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ON THE FIXED POINT PROPERTY FOR INVERSE LIMITS OF FANS

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Abstract. In 1984, M. M. Marsh gave conditions under which the inverse limit of an inverse sequence of fans has the fixed point property. In this paper we give an extension of that result.

1. Definitions and notation

By a *space* we mean a topological space. A *continuum* is a nonempty compact connected metric space. By a *map* we mean a continuous function. A *fixed point* of a map $f : X \to X$ is a point $x \in X$ such that $f(x) = x$. A space X has the *fixed point property* if every map $f : X \to X$ has a fixed point.

The symbols $\mathbb{N}$ and $\mathbb{R}$ denote the set of the positive integers and the set of the real numbers, respectively. The diameter of a set A in a metric space will be denoted by $\operatorname{diam}(A)$.

Given a point x in a product $\prod \{X_n : n \in \mathbb{N}\}$ we will write $x = (x_n)_{n\in\mathbb{N}}$. An *inverse sequence* is a double sequence $\{X_n, f_n\}$ of spaces X_n and maps $f_n : X_{n+1} \to X_n$. The *inverse limit* of the inverse sequence $\{X_n, f_n\}$, which is denoted by $\varprojlim \{X_n, f_n\}$ or by X_∞, is the space of all points $x \in \prod\{X_n : n \in \mathbb{N}\}$ such that $x_n = f_n(x_{n+1})$. If $X_n = X$ for each $n \in \mathbb{N}$, then we write $\{X, f_n\}$ instead of $\{X_n, f_n\}$. The projection map from X_∞ into X_n will be denoted by f_n^∞.

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A finite family $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ of open sets of a space X is a *chain* (*circular chain*) provided that $C_i \cap C_j \neq \emptyset$ if and only if $|i-j| \leq 1$ ($n \geq 3$ and $|i-j| \leq 1$ or $|i-j| = n-1$).

A family of sets $\mathcal{C}$ is *coherent* provided that every proper family $\mathcal{G}$ of $\mathcal{C}$ contains an element intersecting an element of $\mathcal{C} \setminus \mathcal{G}$. A finite coherent family $\mathcal{C}$ of open sets of a space X is a *tree chain* if no subfamily of $\mathcal{C}$ is a circular chain. A *junction link* of a tree chain $\mathcal{C}$ is an element that intersects at least three other elements of $\mathcal{C}$. If the diameter of each element of a chain (circular chain, tree chain) is less than ε, then it is called ε-*chain* (ε-*circular chain*, ε-*tree chain*). The elements of a chain, a circular chain or a tree chain are called *links*.

An ε-*map* is a map $f : X \to Y$ between metric spaces such that $\mathrm{diam}(f^{-1}(y)) < \varepsilon$ for each $y \in Y$.

2. The Theorem

Consider the following well-known result (see [3, (2.33)]):

Lemma 2.1. *Let X and Y be nonempty compact metric spaces. If $f : X \to Y$ is an onto ε-map, then there exists $\delta > 0$ such that* $\mathrm{diam}(f^{-1}[Z]) < \varepsilon$ *whenever $Z \subseteq Y$ and* $\mathrm{diam}(Z) < \delta$.

Note that the previous Lemma implies the following:

Lemma 2.2. *Let $f : X \to Y$ be an onto ε-map between continua and let $\delta > 0$ be a real number given by 2.1. If $\mathcal{D}$ is a δ-chain (δ-circular chain, δ-tree chain) in Y, then the family $f^{-1}[\mathcal{D}] = \left\{f^{-1}[D] : D \in \mathcal{D}\right\}$ is an ε-chain (ε-circular chain, ε-tree chain).*

The main result of this paper is in 2.5. We prove that the inverse limit of an inverse sequence of fans $\{A, f_n\}$ has the fixed point property. Now, we define the space A that we use in 2.5.

Construction 2.3. Given a point $a \in \mathbb{R}^2 \backslash (0,0)$, let $L_a = \{ta : 0 \leq t \leq 1\}$. Let B be a finite subset of $\{(x,y) \in \mathbb{R}^2 : \| (x,y) \| = 1$ and $y > 0\}$ and let $\mathfrak{C}$ be the Cantor set contained in $[0,1] \times \{-1\}$.

For each subset D of B, let $T^D = \bigcup\{L_a : a \in \mathfrak{C} \cup D\}$, let $A = T^B$, and let $T = T^{\emptyset}$.

In the following Lemma, we consider the spaces constructed in 2.3.

Lemma 2.4. *Let $b \in B \cup \mathfrak{C}$. For every positive number δ, there exists a δ-tree chain $\mathcal{D}$ covering A such that:*

(i) *$\mathcal{D}$ has exactly one junction link D_0,*
(ii) $(0,0) \in D_0$,
(iii) *a chain $\mathcal{D}' = \{D_0, \ldots, D_n\}$ of links of $\mathcal{D}$ covers L_b and $D \cap L_b = \emptyset$ for every $D \in \mathcal{D} \setminus \mathcal{D}'$,*

(iv) *if* $b \in B$, *then* $T^{B\setminus\{b\}} \cap D_i = \emptyset$ *for each* $i > 0$.

Proof. Let $n \in \mathbb{N}$, such that $\frac{3}{n} < \delta$. For each $k \in \{0, 1, \dots, n-2, n-1\}$, let:

$$G_k = \{(x, y) \in \mathbb{R}^2 : \tfrac{k}{n} < \| (x, y) \| < \tfrac{k+2}{n}\}$$

and let:

$$F_k = \{(x, y) \in \mathbb{R}^2 : -\tfrac{k+2}{n} < y < -\tfrac{k}{n}\}.$$

From the construction of the Cantor set (see [3, 7.5]), there exists a pairwise disjoint family of closed subintervals $I_1, I_2, \dots, I_m$ of $[0, 1] \times \{-1\}$ such that $\mathfrak{C} \subseteq I_1 \cup I_2 \cup \ldots \cup I_m$ and $\operatorname{diam}(I_l) < \frac{2}{n}$ for every $l \in \{1, \dots, m\}$. Note that each $\mathfrak{C} \cap I_l$ is a clopen subset of $\mathfrak{C}$.

Given $l \in \{1, \dots, m\}$, let $T_l = \bigcup\{L_a : a \in \mathfrak{C} \cap I_l\}$. Let $D_0 = \{(x, y) \in \mathbb{R}^2 : \| (x, y) \| < \frac{1}{n}\} \cap A$, then the family:

$$\begin{aligned} \mathcal{D} = \{D_0\} &\cup \{G_k \cap L_b : k \in \{0, 1, \dots, n-2, n-1\} \text{ and } b \in B\} \\ &\cup \{F_k \cap T_l : k \in \{0, 1, \dots, n-2, n-1\} \text{ and } l \in \{1, \dots, m\}\} \end{aligned}$$

is a tree chain. Clearly, $\mathcal{D}$ has only one junction link; namely D_0. Moreover, the links of $\mathcal{D}$ have diameter less or equal to $\sqrt{\left(\frac{2}{n}\right)^2 + \left(\frac{2}{n}\right)^2} = \sqrt{\frac{8}{n^2}} < \delta$. Therefore, the family $\mathcal{D}$ is a δ-tree chain.

If $b \in B$, we define $D_{k+1} = G_k \cap L_b$ for $k \in \{0, 1, \dots, n-2, n-1\}$.

If $b \in I_l$, then we define $D_{k+1} = F_k \cap T_l$ for $k \in \{0, 1, \dots, n-2, n-1\}$.

So, the families $\mathcal{D}' = \{D_0, D_1, \dots, D_n\}$ and $\mathcal{D}$ satisfy (i), (ii), (iii) and (iv). □

Theorem 2.5. *Consider the spaces built in 2.3. Suppose that* $\{A, f_n\}$ *is an inverse sequence with onto bonding maps such that:*

(C1) $f_n((0, 0)) = (0, 0)$ *for each* $n \in \mathbb{N}$,

(C2) *for every* $c \in \mathfrak{C}$, *there exists* $c' \in \mathfrak{C}$ *such that* $f_n[L_c] \subseteq L_{c'}$ *and,*

(C3) $f_n[L_b] \subseteq T^{\{b\}}$ *for each* $b \in B$.

Then A_∞ *has the fixed point property.*

Proof. We first note that, by (C2):

$$f_n[T] \subseteq T. \tag{2.1}$$

If a point $x \in A_\infty$ has its coordinate $x_k \in L_b \setminus \{(0, 0)\}$ for some $b \in B$, then, by (C3) and (2.1):

$$x_n \in L_b \setminus \{(0, 0)\} \text{ for every } n \geq k. \tag{2.2}$$

By (C1), the point $a = ((0, 0))_{n\in\mathbb{N}}$ ($a_n = (0, 0)$ for each $n \in \mathbb{N}$) belongs to A_∞.

Let d denote the metric on A_∞. Suppose that a map $f : A_\infty \to A_\infty$ has no fixed points. Since A_∞ is compact, there exists $\varepsilon > 0$ such that $d(x, f(x)) \geq \varepsilon$ for each $x \in A_\infty$. Since $f(a) \neq a$, we have that $f_k^\infty(f(a)) \neq (0, 0)$ for some $k \in \mathbb{N}$.

Consider the case $f_k^\infty(f(a)) \in L_b \setminus \{(0,0)\}$ for some $b \in B$. By the continuity of $f_k^\infty \circ f$, there exists $\varepsilon' > 0$, with $\varepsilon' < \frac{\varepsilon}{2}$, such that:

(2.3) $$d(x,a) < \varepsilon' \text{ implies } f_k^\infty(f(x)) \in L_b \setminus \{(0,0)\}.$$

Let $m \geq k$ such that f_m^∞ is an ε'-map (see [3, p. 25]).

We claim that:

(2.4) $$f_m^\infty \circ f[(f_m^\infty)^{-1}((0,0))] \subseteq L_b \setminus \{(0,0)\}.$$

To prove this, let $x \in (f_m^\infty)^{-1}((0,0))$. Since $x, a \in (f_m^\infty)^{-1}((0,0))$ and f_m^∞ is an ε'-map, we have that $d(x,a) < \varepsilon'$. Then, by (2.3), $f_k^\infty(f(x)) \in L_b \setminus \{(0,0)\}$ and, by (2.2), $f_m^\infty(f(x))$ belongs to $L_b \setminus \{(0,0)\}$. So, the inclusion (2.4) holds.

By 2.1, there exists $\delta > 0$ such that $\operatorname{diam}((f_m^\infty)^{-1}[Z]) < \varepsilon'$ whenever $\operatorname{diam}(Z) < \delta$. For δ and b given above, let $\mathcal{D}$ be the δ-tree chain and let $\mathcal{D}' = \{D_0, \ldots, D_l\}$ be the chain covering L_b constructed in 2.4.

Let $\mathcal{C} = \{(f_m^\infty)^{-1}[D] : D \in \mathcal{D}\}$, let $C_i = (f_m^\infty)^{-1}[D_i]$ and let $\mathcal{C}' = \{C_0, \ldots, C_l\}$. By 2.2, the family $\mathcal{C}$ is an ε'-tree chain covering A_∞.

By 2.4(ii) and (2.4), we have that

$$(f_m^\infty)^{-1}((0,0)) \subseteq C_0 \cap (f_m^\infty \circ f)^{-1}[L_b \setminus \{(0,0)\}].$$

Let $U = C_0 \cap (f_m^\infty \circ f)^{-1}[L_b \setminus \{(0,0)\}]$. Let V an open set in A_∞ such that $(f_m^\infty)^{-1}((0,0)) \subseteq V \subseteq \overline{V} \subseteq U$. Let $q \in C_l$ and let K be the component of $A_\infty \setminus V$ containing q. Then, by [1, 6.1.25], K must intersect the boundary of V at some point y.

Since $(f_m^\infty)^{-1}((0,0)) \subseteq V$ and $D \cap L_b = \emptyset$ for every $D \in \mathcal{D} \setminus \mathcal{D}'$ (see Lemma 2.4(iii)), it follows that $K \subseteq \bigcup\{C_i : 0 \leq i \leq l\}$.

Let

$$R = \{x \in K : \text{if } x \in C_i \text{ and } f(x) \in C_j \text{ then } i \leq j\}$$

and let

$$S = \{x \in K : \text{if } x \in C_i \text{ and } f(x) \in C_j \text{ then } j \leq i \text{ or } C_j \notin \mathcal{C}'\}.$$

It is easy to see that $y \in R$, $q \in S$ and $K = R \cup S$. Moreover, R and S are disjoint open sets of K. But this contradicts the connectedness of K. Then $f_k^\infty(f(a)) \in T \setminus \{(0,0)\}$.

We can assume that $f_n^\infty(f(a)) \in T \setminus \{(0,0)\}$ for every $n \geq k$. For each $n \geq k$, let $c_n \in \mathfrak{C}$ such that $f_n^\infty(f(a)) \in L_{c_n}$. Then $f_n[L_{c_{n+1}}] \subseteq L_{c_n}$. Let $K = \varprojlim\{L_{c_n}, f_n|_{L_{c_{n+1}}}\}$. In this case, we choose $\varepsilon' = \frac{\varepsilon}{2}$ and $m \geq k$ such that f_m^∞ is an ε'-map. We also choose $\delta > 0$, $\mathcal{C}$ and $\mathcal{C}'$ as above. Then $K \subseteq \bigcup\{C_i : 0 \leq i \leq l\}$. With similar definitions of R and S, as above, we contradict the connectedness of K.

Hence, such a map does not exist. Therefore, the space A_∞ has the fixed point property. ∎

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