Statistical convergence and rate of convergence of a sequence of positive linear operators

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Abstract. *In the present paper, a modification of positive linear operators which was proposed by O. Agratini is introduced. This modification which preserves function $e_2(x) = x^2$ provides a better estimation than operators given by Agratini. Also, using the concept of statistical convergence, we give the Korovkin type approximation theorem for this modification.*

Key words: *operators given by Agratini, statistical convergence, the Korovkin type approximation theorem, modulus of contiunity*

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1. Introduction

A. Lupaş defined the following operators [7]. Let $\alpha = nx$, $x \geqslant 0$ and consider the linear operators

$$L_n(f;x) = (1-a)^{nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{k!} a^k f\left(\frac{k}{n}\right) \tag{1.1}$$

with $f : [0,\infty) \to \mathbb{R}$ where

$$\frac{1}{(1-a)^{\alpha}} = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} a^k, \quad |a| < 1$$

and

$$(\alpha)_0 = 1, \ (\alpha)_k = \alpha(\alpha+1)\dots(\alpha+k-1), \ k \geq 1.$$

Agratini [1] found $a = \frac{1}{2}$ for $L_n(e_1;x) = e_1(x)$ where $e_i(x) = x^i$, $i = 0,1,2$, using operators L_n which defined by (1.1). Then, Agratini gave the following operators:

$$L_n(f;x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} f\left(\frac{k}{n}\right), \quad x \geqslant 0. \tag{1.2}$$

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It is known [1] that for the operators given by (1.2),

$$L_n(e_0;x) = e_0(x),\ L_n(e_1;x) = e_1(x) \text{ and } L_n(e_2;x) = e_2(x) + \frac{2e_1(x)}{n}.$$

We fix $b > 0$ and the lattice homomorphism H_b maps $C[0,\infty)$ into $C[0,b]$ defined by $H_b(f) = f\mid_{[0,b]}$. For the operators L_n defined by (1.2), it is known [1] that, $H_b(L_n e_i) \to H_b(e_i)$ uniformly on $[0,b]$, where $i = 0,1,2$. Also, in [1], for the L_n operators given by (1.2), Agratini proved the classical Korovkin theorem:

If L_n is defined by (1.2), then

$$\lim_{n\to\infty} L_n(f;x) = f(x) \text{ uniformly on } [0,b]$$

for any $b > 0$.

Most of approximating operators, L_n, preserve e_0 and e_1, i.e., $L_n(e_0;x) = e_0(x)$ and $L_n(e_1;x) = e_1(x)$, $n \in \mathbb{N}$. These conditions hold for Bernstein polynomials and the Szász-Mirakjan operators (see, e.g. [6]). For each of these operators, $L_n(e_2;x) \neq e_2(x)$. Recently, King [5] presented a non-trivial sequence $\{V_n\}$ of positive linear operators which approximate each continuous function on $[0,1]$ while preserving the functions e_0 and e_2.

In this paper we give a modification of positive linear operators which L_n is defined by (1.2) and show that this modification which preserve $e_0(x)$ and $e_2(x)$ is a better estimation than operators given by (1.2) . Finally, we study statistical convergence of this modification.

2. Construction of the operators

Let $\{r_n(x)\}$, $[0,\infty)$ into itself, be a sequence of continuous functions. Let

$$R_n(f;x) = 2^{-nr_n(x)} \sum_{k=0}^{\infty} \frac{(nr_n(x))_k}{2^k k!} f\left(\frac{k}{n}\right) \tag{2.1}$$

for $f \in C[0,\infty)$. Hence, in the special case $r_n(x) = x$, $n = 1,2,\dots$, reduce to operators given by (1.2).

It is clear that R_n are positive and linear. Also, we have

$$R_n(e_0;x) = e_0(x),\ R_n(e_1;x) = r_n(x) \text{ and } R_n(e_2;x) = r_n^2(x) + \frac{2r_n(x)}{n}. \tag{2.2}$$

Theorem 1. *Let R_n denote the sequence of the positive linear operators given by (2.1). If*

$$\lim_{n\to\infty} r_n(x) = x,$$

then

$$\lim_{n\to\infty} R_n(f;x) = f(x) \text{ uniformly on } [0,b]$$

for any $b > 0$.

Furthermore, we present the sequence $\{R_n\}$ of positive linear operators defined on $C[0,\infty)$ that preserve $e_0(x)$ and $e_2(x)$.

It is obvious that the choice $r_n(x) = r_n^*(x)$:

$$r_n^*(x) = -\frac{1}{n} + \sqrt{x^2 + \frac{1}{n^2}}, \quad n = 1, 2, ... \tag{2.3}$$

gives

$$R_n(e_2; x) = e_2(x) = x^2, \ \ n = 1, 2, ... \tag{2.4}$$

Simple calculations show that, for $r_n^*(x)$ given by (2.3),

$$r_n^*(x) \geq 0, \quad n = 1, 2, ..., \quad x \in [0, \infty). \tag{2.5}$$

It is clear that

$$\lim_{n\to\infty} r_n^*(x) = x, \quad x \in [0, \infty). \tag{2.6}$$

Thus, using (2.3), (2.4), (2.5), (2.6), we have the following Korovkin theorem for the operators R_n given by (2.1).

Theorem 2. *Let the sequence $\{R_n\}$ of positive linear operators given by (2.1) and the sequence $\{r_n^*(x)\}$ defined by (2.3). Then,*

(i) R_n is a positive linear operators on $C[0,\infty)$, $n = 1, 2, ...$

(ii) $R_n(e_2; x) = e_2(x) = x^2, \ \ n = 1, 2, ..., x \in [0, \infty)$

(iii) $\lim_{n\to\infty} R_n(f; x) = f(x)$, on $[0, b]$.

3. Rate of convergence

In this section we compute the rates of convergence of the operators $R_n(f; x)$ to $f(x)$ by means of the modulus of continuity. Thus, we show that our estimations are more powerful than the operators given by (1.2) on the interval $[0, \infty)$.

For $f \in C[0, b]$, the modulus of continuity of f, denoted by $\omega(f; \delta)$, is defined to be

$$\omega(f; \delta) = \sup_{|y-x|<\delta, \ x,y\in[0,b]} |f(y) - f(x)|.$$

Then, it is clear that for any $\delta > 0$ and each $x, y \in [0, b]$

$$|f(y) - f(x)| \leq \omega(f; \delta)\left(\frac{|y-x|}{\delta} + 1\right).$$

Now we have the following:

Theorem 3. *If R_n is defined by (2.1), then for $x \in [0, b]$ and any $\delta > 0$, we have*

$$|R_n(f; x) - f(x)| \leq \omega(f, \delta)\left[1 + \frac{1}{\delta}\sqrt{2x(x - R_n(e_1; x))}\right]$$

where $R_n(e_1;x) = r_n^*(x)$ *is given by (2.3).*

Proof. It is known [2] that for $x \in [0,b]$ and any $\delta > 0$

$$|R_n(f;x) - f(x)| \le \omega(f,\delta)\left[R_n(e_0;x) + \frac{1}{\delta}(R_n(e_0;x))^{\frac{1}{2}}(\mu_{n,2}(x))^{\frac{1}{2}}\right] + |f(x)| . |R_n(e_0;x) - e_0(x)| \tag{3.1}$$

where

$$\mu_{n,2}(x) = R_n(\Psi_{x,2};x) \text{ with } \Psi_{x,2}(t) = (t-x)^2 .$$

Then, it is clear that

$$\begin{aligned} \mu_{n,2}(x) &= R_n(\Psi_{x,2};x) \\ &= R_n\left((t-x)^2;x\right) \\ &= R_n(e_2;x) - 2xR_n(e_1;x) + x^2R_n(e_0;x). \end{aligned}$$

For the operators R_n satisfying

$$R_n(e_0;x) = e_0(x),\ R_n(e_2;x) = e_2(x),\ n = 1,2,... \text{ and } x \in [0,b],$$

inequality (3.1) becomes

$$\begin{aligned} |R_n(f;x) - f(x)| &\le \omega(f,\delta)\left[1 + \frac{1}{\delta}\sqrt{x^2 - 2xR_n(e_1;x) + x^2}\right] \\ &= \omega(f,\delta)\left[1 + \frac{1}{\delta}\sqrt{2x(x - R_n(e_1;x))}\right],\ x \in [0,b]. \end{aligned} \tag{3.2}$$

□

Furthermore, when (3.2) holds,

$$2x(x - R_n(e_1;x)) \ge 0 \text{ for } x \in [0,b].$$

For the special case $R_n = L_n$, we get the following inequality:

$$L_n(e_0;x) = e_0(x),\ L_n(e_1;x) = e_1(x) \text{ and } L_n(e_2;x) = e_2(x) + \frac{2e_1(x)}{n}.$$

Hence,

$$|L_n(f;x) - f(x)| \le \omega(f,\delta)\left[1 + \frac{1}{\delta}\sqrt{\frac{2x}{n}}\right]. \tag{3.3}$$

The estimate (3.2) is better than the estimate (3.3) if and only if

$$2x(x - R_n(e_1;x)) \le \frac{2x}{n},\ x \in [0,b]. \tag{3.4}$$

Namely, this is equivalent to

$$R_n(e_1;x) \geq x - \frac{1}{n},\ x \in [0,b]\,. \tag{3.5}$$

Since $R_n(e_1;x) = r_n^*(x) = -\frac{1}{n} + \sqrt{x^2 + \frac{1}{n^2}}$,

$$x^2 + \frac{1}{n^2} \geq x^2, \text{ for } x \geq 0$$

i.e.

$$\sqrt{x^2 + \frac{1}{n^2}} \geq x.$$

(3.5) holds for every $x \geq 0$ and $n \in \mathbb{N}$. Therefore, our estimations are more powerful than the operators given by (1.2) on the interval $[0,\infty)$.

4. Statistical convergence

Gadjiev and Orhan [4] have investigated the Korovkin type approximation theory via statistical convergence. In this section, using the concept of statistical convergence, we give the Korovkin type approximation theorem for the R_n operators given by (2.1). Before we present the new results, we shall recall some notation on the statistical convergence.

Let K be a subset of $\mathbb{N}$, the set of all natural numbers. The density of K, denoted by $\delta(K)$, is defined by

$$\delta(K) := \lim_n \frac{1}{n} \sum_{j=1}^{n} \chi_K(j)$$

provided the limit exists where χ_K is the characteristic function of K. A sequence $x = (x_k)$ is said to be statistical convergence to the number L,

$$\delta\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} = 0$$

for every $\varepsilon > 0$ or equivalently there exists a subset $K \subseteq \mathbb{N}$ with $\delta(K) = 1$ and $n_0(\varepsilon)$ such that $k > n_0$ and $k \in K$ imply that $|x_k - L| < \varepsilon$ ([3]). In this case we write $st - \lim x_k = L$. It is known that any convergent sequence is statistically convergent, but not conversely. For example, for the sequence $x = (x_k)$ is defined as

$$x_k = \begin{cases} 1, & if\ k\ is\ square \\ 0, & otherwise. \end{cases}$$

It is easy to see that $st - \lim x_k = 0$.

The Korovkin type approximation theorem is given as follows:

Theorem 4. *Let R_n denote the sequence of the positive linear operators given by (2.1). If*

$$st - \lim_{n\to\infty} r_n(x) = x,$$

then

$$st - \lim_{n\to\infty} R_n(f;x) = f(x) \ \ \text{on} \ \ [0,b]$$

for any $b > 0$.

Now, we choose a subset K of $\mathbb{N}$ such that $\delta(K) = 1$. Define the function sequence $\{p_n^*\}$ by

$$p_n^*(x) = \begin{cases} 0 & , \quad n \notin K \\ r_n^*(x) & , \quad n \in K \end{cases} \tag{4.1}$$

where $r_n^*(x)$ is given by (2.3).

It is clear that p_n^* is continuous on $[0,\infty)$ and

$$st - \lim_{n\to\infty} p_n^*(x) = x, \; x \in [0,\infty). \tag{4.2}$$

We turn to $\{R_n\}$ given by (2.1) with $\{r_n(x)\}$ replaced by $\{p_n^*(x)\}$ where $p_n^*(x)$ is defined by (4.1). Show that $\{R_n\}$ is a positive linear operator and

$$R_n(e_1;x) = p_n^*(x) \tag{4.3}$$

and

$$R_n(e_2;x) = \begin{cases} e_2(x) & , \quad n \in K \\ 0 & , \quad \text{otherwise} \end{cases} \tag{4.4}$$

where K is any subset of $\mathbb{N}$ such that $\delta(K) = 1$.

Since $\delta(K) = 1$, it is clear that

$$st - \lim_{n\to\infty} R_n(e_2;x) = e_2(x) = x^2, \; x \in [0,\infty). \tag{4.5}$$

Relations (2.2), (4.2), (4.3), (4.4) and *Theorem 4* yield the following:

Theorem 5. *$\{R_n\}$ denote the sequence of positive linear operators given by (2.1) with $\{r_n(x)\}$ replaced by $\{p_n^*(x)\}$ where $p_n^*(x)$ is defined by (4.1). Then*

$$st - \lim_{n\to\infty} R_n(f;x) = f(x) \ \ \text{on} \ \ [0,b]$$

for any $b > 0$.

We denote that $\{R_n\}$ is the sequence of positive linear operators given by (2.1) with $\{r_n(x)\}$ replaced by $\{p_n^*(x)\}$ where $p_n^*(x)$ is defined by (4.1) does not satisfy the condition of the classical Korovkin theorem.

References

[1] O. Agratini, *On a sequence of linear and positive operators*, Facta Universitatis (Niš), Ser. Mat. Inform. **14**(1999), 41–48.

[2] R. A. DeVore, *The Approximation of Continuous Functions by Positive Linear Operators*, Lecture Notes in Mathematics 293, Spinger-Verlag, New York, 1972.

[3] J. A. Fridy, *On statistical convergence*, Analysis **5**(1985), 301–313.

[4] A. D. Gadjiev, C. Orhan, *Some approximation theorems via statistical convergence*, Rocky Mountain J. Math. **32**(2002), 129–138.

[5] J. P. King, *Positive linear operators which preserve x^2*, Acta Math. Hungar. **99**(2003), 203–208.

[6] P. P. Korovkin, *Linear operators and approximation theory*, Hindustan Publ. Co., Delhi, 1960.

[7] A. Lupaş, *The approximation by some positive linear operators*, in: *Proceedings of the International Dortmund Meeting on Approximation Theory* (M.W. Müller et. al., Eds.), Academie Verlag, Berlin, 1995, 201–229.