|^{p'-2} f$ then $u^* = m(w) |f|^{p'-2} f$. □

3. ERROR ESTIMATES

In this paragraph, we establish error estimates for the convergence obtained in Theorem 2.4.

THEOREM 3.1. *Assume that (f_ε) is a bounded sequence in $W_0^{1,\infty}(\Omega)$ such that*

$$\|f_\varepsilon - f\|_\infty \leq c \varepsilon^r \quad (r > 0).$$

Then we have

$$\begin{aligned} \text{I)} \quad & \left\| \frac{u_\varepsilon}{\varepsilon^{p'}} - w_\varepsilon |f|^{p'-2} f \right\|_{L^p(\Omega)} \leq \begin{cases} c \varepsilon^{r(p-1)}, & \text{if } 0 < r < 1 \\ c \varepsilon^{p-1}, & \text{if } r \geq 1 \end{cases}, \\ \text{II)} \quad & \left| \int_{\Omega} \frac{|\nabla u_\varepsilon|^p}{\varepsilon^{p'}} dx - \int_{\Omega} w_\varepsilon |f|^{p'} dx \right| \leq \begin{cases} c \varepsilon^{r(p-1)}, & \text{if } 0 < r < 1 \\ c \varepsilon^{p-1}, & \text{if } r \geq 1 \end{cases}. \end{aligned}$$

PROOF OF 1). Let $v_\varepsilon = \frac{u_\varepsilon}{\varepsilon^{p'}} - w_\varepsilon g$ with $g = |f|^{p'-2} f$. Since $v_\varepsilon = 0$ on T_ε we have

$$\|v_\varepsilon\|_{L^p(\Omega)} \leq c\varepsilon \|\nabla v_\varepsilon\|_{(L^p(\Omega))^N}$$

(see proof of Proposition 2.1). Let us estimate $\|\nabla v_\varepsilon\|_{(L^p(\Omega))^N}$:

Using the known inequality for $1 < p \leq 2$ (see for example [12, Lemma 3.1])

$$\exists c > 0 \quad |X - Y|^2 \leq c(|X| + |Y|)^{2-p} (|X|^{p-2} X - |Y|^{p-2} Y)(X - Y)$$

for all $X, Y \in \mathbb{R}^N$ such that $|X| + |Y| \neq 0$, we obtain

$$\begin{aligned} \int_{\Omega} |\nabla v_\varepsilon|^p dx &= \int_{\Omega} \left| \nabla \left(\frac{u_\varepsilon}{\varepsilon^{p'}} \right) - \nabla(w_\varepsilon g) \right|^p dx \\ &\leq c \int_{\Omega} \left(\left| \nabla \left(\frac{u_\varepsilon}{\varepsilon^{p'}} \right) \right| + |\nabla(w_\varepsilon g)| \right)^{\frac{p(2-p)}{2}} \left[\left| \nabla \left(\frac{u_\varepsilon}{\varepsilon^{p'}} \right) \right|^{p-2} \nabla \left(\frac{u_\varepsilon}{\varepsilon^{p'}} \right) \right. \\ &\quad \left. - |\nabla(w_\varepsilon g)|^{p-2} \nabla(w_\varepsilon g) \right] \nabla v_\varepsilon \Big]^{\frac{p}{2}} dx. \end{aligned}$$

Then

$$\begin{aligned} \int_{\Omega} |\nabla v_\varepsilon|^p dx &\leq c M_\varepsilon \left[\int_{\Omega} \left| \nabla \left(\frac{u_\varepsilon}{\varepsilon^{p'}} \right) \right|^{p-2} \nabla \left(\frac{u_\varepsilon}{\varepsilon^{p'}} \right) \nabla v_\varepsilon dx \right. \\ &\quad \left. - \int_{\Omega} |\nabla(w_\varepsilon g)|^{p-2} \nabla(w_\varepsilon g) \nabla v_\varepsilon dx \right]^{\frac{p}{2}} \end{aligned}$$

(by using Hölder's inequality) and where

$$M_\varepsilon = \left[\int_{\Omega} \left(\left| \nabla \left(\frac{u_\varepsilon}{\varepsilon^{p'}} \right) \right| + |\nabla(w_\varepsilon g)| \right)^p dx \right]^{\frac{(2-p)}{2}}.$$

But

$$\begin{aligned} M_\varepsilon &\leq c \left[\int_{\Omega} \left(\left| \nabla \left(\frac{u_\varepsilon}{\varepsilon^{p'}} \right) \right|^p + |\nabla(w_\varepsilon g)|^p \right) dx \right]^{\frac{(2-p)}{2}} \\ &\leq c \left[\int_{\Omega} \left(\left| \nabla \left(\frac{u_\varepsilon}{\varepsilon^{p'}} \right) \right|^p + |w_\varepsilon|^p + |\nabla w_\varepsilon|^p \right) dx \right]^{\frac{(2-p)}{2}} \\ &\leq \frac{c}{\varepsilon^{\frac{p(2-p)}{2}}} \quad (\text{by using Proposition 2.1 and Lemma 2.5}). \end{aligned}$$

By definitions of u_ε and w_ε we have

$$\begin{aligned} \int_{\Omega} \left| \nabla \left(\frac{u_\varepsilon}{\varepsilon^{p'}} \right) \right|^{p-2} \nabla \left(\frac{u_\varepsilon}{\varepsilon^{p'}} \right) \nabla v_\varepsilon \, dx &= \frac{1}{\varepsilon^p} \int_{\Omega} f_\varepsilon v_\varepsilon \, dx \\ &= \int_{\Omega} |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon \nabla (f_\varepsilon v_\varepsilon) \, dx \end{aligned}$$

and then

$$\begin{aligned} \int_{\Omega} |\nabla v_\varepsilon|^p \, dx &\leq c M_\varepsilon \left[\int_{\Omega} |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon f_\varepsilon \nabla v_\varepsilon \, dx \right. \\ &\quad \left. + \int_{\Omega} |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon v_\varepsilon \nabla f_\varepsilon \, dx - \int_{\Omega} |\nabla(w_\varepsilon g)|^{p-2} \nabla(w_\varepsilon g) \nabla v_\varepsilon \, dx \right]^{\frac{p}{2}}. \end{aligned}$$

Since (by Hölder's inequality, by Lemma 2.5 and Proposition 2.1)

$$\begin{aligned} \int_{\Omega} |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon v_\varepsilon \nabla f_\varepsilon \, dx &\leq c \|v_\varepsilon\|_{L^p(\Omega)} \|\nabla w_\varepsilon\|_{(L^p(\Omega))^N}^{p-1} \\ &\leq \frac{c}{\varepsilon^{p-2}} \|\nabla v_\varepsilon\|_{(L^p(\Omega))^N} \end{aligned}$$

and

$$\begin{aligned} &\int_{\Omega} |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon f_\varepsilon \nabla v_\varepsilon \, dx - \int_{\Omega} |\nabla(w_\varepsilon g)|^{p-2} \nabla(w_\varepsilon g) \nabla v_\varepsilon \, dx \\ &= \int_{\Omega} |g_\varepsilon \nabla w_\varepsilon|^{p-2} g_\varepsilon \nabla w_\varepsilon \nabla v_\varepsilon \, dx - \int_{\Omega} |\nabla(w_\varepsilon g)|^{p-2} \nabla(w_\varepsilon g) \nabla v_\varepsilon \, dx \end{aligned}$$

(where $g_\varepsilon = |f_\varepsilon|^{p'-2} f_\varepsilon$) and

$$\begin{aligned} &\int_{\Omega} |g_\varepsilon \nabla w_\varepsilon|^{p-2} g_\varepsilon \nabla w_\varepsilon \nabla v_\varepsilon \, dx - \int_{\Omega} |\nabla(w_\varepsilon g)|^{p-2} \nabla(w_\varepsilon g) \nabla v_\varepsilon \, dx \\ &= \int_{\Omega} |g_\varepsilon \nabla w_\varepsilon|^{p-2} g_\varepsilon \nabla w_\varepsilon \nabla v_\varepsilon \, dx - \int_{\Omega} |g \nabla w_\varepsilon|^{p-2} g \nabla w_\varepsilon \nabla v_\varepsilon \, dx \\ &\quad + \int_{\Omega} |g \nabla w_\varepsilon|^{p-2} g \nabla w_\varepsilon \nabla v_\varepsilon \, dx - \int_{\Omega} |\nabla(w_\varepsilon g)|^{p-2} \nabla(w_\varepsilon g) \nabla v_\varepsilon \, dx \\ &\leq c \left(\int_{\Omega} |(g_\varepsilon - g) \nabla w_\varepsilon|^{p-1} |\nabla v_\varepsilon| \, dx + \int_{\Omega} |w_\varepsilon \nabla g|^{p-1} |\nabla v_\varepsilon| \, dx \right) \end{aligned}$$

(we obtain latest inequality by using again inequality (I))

$$\leq c \left(\varepsilon^{r(p-1)} \int_{\Omega} |\nabla w_{\varepsilon}|^{p-1} |\nabla v_{\varepsilon}| dx + \int_{\Omega} |w_{\varepsilon}|^{p-1} |\nabla v_{\varepsilon}| dx \right)$$

(because $\|f_{\varepsilon} - f\|_{\infty} \leq c\varepsilon^r$ then $\|g_{\varepsilon} - g\|_{\infty} \leq c\varepsilon^r$ too, and this from known inequality for $p' \geq 2$ (see for example Lemma 3.1 in [12])

$$\forall X, Y \in \mathbb{R}^N, \exists c > 0 \left| |X|^{p'-2} X - |Y|^{p'-2} Y \right| \leq c |X - Y| (|X| + |Y|)^{p'-2}$$

then

$$\begin{aligned} & \int_{\Omega} |\nabla w_{\varepsilon}|^{p-2} \nabla w_{\varepsilon} f_{\varepsilon} \nabla v_{\varepsilon} dx - \int_{\Omega} |\nabla(w_{\varepsilon} g)|^{p-2} \nabla(w_{\varepsilon} g) \nabla v_{\varepsilon} dx \\ & \leq c \left(\varepsilon^{r(p-1)} \|\nabla w_{\varepsilon}\|_{(L^p(\Omega))^N}^{p-1} \|\nabla v_{\varepsilon}\|_{(L^p(\Omega))^N} + \|w_{\varepsilon}\|_{L^p(\Omega)}^{p-1} \|\nabla v_{\varepsilon}\|_{(L^p(\Omega))^N} \right) \\ & \leq c \left(\frac{\varepsilon^{r(p-1)}}{\varepsilon^{p-1}} + 1 \right) \|\nabla v_{\varepsilon}\|_{(L^p(\Omega))^N} \end{aligned}$$

(by using again the Lemma 2.5). It follows that

$$\begin{aligned} \int_{\Omega} |\nabla v_{\varepsilon}|^p dx & \leq \frac{c}{\varepsilon^{\frac{p(2-p)}{2}}} \left[\left(\frac{1}{\varepsilon^{p-2}} + \frac{\varepsilon^{r(p-1)}}{\varepsilon^{p-1}} + 1 \right) \|\nabla v_{\varepsilon}\|_{(L^p(\Omega))^N} \right]^{\frac{p}{2}} \\ & \leq \frac{c}{\varepsilon^{\frac{p(2-p)}{2}}} \left[\left(\frac{\varepsilon^{r(p-1)}}{\varepsilon^{p-1}} + 1 \right) \|\nabla v_{\varepsilon}\|_{(L^p(\Omega))^N} \right]^{\frac{p}{2}} \end{aligned}$$

and

$$\|\nabla v_{\varepsilon}\|_{(L^p(\Omega))^N} \leq \frac{c}{\varepsilon^{2-p}} \left(\varepsilon^{(p-1)(r-1)} + 1 \right).$$

Hence,

$$\|v_{\varepsilon}\|_{L^p(\Omega)} \leq \frac{c}{\varepsilon^{1-p}} \left(\varepsilon^{(p-1)(r-1)} + 1 \right) \leq \begin{cases} c\varepsilon^{r(p-1)}, & \text{if } 0 < r < 1, \\ c\varepsilon^{p-1}, & \text{if } r \geq 1. \end{cases}$$

PROOF OF II). Since

$$\frac{1}{\varepsilon^{p'}} \int_{\Omega} |\nabla u_{\varepsilon}|^p dx = \frac{1}{\varepsilon^{p'}} \int_{\Omega} f_{\varepsilon} u_{\varepsilon} dx = \int_{\Omega} f_{\varepsilon} v_{\varepsilon} dx + \int_{\Omega} f_{\varepsilon} w_{\varepsilon} g dx$$

then

$$\begin{aligned} \left| \frac{1}{\varepsilon^{p'}} \int_{\Omega} |\nabla u_{\varepsilon}|^p dx - \int_{\Omega} w_{\varepsilon} |f|^{p'} dx \right| & = \left| \int_{\Omega} f_{\varepsilon} v_{\varepsilon} dx + \int_{\Omega} w_{\varepsilon} (f_{\varepsilon} g - |f|^{p'}) dx \right| \\ & \leq c \|v_{\varepsilon}\|_{L^p(\Omega)} + c\varepsilon^r, \end{aligned}$$

finally we obtain II) by using I). □

REFERENCES

- [1] H. Attouch, Variational convergence for functions and operators, Applicable Mathematics Series, Pitman, London, 1984.
- [2] L. Boukrim, A. Hakim and T. Mekkaoui, *Limiting behaviour and error estimates for the Dirichlet problem with p -Laplacian in a periodically perforated domain*, in preparation.
- [3] D. Cioranescu and F. Murat, *Un terme étrange venu d'ailleurs*, Collège de France Seminar, Research Notes in Mathematics, Pitman, London, 1982, No. 60, pp. 98-138, No. 70, pp. 154-178.
- [4] D. Cioranescu and P. Donato, An Introduction to Homogenization, Oxford Univ. Press, 1999.
- [5] G. Dal Maso, An Introduction to Γ -convergence, Birkhäuser, Boston, 1993.
- [6] G. Dal Maso and P. Longo, Γ -limits of obstacles, Ann. Mat. Pura Appl. **128** (1981), 1-50.
- [7] E. De Giorgi and T. Franzoni, *Su un tipo di convergenza variazionale*, Rend. Acad. Naz. Lincei **58** (1975), 842-850.
- [8] R. Glowinski and A. Marroco, *Sur l'approximation par éléments finis d'ordre un, et la résolution, par pénalisation -dualité, d'une classe de problèmes de Dirichlet non linéaires*, RAIRO **9** (1975), 41-76.
- [9] N. Labani and C. Picard, *Homogenization of nonlinear Dirichlet problem in a perforated domain*, in "Recent advances in nonlinear elliptic and parabolic problems", Pitman Res. Notes Math. Ser. **208** (1989), 296-305.
- [10] T. Mekkaoui and C. Picard, *Error estimates for the homogenization of a quasilinear Dirichlet problem in a periodically perforated domain*, Progress in P.D.E: the Metz Survey 2, M.Chipot Ed., Pitman Res. Notes in Math. Series **296** (1993), 185-193.
- [11] C. Picard, *Analyse limite d'équations variationnelles dans un domaine contenant une grille*, Math. Model. Numer. Anal. **21** (1987), 293-326.
- [12] D. Sandri, *Sur l'approximation numérique des écoulements quasi-newtoniens dont la viscosité suit la loi puissance ou la loi de carreau*, Math. Model. Numer. Anal. **27** (1993), 131-155.
- [13] L. Tartar, Cours Peccot au collège de France, mars 1977, non publié.

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