An improved Altman type generalization of the Brézis–Browder ordering principle

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Abstract. *By using a modified argument, we prove an improvement of our former Altman type generalization of the Brézis–Browder ordering principle which yields a stronger maximum principle.*

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Introduction

In 1976, to unify a number of diverse results in nonlinear functional analysis, H. Brézis and F. E. Browder [3] proved the following general ordering principle.

Theorem 1. *Let X be an ordered set; for $x \in X$ denote $S(x) = \{ y \in X;\ y \geq x \}$. Let $\phi : X \to \mathbb{R}$ be a function satisfying*

(1) $x \leq y$ implies $\phi(x) \leq \phi(y)$;

(2) for any increasing sequence $\{x_n\}$ in X such that $\phi(x_n) \leq C < \infty$ for all n, there exists some $y \in X$ such that $x_n \leq y$ for all n;

(3) for every $x \in X$ there exists $u \in X$ such that $x \leq u$ and $\phi(x) < \phi(u)$.

Then, for each $x \in X$, $\phi\big(S(x)\big)$ is unbounded.

As a direct consequence of this theorem, the above authors derived the following maximum principle.

Corollary 1. *Let $\phi X \to \mathbb{R}$ be a function, bounded above, and satisfying*

(1') $x \leq y$ and $x \neq y$ imply $\phi(x) < \phi(y)$;

(4) for any increasing sequence $\{x_n\}$ in X, there exists some $y \in X$ such that $x_n \leq y$ for all n.

Then, for each $a \in X$, there exists some $\bar{a} \in X$ such that $a \leq \bar{a}$ and $\bar{a}$ is maximal (i. e., $S(\bar{a}) = \{\bar{a}\}$).

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The importance of this corollary lies mainly in the fact that it easily yields a simplified version of Ekeland's variational principle and hence also of Caristi's fixed point theorem. Moreover, it can also be used to prove Danes' drop theorem [3].

In 1982, having in mind the function Φ, defined by $\Phi(x, y) = \phi(x) - \phi(y)$ for all $x, y \in X$, M. Altman [1] generalized the above theorem in the following less satisfactory form.

Theorem 2. *Let* $(X \ \leq)$ *be an ordered set such that every totally ordered sequence* $\{x_n\} \subset X$ *such that* $x_{n+1} \leq x_n$ *for* $n = 1, 2, \ldots$ *has a minorant, i. e., there exists an element* $y \in X$ *such that*

(i) $y \leq x_n$ *for* $n = 1, 2, \ldots$.

Let $w = \Phi(x, y)$ *be a real-valued function defined for all* $x, y \in X$ *such that for each given* y, $\Phi(\cdot, y)$ *is bounded from below on* $S(y) = [z \in X \mid z \leq y]$*;*

(ii) $\Phi(x, y) \leq 0$ *if* $x \leq y$ *for all* $x, y \in X$*;*

(iii) Φ *is non-increasing in the second variable, i. e., for any given* $x \in X$, $\Phi(x, y_2) \leq \Phi(x, y_1)$ *if* $y_1 \leq y_2$ *for all* $y_1, y_2 \in X$*;*

(iv) $\liminf \Phi(x_{n+1}, x_n) = 0$.

Then for each $x \in X$ *there exists a* $y \in X$ *such that* $y \leq x$ *and* $z \leq y$ *implies* $\Phi(z, y) = 0$.

As a direct consequence of this *Theorem 2*, the above author derived the following

Corollary 2. *Suppose that the hypotheses of* Theorem 2 *are satisfied with the assumption (ii) replaced by the stronger one*

(iib) $x \leq y$ *and* $x \neq y$ *imply* $\Phi(x, y) < 0$.

Then for each $x \in X$ *there exists* $\bar{x} \in X$ *such that* $\bar{x} \leq x$ *and* $\bar{x}$ *is minimal, i. e.,* $z \leq \bar{x}$ *implies* $z = \bar{x}$.

In 1984, M. Turinici [19] gave a better formulation and a metric generalization of the above theorem which also yields a maximum principle. Altman's theorem, in a somewhat improved form, has also been included in Zeidler [23, p. 515].

In 2001, not being aware of the works of M. Turinici, the present author also proved a generalization of Altman's theorem and derived a maximum principle. However, it has turned out that this theorem also contained several superfluous hypotheses.

Therefore, in the present paper we shall show that, by using a somewhat modified argument, we can actually prove a stronger result which may have a wider range of applications. For this, it is convenient to introduce some particular terminology.

1. Some general definitions

Definition 1. *If* X *is a set, then a function* Φ *of* X^2 *into* $\overline{\mathbb{R}}$ *will be called an écart on* X.

Example 1. *If φ and ψ are functions of X into $\mathbb{R}$, then the function Φ, defined by $\Phi(x, y) = \varphi(y) - \psi(x)$ for all $x, y \in X$, is a natural écart on X.*

Definition 2. *A set X equipped with a relation $\leq$ will be called a goset (generalized ordered set).*

Remark 1. *A goset X will be called reflexive, symmetric and transitive, if the relation in it has the corresponding property.*

Definition 3. *If Φ is an écart on a goset X, then the function γ_Φ, defined by*

$$\gamma_\Phi(x) = \sup_{y \geq x} \Phi(x, y)$$

for all $x \in X$, will be called the gauge of Φ.

Remark 2. *Note that if X is a reflexive goset and Φ is as in* Example 1, *then $-\infty < \gamma_\Phi(x)$ for all $x \in X$. Moreover, if $a \in X$ is such that φ is bounded above on $[a, +\infty[= \{x \in X : \ a \leq x\}$, then $\gamma_\Phi(a) < +\infty$.*

Concerning the function γ_Φ, it is also worth noticing the following

Proposition 1. *If Φ is an écart on a goset X such that for any $x_1, x_2, y \in X$, with $x_1 \leq x_2$ and $x_2 \leq y$, there exists $z \in X$, with $x_1 \leq z$, such that $\Phi(x_2, y) \leq \Phi(x_1, z)$, then γ_Φ is decreasing.*

Proof. Suppose that $x_1, x_2 \in X$ such that $x_1 \leq x_2$. If $x_2 \not\leq y$ for all $y \in X$, then because of $\sup(\emptyset) = -\infty$ we have $\gamma_\Phi(x_2) = -\infty$. Therefore, $\gamma_\Phi(x_2) \leq \gamma_\Phi(x_1)$ automatically holds.

If $y \in X$ such that $x_2 \leq y$, then by the assumption of the theorem there exists $z \in X$, with $x_1 \leq z$ such that $\Phi(x_2, y) \leq \Phi(x_1, z)$. Hence, by the definition of the supremum, it is clear that

$$\Phi(x_2, y) \leq \Phi(x_1, z) \leq \sup_{w \geq x_1} \Phi(x_1, w) = \gamma_\Phi(x_1).$$

Therefore, by the definition of the supremum, $\gamma_\Phi(x_2) = \sup_{y \geq x_2} \Phi(x_2, y) \leq \gamma_\Phi(x_1)$ also holds. □

Now, as an immediate consequence of the above proposition, we can also state

Corollary 3. *If Φ is an écart on a transitive goset X such that for any $x_1, x_2, y \in X$, with $x_1 \leq x_2$ and $x_2 \leq y$, we have $\Phi(x_2, y) \leq \Phi(x_1, y)$, then γ_Φ is decreasing.*

Remark 3. *Note that if X is a transitive goset and Φ is as in* Example 1 *such that ψ is increasing, then γ_Φ is already decreasing by the above corollary.*

2. A generalized ordering principle

The importance of the above observations on γ_Φ lies mainly in the following

Lemma 1. *If Φ is an écart on a goset X such that*

(1) γ_Φ is decreasing;

(2) $-\infty < \gamma_\Phi(x)$ for all $x \in X$;

(3) $\gamma_{\Phi}(a) < +\infty$ *for some* $a \in X$ *;*

then there exists an increasing sequence $(x_n)_{n=1}^{\infty}$ *in* X*, with* $x_1 = a$*, such that*

$$\lim_{n\to\infty} \gamma_{\Phi}(x_n) = \lim_{n\to\infty} \Phi(x_n, x_{n+1}).$$

Proof. Define $x_1 = a$. Then, by (2) and (3), we have $-\infty < \gamma_{\Phi}(x_1) < +\infty$. Therefore,

$$\gamma_{\Phi}(x_1) - 1 < \gamma_{\Phi}(x_1) = \sup_{y \geq x_1} \Phi(x_1, y).$$

Hence, by the definition of the supremum, it is clear that there exists $x_2 \in X$, with $x_1 \leq x_2$, such that

$$\gamma_{\Phi}(x_1) - 1 < \Phi(x_1, x_2).$$

Moreover, by using (2) and (1), we can also note that $-\infty < \gamma_{\Phi}(x_2) \leq \gamma_{\Phi}(x_1) < +\infty$. Therefore,

$$\gamma_{\Phi}(x_2) - 2^{-1} < \gamma_{\Phi}(x_2) = \sup_{y \geq x_2} \Phi(x_2, y).$$

Hence, by the definition of the supremum, it is clear that there exists $x_3 \in X$, with $x_2 \leq x_3$, such that

$$\gamma_{\Phi}(x_2) - 2^{-1} < \Phi(x_2, x_3).$$

Moreover, by using (2) and (1), we can note that $-\infty < \gamma_{\Phi}(x_3) \leq \gamma_{\Phi}(x_2) < +\infty$.

Now, by induction, it is clear that there exists an increasing sequence $(x_n)_{n=1}^{\infty}$ in X, with $x_1 = a$, such that

$$\gamma_{\Phi}(x_n) - n^{-1} < \Phi(x_n, x_{n+1})$$

for all $n \in \mathbb{N}$. Moreover, we can also note that

$$\Phi(x_n, x_{n+1}) \leq \sup_{y \geq x_n} \Phi(x_n, y) = \gamma_{\Phi}(x_n)$$

for all $n \in \mathbb{N}$. Therefore, we actually have

$$\gamma_{\Phi}(x_n) - n^{-1} < \Phi(x_n, x_{n+1}) \leq \gamma_{\Phi}(x_n)$$

for all $n \in \mathbb{N}$. Hence, by using the monotonicity of the sequence $\big(\gamma_{\Phi}(x_n)\big)_{n=1}^{\infty}$ and some basic theorems on the limits of sequences in $\overline{\mathbb{R}}$, we can infer that

$$\lim_{n\to\infty} \gamma_{\Phi}(x_n) = \lim_{n\to\infty} \Phi(x_n, x_{n+1}).$$

□

Now, by using the above lemma, we can easily prove the following generalized ordering principle.

Theorem 3. *If* Φ *is as in* Lemma 1 *and* $\alpha \in \overline{\mathbb{R}}$ *such that*

(4) each increasing sequence $(x_n)_{n=1}^{\infty}$ *in* X*, with* $x_1 = a$ *is bounded above and satisfies*

$$\underline{\lim}_{n\to\infty} \Phi(x_n, x_{n+1}) \leq \alpha;$$

then there exists $b \in X$*, with* $a \leq b$*, such that* $\gamma_{\Phi}(b) \leq \alpha$*.*

Proof. If $(x_n)_{n=1}^{\infty}$ is as *Lemma 1*, then by (4) we have

$$\lim_{n\to\infty} \gamma_{\Phi}(x_n) = \lim_{n\to\infty} \Phi(x_n, x_{n+1}) = \underline{\lim}_{n\to\infty} \Phi(x_n, x_{n+1}) \leq \alpha.$$

Moreover, by (4), there exists $b \in X$ such that $x_n \leq b$ for all $n \in \mathbb{N}$. Thus, in particular $a = x_1 \leq b$. Moreover, by (1) it is clear that $\gamma_{\Phi}(b) \leq \gamma_{\Phi}(x_n)$ for all $n \in \mathbb{N}$, and thus

$$\gamma_{\Phi}(b) \leq \lim_{n\to\infty} \gamma_{\Phi}(x_n) \leq \alpha.$$

□

3. Applications of the generalized ordering principle

Theorem 3 easily yields the following extension of the main ordering principle of our former paper [13].

Theorem 4. *Assume that* Φ *is an écart on a goset* X *such that* γ_{Φ} *is decreasing. Moreover, assume that there exists* $\alpha \in \overline{\mathbb{R}}$ *such that*

(a) $\alpha < \gamma_{\Phi}(x)$ *for all* $x \in X$*;*

(b) each increasing sequence $(x_n)_{n=1}^{\infty}$ *in* X*, with* $\sup_{x_n \geq x_1} \Phi(x_1, x_n) < +\infty$*, is bounded above and satisfies* $\underline{\lim}_{n\to\infty} \Phi(x_n, x_{n+1}) \leq \alpha$*.*

Then, we have $\gamma_{\Phi}(x) = +\infty$ *for all* $x \in X$*.*

Proof. If the required assertion is not true, then there exists $a \in X$ such that $\gamma_{\Phi}(a) < +\infty$. Hence, it is clear that for any sequence $(x_n)_{n=1}^{\infty}$ in X, with $x_1 = a$, we have

$$\sup_{x_n \geq x_1} \Phi(x_1, x_n) \leq \sup_{y \geq x_1} \Phi(x_1, y) = \gamma_{\Phi}(x_1) = \gamma_{\Phi}(a) < +\infty.$$

Therefore, by condition (b) and *Theorem 3*, there exists $b \in X$ such that $\gamma_{\Phi}(b) \leq \alpha$. Moreover, by condition (a), we have $\alpha < \gamma_{\Phi}(b)$. This contradiction proves the required assertion. □

By using *Theorem 3*, we can also easily establish an extension of the main maximum principle of our former paper [13]. For this, it seems convenient to introduce the following

Definition 4. *An écart* Φ *on a goset* X*, satisfying (1)–(3), will be called admissible at the point* a *if there exists* $\alpha \in \overline{\mathbb{R}}$ *such that, in addition to (4), we also have*

(5) $\alpha < \Phi(x, y)$ *for all* $x, y \in X$ *with* $x < y$.

Now, by calling an element x of a goset X maximal if $x \leq y$ implies $x = y$ for all $y \in X$, we can easily state and prove the following generalized maximum principle.

Theorem 5. *If* X *is a goset and* $a \in X$ *such that there exists an écart* Φ *on* X *which is admissible at* a*, then there exists a maximal element* b *of* X *with* $a \leq b$.

Proof. By *Definition 4*, there exists $\alpha \in \overline{\mathbb{R}}$ such that, in addition to $(1)-(3)$, we also have (4) and (5). Thus, in particular by *Theorem 3* there exists $b \in X$, with $a \leq b$, such that $\gamma_{\Phi}(b) \leq \alpha$, and thus $\Phi(b, y) \leq \alpha$ for all $y \in X$ with $b \leq y$.

Now, it remains only to show that b is maximal. For this, note that if this not the case, then there exists $y \in X$, with $b \leq y$, such that $b \neq y$, and thus $b < y$. Then, by the above property of b, we have $\Phi(b, y) \leq \alpha$. Moreover, by condition (5), we also have $\alpha < \Phi(b, y)$. This contradiction proves the maximality of b. □

Remark 4. *By making some obvious modifications in conditions (4) and (5), we can also easily establish the existence of an element* b *of* X*, with* $a \leq b$*, which is quasi-maximal in the sense that* $b \leq y$ *implies* $y \leq b$ *for all* $y \in X$.

Note that if the goset X *is reflexive, then every maximal element of* X *is quasi-maximal. While, if the goset* X *is antisymmetric, then the converse statement holds. Therefore, in a reflexive and antisymmetric goset the two notions coincide.*

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References

[1] M. Altman, *A generalization of the Brézis-Browder principle on ordered sets*, Nonlinear Anal. **6**(1982), 157–165.

[2] E. Bishop, R. R. Phelps, *The support functionals of a convex set*, In: *Convexity, Proc. Symp. Pure Math. VII* (V. Klee, Ed.), Amer. Math. Soc., 1963, 27–35.

[3] H. Brézis, F. E. Browder, *A general principle on ordered sets in nonlinear functional analysis*, Adv. Math. **21**(1976), 355–364.

[4] A. Brøndsted, *On a lemma of Bishop and Phelps*, Pacific J. Math. **55**(1974), 335–341.

[5] A. Brøndsted, *Common fixed points and partial order*, Proc. Amer. Math. Soc. **77**(1979), 365–368.

[6] S. DANCS, M. HEGEDŰS, P. MEDVEGYEV, *A general ordering and fixed point principle in complete metric space*, Acta Sci. Math. (Szeged) **46**(1983), 381–388.

[7] I. EKELAND, *On the variational principle*, J. Math. Anal. Appl. **47**(1974), 324–353.

[8] I. EKELAND, *Nonconvex minimization problems*, Bull. Amer. Math. Soc. (N.S.) **1**(1979), 443–474.

[9] A. HAMEL, *Variational Principles on Metric and Uniform Spaces* (Habilitation Thesis), Martin-Luther University, Halle-Wittenberg (Germany), 2005.

[10] T. L. HICKS, *Some fixed point theorems*, Radovi Mat. **5**(1989), 115–119.

[11] B. G. KANG, S. PARK, *On generalized ordering principles in nonlinear analysis*, Nonlinear Anal. **14**(1990), 159–165.

[12] N. MIZOGUCHI, *A generalization of Brøndsted results and its applications*, Proc. Amer. Math. Soc. **108**(1990), 707–714.

[13] Á. SZÁZ, *An Altman type generalization of the Brézis–Browder ordering principle*, Math. Moravica **5**(2001), 1–6.

[14] Á. SZÁZ, *Lower and upper bounds in ordered sets without axioms*, Tech. Rep., Inst. Math., Univ. Debrecen **2004/1**, 1–11.

[15] Á. SZÁZ, *Some easy to remember abstract forms of Ekeland's variational principle and Caristi's fixed point theorem*, Applicable Analysis and Discrete Mathematics **1**(2007), 335–339

[16] M. TURINICI, *Maximal elements in a class of order complete metric spaces*, Math. Japon. **25**(1980), 511–517.

[17] M. TURINICI, *Maximality principles and mean value theorems*, An. Acad. Bras. Ciênc. **53**(1981), 653–655.

[18] M. TURINICI, *A generalization of Brézis-Browder's ordering principle*, A. Stiint. Univ. Al. I. Cuza Iasi, Ser. Nouă, Mat. **28**(1982), 11–16.

[19] M. TURINICI, *A generalization of Altman's ordering principle*, Proc. Amer. Math. Soc. **90**(1984), 128–132.

[20] M. TURINICI, *Metric variants of the Brézis-Browder ordering principle*, Demonstr. Math. **22**(1989), 213–228.

[21] M. TURINICI, *Relational Brézis-Browder principles*, Fixed Point Theory **7**(2006), 111–126.

[22] I. VÁLYI, *A general maximality principle and a fixed point theorem in uniform space*, Period. Math. Hung. **16**(1985), 127–134.

[23] E. ZEIDLER, *Nonlinear Functional Analysis and its Applications I: Fixed–Point Theorems*, Spinger-Verlag, New York, 1986.