GLASNIK MATEMATIČKI
Vol. 42(62)(2007), 389 – 399

SEQUENCES OF ITERATES OF RANDOM-VALUED VECTOR FUNCTIONS AND CONTINUOUS SOLUTIONS OF RELATED EQUATIONS

Rafał Kapica
Silesian University, Poland

Abstract. Given a probability space $(\Omega, \mathcal{A}, P)$, a separable metric space X, and a random-valued vector function $f : X \times \Omega \to X$, we obtain some theorems on the existence and on the uniqueness of continuous solutions $\varphi : X \to \mathbb{R}$ of the equation $\varphi(x) = \int_\Omega \varphi(f(x, \omega))P(d\omega)$.

1. Introduction

The basic technique for getting a solution of functional equations in a single variable is iteration. However it may happen that instead of the exact value of a function at a point we know only some parameters of this value. The iterates of such functions were defined independently by K. Baron and M. Kuczma [4] and Ph. Diamond [5]. In [3] and [6, 8] these iterates were applied (for the first time in [3]) to equations of the form

$$\varphi(x) = \int_\Omega \varphi\big(f(x, \omega)\big)P(d\omega). \tag{1.1}$$

Equation (1.1) appears in many branches of mathematics and its solutions φ are extensively studied (see [2, Part 4] and [1, Part 3]). A very particular case of (1.1) was studied by W. Sierpiński in [15] (cf. [9, Theorem 11.11]) to characterize Cantor's function. A more general equation, but still much less general then (1.1), was considered by S. Paganoni Marzegalli [14]. J. Morawiec elaborated on her method in [12] and [13] to the case of (1.1) but on the real line only. The aim of this paper is to enlarge the procedure of J. Morawiec to

2000 *Mathematics Subject Classification.* 39B12, 39B52, 60B12.
Key words and phrases. Random-valued vector functions, sequences of iterates, iterative functional equations, continuous solutions.

get the continuity of the solution given via probability distribution of a limit of the sequence of iterates $\big(f^n(x,\cdot)\big)$ of the given function f in the vector case.

2. Random-valued functions and their iterates

Fix a probability space $(\Omega, \mathcal{A}, P)$ and a separable metric space X. Let $\mathcal{B}(X)$ denote the σ-algebra of all Borel subsets of X. We say that $f : X \times \Omega \to X$ is a random-valued function if it is measurable with respect to the product σ-algebra $\mathcal{B}(X) \otimes \mathcal{A}$. The iterates of such a function f are defined by

$$f^1(x, \omega_1, \omega_2, \dots) = f(x, \omega_1),\ f^{n+1}(x, \omega_1, \omega_2, \dots) = f(f^n(x, \omega_1, \omega_2, \dots), \omega_{n+1})$$

for x from X and $(\omega_1, \omega_2, \dots)$ from Ω^∞ defined as $\Omega^{\mathbb{N}}$. Note that $f^n : X \times \Omega^\infty \to X$ is a random-valued function on the product probability space $(\Omega^\infty, \mathcal{A}^\infty, P^\infty)$. More exactly, the n-th iterate f^n is $\mathcal{B}(X) \otimes \mathcal{A}_n$-measurable, where $\mathcal{A}_n$ denotes the σ-algebra of all the sets of the form

$$\{(\omega_1, \omega_2, \dots) \in \Omega^\infty : (\omega_1, \omega_2, \dots, \omega_n) \in A\}$$

with A from the product σ-algebra $\mathcal{A}^n$. (See [4, 7]; also [10, Sec. 1.4]). Since, in fact, $f^n(\cdot, \omega)$ depends only on the first n coordinates of ω, instead of $f^n(x, \omega_1, \omega_2, \dots)$ we will write also $f^n(x, \omega_1, \dots, \omega_n)$.

3. Main results

Being motivated by the paper [3] (especially by [3, Proposition 2.2]) we will get continuity of the solution of (1.1) given via the probability distribution of the limit of $\big(f^n(x,\cdot)\big)$ (cf. also [8]). For this purpose we will obtain the vector counterparts of [12, Proposition 1, Theorem 1] adopting methods of S. Paganoni Marzegalli and J. Morawiec.

Fix a nonempty set S, and for every $s \in S$ fix a nonempty subset X_s of X and a function $u_s : X_s \to \mathbb{R}$. We are interested in solutions $\varphi : X \to \mathbb{R}$ of (1.1) in the class $\mathcal{F}$ defined by

$$\mathcal{F} = \{\varphi : X \to \mathbb{R} \quad | \quad \varphi \text{ is a bounded function, } \varphi(x) = u_s(x) \text{ for } x \in X_s \text{ and } s \in S\}.$$

First we prove a theorem on the existence and uniqueness of such solutions accepting the following assumptions:

(A) For every $s \in S$ there exist: an open set $U_s \subset X$, an event $A_s \in \mathcal{A}$ of positive probability and a positive integer m such that

$$f^m(U_s \times A_s^{\mathbb{N}}) \subset X_s; \tag{3.1}$$

moreover, for some $s_0 \in S$ the function $f(\cdot, \omega)$ is continuous for $\omega \in A_{s_0}$ and there exists an $m_0 \in \mathbb{N}$ such that

$$f^{m_0}\big((X \setminus \bigcup_{s \in S} U_s) \times A_{s_0}^{\mathbb{N}}\big) \subset \bigcup_{s \in S} U_s. \tag{3.2}$$

The following theorem is an extension of [12, Proposition 1].

THEOREM 3.1. *Assume* (A). *If the closure of* $X \setminus \bigcup_{s\in S} X_s$ *is compact, then equation (1.1) has in the class* $\mathcal{F}$ *at most one solution.*

PROOF. Assume that $\varphi_1, \varphi_2 \in \mathcal{F}$ are solutions of (1.1) and put $\varphi = \varphi_1 - \varphi_2$. Clearly φ is a solution of (1.1) and

$$\varphi(x) = 0 \qquad \text{for } x \in \bigcup_{s\in S} X_s. \tag{3.3}$$

Suppose that

$$M := \sup\{|\varphi(x)| : x \in X\} > 0$$

and consider the set

$$Y = \{x \in X : \text{there exists a sequence } (x_n) \text{ such that } \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} |\varphi(x_n)| = M\}.$$

Since $M > 0$, (3.3) and compactness of $\mathrm{cl}(X \setminus \bigcup_{s\in S} X_s)$ show that the set Y is nonempty. We will prove that $U_s \cap Y = \emptyset$ for every $s \in S$. To get this suppose that $x \in U_s \cap Y$ for some $s \in S$. Then

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} |\varphi(x_n)| = M \tag{3.4}$$

for some sequence (x_n) of points of U_s. Applying (1.1), (3.1) and (3.3) we see that

$$\begin{aligned}
|\varphi(x_n)| &= \Big|\int_\Omega \big(\dots\big(\int_\Omega \varphi(f^m(x_n,\omega_1,\dots,\omega_m))P(d\omega_m)\big)\dots\big)P(d\omega_1)\Big| \\
&\leqslant \int_{A_s} \big(\dots\big(\int_{A_s} |\varphi(f^m(x_n,\omega_1,\dots,\omega_m))|P(d\omega_m)\big)\dots\big)P(d\omega_1) \\
&\qquad + MP^\infty\{(\omega_1,\omega_2,\dots) \in \Omega^\infty : \ (\omega_1,\dots,\omega_m) \notin A_s^m\} \\
&= M\big(1 - P(A_s)^m\big)
\end{aligned}$$

for every $n \in \mathbb{N}$, which is a contradiction. Consequently,

$$Y \subset X \setminus \bigcup_{s\in S} U_s. \tag{3.5}$$

Now fix an $x \in Y$ and an (x_n) satisfying (3.4). Applying Fatou's Lemma and (1.1) we obtain

$$\begin{aligned}
0 &\leqslant \int_\Omega \liminf_{n\to\infty} \big(M - \big|\varphi\big(f(x_n,\omega)\big)\big|\big)P(d\omega) \\
&\leqslant \liminf_{n\to\infty} \int_\Omega \big(M - \big|\varphi\big(f(x_n,\omega)\big)\big|\big)P(d\omega) \\
&\leqslant \liminf_{n\to\infty} \big(M - |\varphi(x_n)|\big) = 0.
\end{aligned}$$

This gives $\liminf_{n\to\infty}\big(M-\big|\varphi\big(f(x_n,\omega)\big)\big|\big)=0$ a.e. In particular,

$$\limsup_{n\to\infty}\big|\varphi\big(f(x_n,\omega_1)\big)\big|=M$$

for some $\omega_1\in A_{s_0}$. By the continuity of $f(\cdot,\omega_1)$ we have $f(x,\omega_1)\in Y$. Replacing x by $f(x,\omega_1)$ we can find $\omega_2\in A_{s_0}$ such that $f(f(x,\omega_1),\omega_2)\in Y$, i.e. $f^2(x,\omega_1,\omega_2)\in Y$. After m_0 steps we obtain a sequence $\omega_1,\dots,\omega_{m_0}$ of elements of A_{s_0} such that

$$f^{m_0}(x,\omega_1,\dots,\omega_{m_0})\in Y.$$

On the other hand, on account of (3.5) and (3.2), $f^{m_0}(x,\omega_1,\dots,\omega_{m_0})$ belongs to $\bigcup_{s\in S}U_s$ which is a contradiction. ∎

Now fix a family $\mathcal{F}_0\subset\mathcal{F}$. We will prove a theorem on the existence and on the uniqueness of solutions of (1.1) in the class $\mathcal{F}_0$ under the following assumptions:

(B) There exist an $m\in\mathbb{N}$ and $U_s\subset X$, $A_s\in\mathcal{A}$ for $s\in S$ such that

$$\inf\{P(A_s):s\in S\}>0,$$

condition (3.1) holds for every $s\in S$, and for some $s_0\in S$ we have

$$f^m\big((X\setminus\bigcup_{s\in S}U_s)\times A_{s_0}^{\mathbb{N}}\big)\subset\bigcup_{s\in S}X_s. \tag{3.6}$$

(C) For every $\varphi\in\mathcal{F}_0$ the function $\varphi\circ f(x,\cdot)$ is measurable for $x\in X$, and the function ψ given by

$$\psi(x)=\int_\Omega\varphi\big(f(x,\omega)\big)P(d\omega) \tag{3.7}$$

belongs to $\mathcal{F}_0$.

In the proof of the next theorem we will integrate nonnegative functions possibly nonmeasurable. If $A\in\mathcal{A}$ and $h:A\to[0,\infty)$, then

$$\int_A h(\omega)P(d\omega)=\sup_\Pi\sum_{E\in\Pi}P(E)\inf h(E)$$

where the supremum is taken over all partitions Π of A into a countable number of pairwise disjoint members of $\mathcal{A}$ (cf. [11, p. 117]).

THEOREM 3.2. *Assume* (B) *and* (C). *If $\mathcal{F}_0$ is nonempty and closed in uniform convergence, then equation (1.1) has in $\mathcal{F}_0$ exactly one solution.*

PROOF. Consider the operator $L:\mathcal{F}_0\to\mathcal{F}_0$ given by

$$L\varphi(x)=\int_\Omega\varphi\big(f(x,\omega)\big)P(d\omega).$$

It is enough to prove that $L^m : \mathcal{F}_0 \to \mathcal{F}_0$ is a contraction in the supremum metric τ. To this end we will show (by induction) that for every $n \in \mathbb{N}$, $\varphi_1, \varphi_2 \in \mathcal{F}_0$, $x \in X$ and $A \in \mathcal{A}$ the following inequality holds:

$$|L^n\varphi_1(x) - L^n\varphi_2(x)| \leqslant \tau(\varphi_1,\varphi_2)\big(1 - P(A)^n\big) + \int_A \big(\dots\big(\int_A \big|(\varphi_1 - \varphi_2)\big(f^n(x,\omega_1,\dots,\omega_n)\big)\big|P(d\omega_n)\big)\dots\big)P(d\omega_1). \tag{3.8}$$

In fact, if $\varphi_1, \varphi_2 \in \mathcal{F}_0$, then putting $\varphi = \varphi_1 - \varphi_2$, for every $x \in X$ and $A \in \mathcal{A}$ we have

$$\begin{aligned}|L\varphi_1(x) - L\varphi_2(x)| &\leqslant \int_{\Omega\setminus A} \big|\varphi\big(f(x,\omega)\big)\big|P(d\omega) + \int_A \big|\varphi\big(f(x,\omega)\big)\big|P(d\omega)\\ &\leqslant \tau(\varphi_1,\varphi_2)(1 - P(A)) + \int_A \big|\varphi\big(f(x,\omega)\big)\big|P(d\omega)\end{aligned}$$

and

$$\begin{aligned}&|L^{n+1}\varphi_1(x) - L^{n+1}\varphi_2(x)| = |L^nL\varphi_1(x) - L^nL\varphi_2(x)|\\ &\quad\leqslant \tau(L\varphi_1, L\varphi_2)\big(1 - P(A)^n\big)\\ &\qquad+ \int_A \big(\dots\big(\int_A \big|(L\varphi_1 - L\varphi_2)\big(f^n(x,\omega_1,\dots,\omega_n)\big)\big|P(d\omega_n)\big)\dots\big)P(d\omega_1)\\ &\quad\leqslant \tau(\varphi_1,\varphi_2)\big(1 - P(A)^n\big)\\ &\qquad+ \int_A \big(\dots\big(\int_A \big\{\tau(\varphi_1,\varphi_2)(1 - P(A))\\ &\qquad+ \int_A \big|\varphi\big(f(f^n(x,\omega_1,\dots,\omega_n),\omega_{n+1})\big)\big|P(d\omega_{n+1})\big\}P(d\omega_n)\big)\dots\big)P(d\omega_1)\\ &\quad= \tau(\varphi_1,\varphi_2)\big(1 - P(A)^n\big) + \tau(\varphi_1,\varphi_2)\big(1 - P(A)\big)P(A)^n\\ &\qquad+ \int_A \big(\dots\big(\int_A \big|\varphi\big(f^{n+1}(x,\omega_1,\dots,\omega_{n+1})\big)\big|P(d\omega_{n+1})\big)\dots\big)P(d\omega_1).\end{aligned}$$

Fix $\varphi_1, \varphi_2 \in \mathcal{F}_0$ and, using (B), fix also an $m \in \mathbb{N}$ satisfying (3.1) and (3.6). If $s \in S$ and $x \in U_s$, then by (3.8) and (3.1) we have

$$|L^m\varphi_1(x) - L^m\varphi_2(x)| \leqslant \tau(\varphi_1,\varphi_2)\big(1 - P(A_s)^m\big),$$

whilst if $x \in X \setminus \bigcup_{s\in S} U_s$, then (3.8) and (3.6) give

$$|L^m\varphi_1(x) - L^m\varphi_2(x)| \leqslant \tau(\varphi_1,\varphi_2)\big(1 - P(A_{s_0})^m\big).$$

By this we obtain

$$|L^m\varphi_1(x) - L^m\varphi_2(x)| \leqslant \tau(\varphi_1,\varphi_2)\sup\{1 - P(A_s)^m : s \in S\}$$

for every $x \in X$ and, consequently,

$$\tau(L^m\varphi_1, L^m\varphi_2) \leqslant \tau(\varphi_1,\varphi_2)\sup\{1 - P(A_s)^m : s \in S\}.$$

□

REMARK 3.3. Under the assumptions of Theorems 3.1 and 3.2 equation (1.1) has in $\mathcal{F}$ exactly one solution and this solution belongs to $\mathcal{F}_0$.

Now we proceed to the case where

$$\mathcal{F}_0 = \{\varphi : X \to \mathbb{R} \mid \varphi \text{ is a bounded continuous function}, \\ \varphi(x) = 0 \text{ for } x \in X_1,\ \varphi(x) = 1 \text{ for } x \in X_2\}$$

for some Borel subsets $X_1, X_2 \subset X$, assuming the following:

(D) There exist open sets $U_1, U_2 \subset X$, events A_1, A_2 of positive probability, and an $m \in \mathbb{N}$ such that (3.1) holds for $s \in \{1, 2\}$,

$$f^m\big((X \setminus (U_1 \cup U_2)) \times A_1^{\mathbb{N}}\big) \subset (X_1 \cup X_2) \cap (U_1 \cup U_2),$$

$$f(X_1 \times \Omega) \subset X_1, \quad f(X_2 \times \Omega) \subset X_2, \tag{3.9}$$

$f(\cdot, \omega)$ is continuous for every $\omega \in A_1$ and f is P-continuous (i.e., if $x_n \to x$, then $f(x_n, \cdot) \to f(x, \cdot)$ in probability).

The main result of this paper, which is a generalization of [3, Proposition 2.2], reads as follows.

THEOREM 3.4. *Assume* (D), $\operatorname{dist}(X_1, X_2) > 0$ *and that* $\operatorname{cl}\big(X \setminus (X_1 \cup X_2)\big)$ *is compact. Then*:

(i) *Equation (1.1) has exactly one bounded solution* $\varphi : X \to \mathbb{R}$ *such that*

$$\varphi(x) = 0 \quad \text{for } x \in X_1, \qquad \varphi(x) = 1 \quad \text{for } x \in X_2; \tag{3.10}$$

this solution is a continuous function.

(ii) *If* X *is complete and the function* $\pi : X \times \mathcal{B}(X) \to [0, 1]$ *given by*

$$\pi(x, B) = P^\infty\big(\{\omega \in \Omega^\infty : \text{ the sequence } \big(f^n(x, \omega)\big) \\ \text{converges and its limit belongs to } B\}\big) \tag{3.11}$$

satisfies

$$\pi(x, X_2) = 0 \quad \text{for } x \in X_1, \qquad \pi(x, X_2) = 1 \quad \text{for } x \in X_2,$$

then $\pi(\cdot, X_2)$ *is a continuous solution of (1.1).*

(iii) *If for every* $x \in X$ *the sequence* $\big(f^n(x, \cdot)\big)$ *converges in probability to a random variable* $\xi(x, \cdot)$, *and the function* $\pi : X \times \mathcal{B}(X) \to [0, 1]$ *given by*

$$\pi(x, B) = P^\infty(\xi(x, \cdot) \in B) \tag{3.12}$$

satisfies

$$\pi(x, X_1) = 1 \quad \text{for } x \in X_1, \qquad \pi(x, X_2) = 1 \quad \text{for } x \in X_2, \tag{3.13}$$

then for every bounded and continuous function $u : X \to \mathbb{R}$ such that

$$u(x) = 0 \quad \text{for } x \in X_1, \qquad u(x) = 1 \quad \text{for } x \in X_2, \tag{3.14}$$

the function $\varphi : X \to \mathbb{R}$ defined by

$$\varphi(x) = \int_X u(y)\pi(x, dy) = \int_{\Omega^\infty} u\big(\xi(x,\omega)\big) P^\infty(d\omega) \tag{3.15}$$

is a continuous solution of equation (1.1) and has property (3.10).

Proof. Since $\mathrm{cl}X_1$ and $\mathrm{cl}X_2$ are disjoint, the family $\mathcal{F}_0$ is nonempty. It is also closed in the uniform convergence. Fix a $\varphi \in \mathcal{F}_0$. By the continuity of φ the function $\varphi \circ f(x, \cdot)$ is measurable for every $x \in X$. Consider the function $\psi : X \to \mathbb{R}$ defined by (3.7). Obviously ψ is a bounded function, $\psi(x) = 0$ for $x \in X_1$ and $\psi(x) = 1$ for $x \in X_2$. We will prove that ψ is continuous. If the sequence (x_n) of points of X converges to an x, then the sequence $\big(\varphi \circ f(x_n, \cdot)\big)$ of uniformly bounded functions converges in probability to $\varphi \circ f(x, \cdot)$ and on account of the Lebesgue-Vitali Dominated Convergence Theorem the sequence $\big(\psi(x_n)\big)$ converges to $\psi(x)$. This shows (C) with

$$S = \{1, 2\}, \quad u_1 = 0, \quad u_2 = 1.$$

Clearly, conditions (A) and (B) are fulfilled. Applying Remark 3.3 we get the first assertion.

To prove the second one it is enough to observe that by [8, Theorem 1] (for $u = \mathbf{1}_{X_2}$) the function $\pi(\cdot, X_2)$ is a (bounded) solution of (1.1) and to apply (i).

Passing to a proof of the third assertion fix a $u \in \mathcal{F}_0$. According to [8, Theorem 2.(i)] the function $\varphi : X \to \mathbb{R}$ given by (3.15) is a bounded solution of (1.1). In view of the first part of Theorem 3.4 it is enough to verify that φ satisfies (3.10). This however follows immediately from (3.13) and (3.14): if $x \in X_1$, then

$$\varphi(x) = \int_{X_1} u(y)\pi(x, dy) = 0,$$

and for $x \in X_2$ we have

$$\varphi(x) = \int_{X_2} u(y)\pi(x, dy) = 1.$$

□

4. Examples

The following shows a possible application of Theorem 3.4.

Fix an $N \in \mathbb{N}$ and let $X = [0,1]^N$.

Denoting the set $\{1, \dots, N\}$ by I, define the subsets X_1, X_2 and U_1, U_2 of X as follows:

$$X_1 = \{0\}, \quad X_2 = \{x \in X : x_n = 1 \text{ for some } n \in I\},$$

$$U_1 = \{x \in X : x_n < b \text{ for } n \in I\}, \quad U_2 = \{x \in X : x_n > a \text{ for some } n \in I\},$$

where $0 < b < a < 1$ are fixed. Assume that $\alpha_1, \ldots, \alpha_N : [0,1] \to [0,1]$ are nondecreasing continuous functions such that

$$\alpha_n(t) = 0 \quad \text{for } t \in [0,b], \qquad \alpha_n(1) = 1 \qquad \text{and} \qquad \alpha_n(t) < t \quad \text{for } t \in (0,1), \tag{4.1}$$

and let $v_1, \ldots, v_N, w_1, \ldots, w_N : X \to [0,1]$ be continuous functions. Given $p_1 > 0$ and $p_2 > 0$ summing up to 1, consider also $\Omega = \{\omega_1, \omega_2\}$ and define the function $f : X \times \Omega \to X$ by

$$f(x, \omega_i) = f_i(x),$$

where

$$f_1(x) = \big(\alpha_1(v_1(x)), \ldots, \alpha_N(v_N(x))\big), \quad f_2(x) = \big(w_1(x), \ldots, w_N(x)\big).$$

Since f_1, f_2 are continuous, it follows that f is random-valued. Equation (1.1) takes the form

$$\varphi(x) = p_1\varphi\big(\alpha_1(v_1(x)), \ldots, \alpha_N(v_N(x))\big) + p_2\varphi\big(w_1(x), \ldots, w_N(x)\big). \tag{4.2}$$

(I) Assume that

$$v_1(x), \ldots, v_N(x) \leqslant \max\{x_1, \ldots, x_N\} \qquad \text{for } x \in X \setminus U_2, \tag{4.3}$$

$$\max\{v_1(x), \ldots, v_N(x)\} = 1 \qquad \text{for } x \in X_2, \tag{4.4}$$

$$\max\{w_1(x), \ldots, w_N(x)\} = 1 \qquad \text{for } x \in U_2, \tag{4.5}$$

$$w_1(0) = \ldots = w_N(0) = 0. \tag{4.6}$$

We will show that:

(i) *Equation* (4.2) *has exactly one bounded solution* $\varphi : X \to [0,1]$ *satisfying*

$$\varphi(0) = 0 \qquad \text{and} \qquad \varphi(x) = 1 \quad \text{for } x \in X_2; \tag{4.7}$$

this solution is a continuous function.

(ii) *If the function* π *given by* (3.11) *fullfils*

$$\pi(x, X_2) = 1 \qquad \text{for } x \in X_2, \tag{4.8}$$

then $\pi(\cdot, X_2)$ *is a continuous solution of* (4.2).

Proof. First we show that (D) holds. Let $A_1 = \{\omega_1\}, A_2 = \{\omega_2\}$. We claim that

$$f_1(U_1) \subset X_1, \quad f_2(U_2) \subset X_2. \tag{4.9}$$

If $x \in U_1$, then $x_n < b$ for $n \in I$ and according to (4.3) we have $v_n(x) < b$ for $n \in I$, hence by (4.1) we see that $\alpha_n(v_n(x)) = 0$ for $n \in I$, i.e. $f_1(x) = 0$. If $x \in U_2$, then (4.5) gives $f_2(x) \in X_2$. From this (4.9) follows, and since $X_1 \subset U_1$ and $X_2 \subset U_2$, we have (3.1) for every $m \in \mathbb{N}$ and $s \in \{1, 2\}$.

Similarly we verify that (3.9) holds. The task is now to find a positive integer m with

$$f_1^m(x) = 0 \quad \text{for } x \in X \setminus U_2.$$

Put $\alpha(t) = \max\{\alpha_1(t), \dots, \alpha_N(t)\}$ for $t \in [0,1]$. Clearly, α is a continuous nondecreasing function,

$$\alpha(t) = 0 \quad \text{for } t \in [0,b] \qquad \text{and} \qquad \alpha(t) < t \quad \text{for } t \in (0,1).$$

In particular, $\lim_{m\to\infty} \alpha^m(a) = 0$. Hence $\alpha^m(a) = 0$ for some $m \in \mathbb{N}$. Fix an $x \in X \setminus U_2$. By the monotonicity of α and (4.3) we have

$$\begin{aligned} f_1(x) &\leqslant \big(\alpha(v_1(x)), \dots, \alpha(v_N(x))\big) \leqslant \\ &\leqslant \big(\alpha(\max\{x_1, \dots, x_N\}), \dots, \alpha(\max\{x_1, \dots, x_N\})\big), \end{aligned}$$

whence

$$f_1(x) \leqslant \big(\alpha(a), \dots, \alpha(a)\big) \leqslant (a, \dots, a).$$

In particular, $f_1(x) \in X \setminus U_2$ and since $x \in X \setminus U_2$ was arbitrarily fixed we can replace it by $f_1(x)$ to get

$$\begin{aligned} f_1^2(x) &\leqslant \big(\alpha(\max\{(f_1(x))_n : n \in I\}), \dots, \alpha(\max\{(f_1(x))_n : n \in I\})\big) \\ &\leqslant \big(\alpha^2(\max\{x_1, \dots, x_N\}), \dots, \alpha^2(\max\{x_1, \dots, x_N\})\big) \\ &\leqslant \big(\alpha^2(a), \dots, \alpha^2(a)\big). \end{aligned}$$

After m steps

$$f_1^m(x) \leqslant \big(\alpha^m(a), \dots, \alpha^m(a)\big)$$

and $f_1^m(x) = 0$. This ends the proof of (D).

Consequently Theorem 3.4(i) yields part (i) of our example.

Since $f_1(0) = f_2(0) = 0$, we conclude that for π given by (3.11) we have $\pi(0, X_2) = 0$. The continuity of $\pi(\cdot, X_2)$ follows from (4.8) and Theorem 3.4(ii). ∎

Consider now continuous functions $\beta_1, \dots, \beta_N : [0,1] \to [0,1]$ such that

$$\beta_n(0) = 0, \qquad \beta_n(t) = 1 \quad \text{for } t \in [a,1],\ n \in I.$$

(II) The functions $v_1, \dots, v_N, w_1, \dots, w_N$ defined by

$$v_n(x) = \max\{x_1, \dots, x_N\}, \quad w_n(x) = \beta_n(\min\{x_1 + \ldots + x_N, 1\}) \qquad \text{for } x \in X$$

satisfy (4.3) - (4.6). By Example (I).(i) the equation

$$\begin{aligned} \varphi(x) \ &= \ p_1\varphi\big(\alpha_1(\max\{x_1, \dots, x_N\}), \dots, \alpha_N(\max\{x_1, \dots, x_N\})\big) \\ &\quad +\varphi\big(\beta_1(\min\{x_1 + \ldots + x_N, 1\}), \dots, \beta_N(\min\{x_1 + \ldots + x_N, 1\})\big) \end{aligned} \tag{4.10}$$

has exactly one bounded solution $\varphi : X \to \mathbb{R}$ satisfying (4.7) and this solution is a continuous function. We will show that it equals to

$$x \mapsto P^\infty\big(\lim_{n\to\infty} f^n(x, \cdot) = (1, \dots, 1)\big), \quad x \in X. \tag{4.11}$$

In fact, according to [8, Theorem 1 (with $u = \mathbf{1}_{\{(1,\dots,1)\}}$)] the function (4.11) is a (bounded) solution of (4.10). If $x \in X_2$, then

$$v_n(x) = 1 = \min\{x_1 + \ldots + x_N, 1\} \qquad \text{for } n \in I,$$

whence $f(x, \omega_i) = (1, \ldots, 1) \in X_2$ for $i = 1, 2$. Consequently

$$f^n(x, \omega) = (1, \ldots, 1) \qquad \text{for } n \in \mathbb{N},\ x \in X_2 \text{ and } \omega \in \Omega^\infty,$$

and the function (4.11) takes the value 1 on X_2. Moreover, $f(0, \omega_i) = 0$ for $i = 1, 2$, whence $f^n(0, \omega) = 0$ for $n \in \mathbb{N}$ and $\omega \in \Omega^\infty$ and, consequently, $\pi(0, \cdot) = 0$.

(III) Define now the functions $v_1, \ldots, v_N, w_1, \ldots, w_N$ by

$$v_n(x) = x_n, \quad w_n(x) = \beta_n(x_n) \qquad \text{for } x \in X.$$

Clearly (4.3)–(4.6) are fulfilled. Consequently the equation

$$\varphi(x) = p_1\varphi\big(\alpha_1(x_1), \ldots, \alpha_N(x_N)\big) + p_2\varphi\big(\beta_1(x_1), \ldots, \beta_N(x_N)\big) \tag{4.12}$$

has exactly one bounded solution $\varphi : X \to \mathbb{R}$ satisfying (4.7). Assume additionally (cf. [3, Example 2.1]) that $p_2 \leqslant b$ and

$$\alpha_n(t) = 0 \quad \text{for } t \in [0, a], \qquad \alpha_n(t) \leqslant \frac{t - p_2}{p_1} \quad \text{for } t \in [a, 1],$$

$$\beta_n(t) = 1 \quad \text{for } t \in [b, 1], \qquad \beta_n(t) \leqslant \frac{t}{p_2} \quad \text{for } t \in [0, b],$$

for $n \in I$. Then

$$p_1\alpha_n(t) + p_2\beta_n(t) \leqslant t \qquad \text{for } t \in [0, 1] \text{ and } n \in I,$$

and

$$p_1 f_1(x) + p_2 f_2(x) \leqslant x \qquad \text{for } x \in X.$$

Due to [7, Theorem 4] for every $x \in X$ the sequence $\big(f^n(x, \cdot)\big)$ converges a.s. to a measurable function $\xi(x, \cdot) : \Omega^\infty \to X$. In particular, the functions (3.11) and (3.12) coincide. Since $f_1(X_2) \subset X_2, f_2(X_2) \subset X_2$, we have

$$f^n(x, \omega) \in X_2 \qquad \text{for } x \in X_2,\ \omega \in \Omega^\infty,\ n \in \mathbb{N}.$$

This gives (4.8), because X_2 is closed. Thus $\pi(\cdot, X_2)$ is a continuous solution of (4.12).

Acknowledgements.

The research was supported by the Silesian University Mathematics Department (Iterative Functional Equations and Real Analysis program).

References

[1] K. Baron, *Recent results in the theory of functional equations in a single variable*, Seminar LV **15** (2003), 16 pp.

[2] K. Baron, W. Jarczyk, *Recent results on functional equations in a single variable, perspectives and open problems*, Aequationes Math. **61** (2001), 1-48.

[3] K. Baron, W. Jarczyk, *Random-valued functions and iterative functional equations*, Aequationes Math. **67** (2004), 140-153.

[4] K. Baron, M. Kuczma, *Iteration of random-valued functions on the unit interval*, Colloq. Math. **37** (1977), 263-269.

[5] Ph. Diamond, *A stochastic functional equation*, Aequationes Math. **15** (1977), 225-233.

[6] R. Kapica, *Sequences of iterates of random-valued vector functions and continuous solutions of a linear functional equation of infinite order*, Bull. Polish Acad. Sci. Math. **50** (2002), 447-455.

[7] R. Kapica, *Convergence of sequences of iterates of random-valued vector functions*, Colloq. Math. **97** (2003), 1-6.

[8] R. Kapica, *Sequences of iterates of random-valued vector functions and solutions of related equations*, Sitzungsber. Abt. II **213** (2004), 113-118.

[9] M. Kuczma, Functional equations in a single variable, Państwowe Wydawnictwo Naukowe, Warszawa, 1968.

[10] M. Kuczma, B. Choczewski, R. Ger, Iterative functional equations, Cambridge University Press, Cambridge, 1990.

[11] S. Łojasiewicz, An introduction to the theory of real functions, John Wiley and Sons, New York 1988.

[12] J. Morawiec, *On a linear functional equation*, Bull. Polish Acad. Sci. Math. **43** (1995), 131-142.

[13] J. Morawiec, *Some properties of probability distribution solutions of linear functional equations*, Aequationes Math. **56** (1998), 81-90.

[14] S. Paganoni Marzegalli, *One-parameter system of functional equations*, Aequationes Math. **47** (1994), 50-59.

[15] W. Sierpiński, *Sur un système d'equations fonctionnelles, définissant une fonction avec un ensemble dense d'intervalles d'invariabilité*, Bull. Intern. Acad. Sci. Cracovie A (1911), 577-582.

R. Kapica
Institute of Mathematics
Silesian University
Bankowa 14
PL-40-007 Katowice
Poland
E-mail: rkapica@ux2.math.us.edu.pl

Received: 2.1.2006.
Revised: 4.7.2006.