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MORPHISMS OF EXTENSIONS OF HILBERT C^*-MODULES

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Abstract. We consider the condition for a morphism of (between) extensions of Hilbert C^*-modules to exist and give the description of a morphism out of an extension of a Hilbert C^*-module in a general case.

1. Preliminary definitions

The definition of extensions of Hilbert C^*-modules is relatively recent. It was announced in [2, Example 2.10] and developed in a series of papers ([3, 4, 5]). For the sake of completeness we recall here some basic definitions.

Let V be a (right) Hilbert C^*-module over a C^*-algebra A (a Hilbert A-module) and let W be a Hilbert B-module over some C^*-algebra B. A map $\Phi : V \to W$ is called a (φ-)morphism of Hilbert C^*-modules if there is a morphism (i.e. a $*$-homomorphism) $\varphi : A \to B$ of underlying C^*-algebras such that $(\Phi(v_1) \mid \Phi(v_2)) = \varphi((v_1 \mid v_2))$ is satisfied for all $v_1, v_2 \in V$. Each morphism of Hilbert C^*-modules is necessarily linear, contractive and a module map (in a sense that $\Phi(va) = \Phi(v)\varphi(a)$ is valid for all $v \in V, a \in A$).

The ideal submodule V_I of V associated to an ideal $I \subseteq A$ is $V_I = VI = \{vb : v \in V, b \in I\} = \{v \in V : (v \mid v) \in I\}$. By $\Pi : V \to V|_{V_I}$, $\pi : A \to A|_I$ we denote canonical quotient maps. A quotient $V|_{V_I}$ has a natural Hilbert $A|_I$-module structure via operations: $\Pi(v)\pi(a) = \Pi(va)$, $(\Pi(v_1) \mid \Pi(v_2)) = \pi((v_1 \mid v_2))$. Obviously Π is a π-morphism.

A Hilbert A-module V is said to be full if $(V \mid V)$ (the (closed) ideal in A generated by elements $(v_1 \mid v_2)$, $v_1, v_2 \in V$) is all of A. We may freely suppose that V is full because we can always consider V as a full Hilbert $(V \mid V)$-module.

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We say that a Hilbert C^*-module W is an extension of a (full) Hilbert C^*-module V if we have an exact sequence of Hilbert C^*-modules and morphisms of modules

$$0 \longrightarrow V \xrightarrow{\ \Phi\ } W \xrightarrow{\ \Pi\ } W|_{\Phi(V)} \longrightarrow 0.$$

Associated with this sequence there is an exact sequence of underlying C^*-algebras

$$0 \longrightarrow A \xrightarrow{\ \varphi\ } B \xrightarrow{\ \pi\ } B|_{\varphi(A)} \longrightarrow 0.$$

So, we have: W is a Hilbert B-module, Φ is a φ-morphism, $\varphi(A)$ is an ideal in B and $\Phi(V)$ is the ideal submodule of W associated to $\varphi(A)$ i.e. $\Phi(V) = W\varphi(A)$. Another notation for an extension W of V is as a triple (W, B, Φ). An extension W of a Hilbert C^*-module V is called essential if $\varphi(A)$ is an essential ideal in B. An extension (W, B, Φ) of V is called full if W is a full Hilbert B-module.

There is a maximal essential extension of a Hilbert C^*-module V over a C^*-algebra A. Namely this is $M(V) = \mathbf{B}(A, V)$, the Hilbert C^*-module over the multiplier C^*-algebra $M(A)$ of A, consisting of all adjointable maps from A to V with the inner product $(r \mid s) = r^*s$. A map $\Gamma : V \to M(V)$ defined as $\Gamma(v) = l_v$ where $l_v : A \to V$ is given by $l_v(a) = va$ is a γ-morphism (for $\gamma : A \to M(A)$ such that $\gamma(a) = T_a$ where $T_a(b) = ab$, $b \in A$). The triple $(M(V), M(A), \Gamma)$ is evidently an essential extension of V. Further, if (W, B, Φ) is an arbitrary extension of V and $\lambda : B \to M(A)$ the unique morphism of C^*-algebras such that $\lambda|_A = id_A$, then there is the unique λ-morphism $\Lambda : W \to M(V)$ such that $\Lambda\Phi = \Gamma$ ([3, Theorem 1.1]). Let us denote by $Q(V)$ the quotient $M(V)|_{\Gamma(V)}$. After the identification of $\Gamma(V)$ with V (via $l_v \leftrightarrow v$), $Q(V) = M(V)|_V$. So we have the exact sequence

$$0 \longrightarrow V \xrightarrow{\ \Gamma\ } M(V) \xrightarrow{\ \Pi_m\ } Q(V) \longrightarrow 0$$

over the exact sequence of C^*-algebras

$$0 \longrightarrow A \xrightarrow{\ \gamma\ } M(A) \xrightarrow{\ \pi_m\ } M(A)|_A \longrightarrow 0.$$

To justify the notation of $M(V)$ let us mention that a strict topology can be defined on V such that V is strictly dense in $M(V)$ and $M(V)$ is a strict completion of V ([3]). We conclude: $M(V)$ is a Hilbert C^*-module analog of a multiplier C^*-algebra, therefore, we will call $M(V)$ the multiplier module of V.

An arbitrary extension (W, B, Φ) of a full Hilbert A-module V can be compared with the maximal extension $(M(V), M(A), \Gamma)$ i.e. the commutative diagram of the folowing form can be drawn

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V & \xrightarrow{\Phi} & W & \xrightarrow{\Pi} & W|_{\Phi(V)} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\Lambda} & & \downarrow{\scriptstyle\Delta} & & \\
0 & \longrightarrow & V & \xrightarrow{\Gamma} & M(V) & \xrightarrow{\Pi_m} & Q(V) & \longrightarrow & 0.
\end{array}$$

The morphism Δ is called the Busby invariant corresponding to the extension (W, B, Φ) of V. For an arbitrary Hilbert C^*-module Z and a given morphism $\Delta : Z \to Q(V)$ there is (unique up to an equivalency) an extension (W, B, Φ) such that Δ is the Busby invariant for it ([4]). To be precise, W equals restricted direct sum $M(V) \oplus_{Q(V)} Z$. The construction of the pullback $M(V) \oplus_{Q(V)} Z$ goes the same way in which the pullback of C^*-algebras is constructed, see [4].

2. Main results

Definition 2.1. *Let Hilbert C^*-modules V and V_1 be given. Let W and W_1 be extensions of V and V_1 respectively. A morphism $\mathcal{E} : W \to W_1$ is called a morphism between extensions if we have the commutative diagram of Hilbert C^*-modules and morphisms of modules*

$$\begin{array}{ccccc}
V & \xrightarrow{\Phi} & W & \xrightarrow{\Pi} & Z \\
\downarrow{\scriptstyle\mathcal{A}} & & \downarrow{\scriptstyle\mathcal{E}} & & \downarrow{\scriptstyle\mathcal{B}} \\
V_1 & \xrightarrow{\Phi_1} & W_1 & \xrightarrow{\Pi_1} & Z_1.
\end{array}$$

For example, the morphism Λ mentioned above obviously serves as a morphism between arbitrary extension W of a Hilbert C^*-module V and his maximal extension $M(V)$.

In the theory of C^*-algebras Eilers, Loring and Pedersen [8, Theorem 2.2] gave the answer to the question when does a morphism between extensions of C^*-algebras exist. It requires a bit of work to state the analog in a Hilbert C^*-module theory. In proving the existence of a morphism between extensions one has to move to the third dimension: to each one of extensions that we require to form a commutative six-term diagram from a definition we associate the maximal essential extension and we have to know how to relate such two extensions. There appears the question of possibility to extend a morphism of Hilbert C^*-modules to a morphism of their multiplier modules. Fortunately, it is already answered.

THEOREM 2.2 ([5, Proposition 1]). *Let $\Phi : V \to W$ be a surjective morphism of full Hilbert C^*-modules. It can be extended to $\overline{\Phi} : M(V) \to M(W)$, a morphism uniquely extending Φ in a sense that $\overline{\Phi}(l_v) = l_{\Phi(v)}$ is satisfied for all $v \in V$.*

Let us notice that we are able to extend Φ even further, to the quotient modules.

LEMMA 2.3. *Let V and W be full Hilbert C^*-modules and let $\Phi : V \to W$ be a surjective morphism. Φ can be extended to a morphism $\tilde{\Phi} : Q(V) \to Q(W)$.*

PROOF. Denote by $\tilde{\Phi} : Q(V) \to Q(W)$ the morphism given by $\tilde{\Phi}(l+V) = \overline{\Phi}(l) + W$. Notice that $\tilde{\Phi}$ is well-defined. Indeed, if we suppose $l + V = l' + V$, then we have $l - l' \in V$, so $l - l' = l_v$ for some $v \in V$. Now $\overline{\Phi}(l - l')(\varphi(a)) = \overline{\Phi}(l_v(a)) = \overline{\Phi}(va) = \Phi(v)\varphi(a) \in W$, that is to say $\overline{\Phi}(l) + W = \overline{\Phi}(l') + W$. ∎

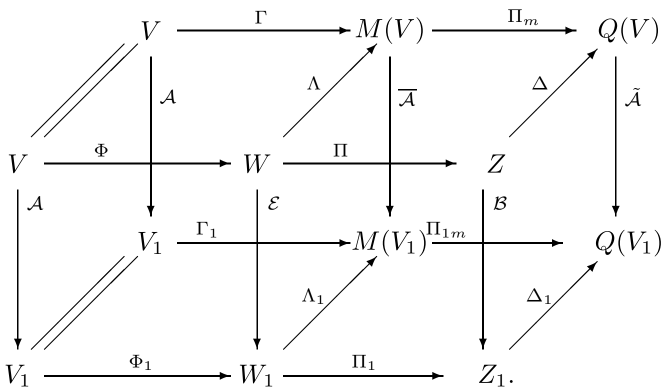
Now we are able to state the main theorem.

THEOREM 2.4. *Let V and V_1 be full Hilbert C^*-modules. Let Δ and Δ_1 be the Busby invariants of extensions W and W_1 of V and V_1, respectively. Let us further have a surjective morphism $\mathcal{A} : V \to V_1$ and a morphism $\mathcal{B} : Z \to Z_1$. The unique morphism $\mathcal{E} : W \to W_1$ such that the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \xrightarrow{\Phi} & W & \xrightarrow{\Pi} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathcal{A}} & & \downarrow{\scriptstyle \mathcal{E}} & & \downarrow{\scriptstyle \mathcal{B}} & & \\ 0 & \longrightarrow & V_1 & \xrightarrow{\Phi_1} & W_1 & \xrightarrow{\Pi_1} & Z_1 & \longrightarrow & 0 \end{array}$$

is commutative exists if and only if $\tilde{\mathcal{A}}\Delta = \Delta_1\mathcal{B}$ is satisfied.

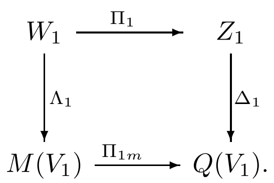
PROOF. Let us add maximal extensions to given ones and draw the 3D diagram with exact rows:



As we can see, the condition $\tilde{\mathcal{A}}\Delta = \Delta_1\mathcal{B}$ can be recognized as the commutativity property of the right hand side square of the 3D diagram. Let us suppose that this equality holds. By the construction top, bottom and back faces are commutative as well as (obviously) the leftmost square. Notice that there are two equal paths from W to $Q(V_1)$ (we make use of known commutativities):

$$\begin{array}{ccccccccc} & W & \xrightarrow{\Pi} & Z & \xrightarrow{\mathcal{B}} & Z_1 & \xrightarrow{\Delta_1} & Q(V_1) & \sim \\ \sim & W & \xrightarrow{\Pi} & Z & \xrightarrow{\Delta} & Q(V) & \xrightarrow{\tilde{\mathcal{A}}} & Q(V_1) & \sim \\ \sim & W & \xrightarrow{\Lambda} & M(V) & \xrightarrow{\Pi_m} & Q(V) & \xrightarrow{\tilde{\mathcal{A}}} & Q(V_1) & \sim \\ \sim & W & \xrightarrow{\Lambda} & M(V) & \xrightarrow{\mathcal{A}} & M(V_1) & \xrightarrow{\Pi_{1m}} & Q(V_1). & \end{array}$$

But, W_1 is the pullback



We have a coherent pair of morphisms $\mathcal{B}\Pi : W \to Z_1$ and $\overline{\mathcal{A}}\Lambda : W \to M(V_1)$ because $\Delta_1\mathcal{B}\Pi = \Pi_{1m}\overline{\mathcal{A}}\Lambda$. By the universal property for the pullback W_1 there is a unique morphism $\mathcal{E} : W \to W_1$ such that

1. $\Pi_1\mathcal{E} = \mathcal{B}\Pi$ i. e. a morphism $\mathcal{E}$ makes the right-hand side square of the front face commutative,

2. $\Lambda_1\mathcal{E} = \overline{\mathcal{A}}\Lambda$ (commutativity of the center square).

It remains to see that a morphism $\mathcal{E}$ makes the left-hand side square of the front face commutative. First we make use of commutativity of the center diagram to get $\Lambda_1\mathcal{E}\Phi = \overline{\mathcal{A}}\Lambda\Phi = \overline{\mathcal{A}}\Gamma = \Gamma_1\mathcal{A} = \Lambda_1\Phi_1\mathcal{A}$. As we know Λ_1 is an injective morphism if and only if W_1 is a full essential extension of V_1 ([3]). Then the claim follows. What about underlying morphisms of C^*-algebras? The existence of a morphism $\varepsilon : B \to B_1$ of C^*-algebras is guaranteed by [8] and from the pullback construction we know that a morphism $\mathcal{E}$ is an ε-morphism.

Conversely, let us have the commutative diagram

$$\begin{array}{ccccc} V & \xrightarrow{\Phi} & W & \xrightarrow{\Pi} & Z \\ \downarrow{\scriptstyle \mathcal{A}} & & \downarrow{\scriptstyle \mathcal{E}} & & \downarrow{\scriptstyle \mathcal{B}} \\ V_1 & \xrightarrow{\Phi_1} & W_1 & \xrightarrow{\Pi_1} & Z_1. \end{array}$$

Now form the 3D diagram as above. As we have (1) $\mathcal{E}\Phi = \Phi_1\mathcal{A}$ and (2) $\mathcal{B}\Pi = \Pi_1\mathcal{E}$, the front face of the 3D diagram is commutative: indeed, (2) forces $\mathcal{B}\Pi\Phi = \Pi_1\mathcal{E}\Phi = \Pi_1\Phi_1\mathcal{A}$. The central diagram is commutative as well: $\Lambda_1\mathcal{E}\Phi \overset{(1)}{=} \Lambda_1\Phi_1\mathcal{A} = \Gamma_1\mathcal{A} = \tilde{\mathcal{A}}\Gamma = \tilde{\mathcal{A}}\Lambda\Phi$ leads to $\Lambda_1\mathcal{E} = \tilde{\mathcal{A}}\Lambda$. Now, as we know, all faces in the right-hand side cube of the 3D diagram except of the right-most one are commutative, then we have:

$$\Delta_1\mathcal{B}\Pi \overset{(2)}{=} \Delta_1\Pi_1\mathcal{E} = \Pi_{1m}\Lambda_1\mathcal{E} = \Pi_{1m}\overline{\mathcal{A}}\Lambda = \tilde{\mathcal{A}}\Pi_m\Lambda = \tilde{\mathcal{A}}\Delta\Pi.$$

It follows that $\Delta_1\mathcal{B} = \tilde{\mathcal{A}}\Delta$ is valid. □

We consider now the question of describing morphisms out of an extension of a Hilbert C^*-module i. e. out of a middle object in a given exact sequence. It is done in [9] for the case of split extensions. In order to do it in a general case, one has to define an idealizer of a Hilbert C^*-module.

DEFINITION 2.5. *Let V and W be Hilbert C^*-modules over C^*-algebras A and B, respectively. Let $\Phi : V \to W$ be a φ-morphism for a morphism $\varphi : A \to B$ of C^*-algebras. The idealizer of $\Phi(V)$ in W is defined as*

$$I(\Phi(V); W) = \{w \in W : (w \mid \Phi(v)) \in I(\varphi(A)), w\varphi(a) \in \Phi(V), a \in A, v \in V\}.$$

LEMMA 2.6. *$\Phi(V)$ is the ideal submodule of $I(\Phi(V); W)$ associated to the ideal $\varphi(A) \subseteq I(\varphi(A); B)$ i. e. $\Phi(V) = I(\Phi(V); W)\varphi(A)$.*

PROOF. It is obvious from the definition of the idealizer that one has $I(\Phi(V); W)\varphi(A) \subseteq \Phi(V)$. Let us check the opposite inclusion: let (e_j) be the approximate identity for A, we know that for $v \in V$ we can write $v = \lim_j ve_j$. Now, using the fact that Φ is a continuous and a φ-linear morphism, we have:

$$\Phi(v) = \Phi(\lim_j ve_j) = \lim_j \Phi(ve_j) = \lim_j \Phi(v)\varphi(e_j) \in I(\Phi(V); W)\varphi(A).$$

□

We obtain the eigenmodule $E(\Phi(V); W)$ as a quotient $I(\Phi(V); W)|_{\Phi(V)}$, it is a Hilbert C^*-module over the eigenalgebra $E(\varphi(A); B)$. In what follows, we shall write $I(\Phi(V))$, $E(\Phi(V))$ for short. Notice there is the exact sequence of Hilbert C^*-modules

$$0 \longrightarrow \Phi(V) \longrightarrow I(\Phi(V)) \longrightarrow E(\Phi(V)) \longrightarrow 0.$$

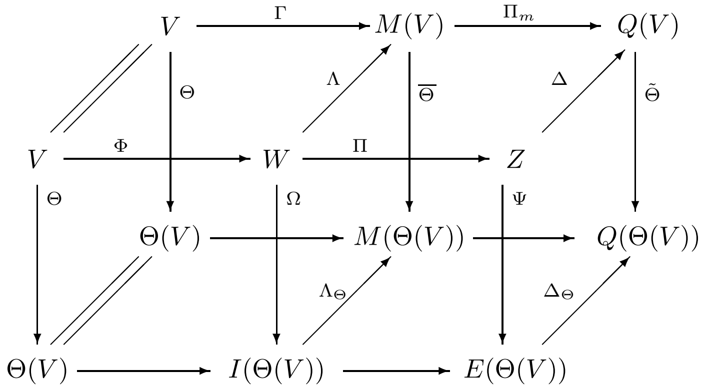
As for any other extension, there is the Busby invariant for this one too, let us denote it by Δ_Φ. It is obvious that if $W' \subseteq W$ is a submodule of W such that $\Phi(V)$ is an ideal submodule of W', then we have $W' \subseteq I(\Phi(V))$. We can state now an analog of the Theorem 2.4 stated for a C^*-algebra case in [8].

THEOREM 2.7. *Let Δ be the Busby invariant of the extension W of a full Hilbert A-module V:*

$$0 \longrightarrow V \xrightarrow{\ \Phi\ } W \xrightarrow{\ \Pi\ } Z \longrightarrow 0.$$

Given a Hilbert C^*-module U over a C^*-algebra D there is a bijective correspondence between a morphism $\Omega : W \to U$ and a pair of morphisms (Θ, Ψ) where $\Theta : V \to U$, $\Psi : Z \to E(\Theta(V); U)$ are such that $\Delta_\Theta \Psi = \tilde{\Theta}\Delta$ is satisfied. Here Δ_Θ is the Busby invariant for the extension of $\Theta(V)$ by its idealizer and $\tilde{\Theta}$ is the surjective morphism $\Theta : V \to \Theta(V)$ extended to $Q(V)$.

PROOF. Let a morphism $\Omega : W \to U$ be given, we define $\Theta : V \to U$ by $\Theta = \Omega\Phi$. Θ is a θ-morphism over a morphism of C^*-algebras $\theta : A \to C$ given by $\theta = \omega\varphi$. As we know from the definition of an extension (W, B, Φ) of a Hilbert A-module V, $\Phi(V)$ is the ideal submodule in W associated to the ideal $\varphi(A) \subseteq B$, so we have $\Phi(V) = W\varphi(A)$. Then $\Theta(V) = \Omega(\Phi(V)) = \Omega(W\varphi(A)) = \Omega(W)\omega(\varphi(A)) = \Omega(W)\theta(A)$. Equality shows that $\Theta(W)$ is the ideal submodule of $\Omega(W)$ associated to the ideal $\theta(A) \subseteq \omega(B)$. Therefore we conclude that Ω actually maps to $I(\Theta(V); U)$, i. e. $\Omega(W) \subseteq I(\Theta(V); U)$. We get the 3D diagram with exact rows:



As a morphism $\Omega : W \to I(\Phi(V))$ for which the front face of the above diagram is commutative exists, the Theorem 2.4 says that $\Delta_\Theta \Psi = \tilde{\Theta}\Delta$.

Conversely, if there is a pair (Θ, Ψ) of morphisms such that $\Delta_\Theta \Psi = \tilde{\Theta}\Delta$ is satisfied, by the same Theorem 2.4 there is a unique morphism, call it Ω, $\Omega : W \to I(\Theta(V))$, associated to the given pair. ∎

References

[1] W. Arveson, *Notes on extensions of C^*-algebras*, Duke Math. J. **44** (1977), 329-355.
[2] D. Bakić, B. Guljaš, *On a class of module maps of Hilbert C^*-modules*, Math. Commun. **7** (2002), 177-192.
[3] D. Bakić, B. Guljaš, *Extensions of Hilbert C^*-modules*, Houston J. of Math. **30** (2004), 537-558.
[4] D. Bakić, B. Guljaš, *Extensions of Hilbert C^*-modules II*, Glas. Mat. Ser. III **38** (2003), 341-357.
[5] D. Bakić, *Tietze extension theorem for Hilbert C^*-modules*, Proc. Amer. Math. Soc. **133** (2005), 441-448.
[6] L. G. Brown, R. G. Douglas, P. A. Fillmore, *Unitary equivalence modulo the compact operators and extensions of C^*-algebras*, Lecture Notes in Mathematics **345**, Springer-Verlag, 1973, 58-128.
[7] R. C. Busby, *Double cenralizers and extensions of C^*-algebras*, Trans. Amer. Math. Soc. **132** (1968), 79-99.
[8] S. Eilers, T. A. Loring, G. K. Pedersen, *Morphisms of extensions of C^*-algebras: Pushing forward the Busby invariant*, Adv. Math **147** (1999), 74-109.
[9] B. Kolarec, *Morphisms out of a split extension of a Hilbert C^*-module*, Glas. Mat. Ser. III **41** (2006), 309-315.
[10] E. C. Lance, Hilbert C^*-modules - a toolkit for operator algebraists, Lecture Note Series 210, Cambridge University Press, 1995.
[11] S. MacLane, Categories for the Working Mathematician, Graduate Texts in Math., vol. 5, Springer-Verlag, 1971.
[12] T. A. Loring, Lifting Solutions to Perturbing Problems in C^*-algebras, Fields Institute Monographs, 1997.
[13] G. K. Pedersen, *Extensions of C^*-algebras*, preprint.

[14] G. K. Pedersen, *Pullback and pushout construction in C^*-algebra theory*, J. Funct. Anal. **167** (1999), 143-344.

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