On I and I*−convergence of double sequences

VIJAY KUMAR*

Abstract. *The idea of I-convergence for single sequences was introduced by Kostyrko, Salat and Wilczynski [7] in 2000/2001 and developed in [1], [2], [3], [6], [8], [9], and [15]. Nowaday it has become one of the most active areas of research in classical analysis. Recently Tripathy and Tripathy [15] extended the concept of I-Convergence from single sequences to double sequences. In this paper we introduce the concept of I^*-convergence for double sequences and prove some results for I and I^*-convergence of double sequences.*

Key words: *statistical convergence, I-convergence, I-Cauchy sequences, double sequences*

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1. Introduction

The notion of the statistical convergence was first independently introduced by Fast [4] and Schonenberg [14]. Later on it was further investigated from a sequence space point of view and linked with summability theory by Fridy [5], Salat [13], and many others. In [10] and [11] the above concept is extended to double sequences by using the idea of a two dimensional analogue of natural density. Kostyrko, Salat and Wilczynski [7] defined I-convergence for single sequences which is a natural generalization of statistical convergence. The idea of I-convergence is based on the notion of the ideal I of subsets of N, the set of positive integers. Tripathy and Tripathy [15] introduced the concept of I-convergence and I-Cauchy sequence for double sequences and proved some properties related to the solidity, symmetricity, completeness and denseness. In the present paper we introduce the concept of I^*-convergence of double sequences and prove some results for I and I^*-convergence in a more natural way.

*National Institue of Technology, Deemed University, Kurukshetra-136119, Haryana State, India, e-mail: `vjy_kaushik@yahoo.com`. Presently working in Haryana College of Technology & Mgt., Kaithal 136027, Haryana State, India.

2. Known definitions and theorems

Throughout the paper, N will denote the set of positive integers whereas N^2; the usual product set N × N. For any set X, P(X) stands for the power set of X and A^c will denote the complement of the set A.

Definition 2.1 ([7]) *If X is a non-empty set. A family of sets $I \subset P(X)$ is called an ideal in X if and only if (i) $\emptyset \notin I$; (ii) For each A, $B \in I$ we have $A \cup B \in I$; (iii) For each $A \in I$ and $B \subset A$ we have $B \in I$.*

Definition 2.2 ([7]) *Let X be a non-empty set. A non-empty family of sets $F \subset P(X)$ is called a filter on X if and only if (i) $\emptyset \in F$; (ii) For each A, $B \in F$ we have $A \cap B \in F$; (iii) For each $A \in F$ and $B \supset A$ we have $B \in$F.*

An ideal I is called non-trivial if I$\neq \emptyset$ and X $\notin$ I. It immediately follows that$I \subset P(X)$ is a non-trivial ideal if and only if the class F = F (I) = $\{X - A : A \in I\}$ is a filter on X. The filter F = F (I) is called the filter associated with the ideal I.

Definition 2.3 ([7]) *A non-trivial ideal $I \subset P(X)$ is called an admissible ideal in X if and only if it contains all singletons i.e., if it contains $\{\{\ x\} : x \in X\}$.*

Definition 2.4 ([7]) *Let I be a non trivial ideal of subsets of N. A sequence x $= (x_n)$ of numbers is said to be I-convergent to a number ξ if and only if for each $\epsilon > 0$, the set $A\ (\epsilon) = \{n \in N : |x_n - \xi| \geq \epsilon\}$ belongs to I. The number ξ is called the I-limit of the sequence $x = (x_n)$ and we write =I-$lim_{n\to\infty}x_n = \xi$.*

I-convergence generates another type of convergence which we call I*-convergence.

Definition 2.5 ([7]) *A sequence $x = (x_n)$ of numbers is said to be I*-convergent to a number ξ if and only if there exists a set $M = \{m_1 < m_2 < \ldots < m_k < \ldots\}$ in F (I) such that $lim_{k\to\infty}x_{m_k} = \xi$.*

Definition 2.6 ([3]) *A sequence $x = (x_n)$ is said to be I-Cauchy sequence if for each $\epsilon > 0$ there exists a positive integer k such that, the set $\{n \in N : |x_n - x_k| \geq \epsilon\}$ belongs to I.*

Definition 2.7 *A double sequence $x = (x_{ij})$ is said to be convergent to a number ξ in the Pringsheim's sense [12] if for each $\epsilon > 0$ there exists a positive integer m such that $|x_{ij} - \xi| < \epsilon$ whenever i, $j \geq m$. The number ξ is called the Pringsheim limit of the sequence x and we abbreviate it as P-$lim_{i,j\to\infty}x_{ij} = \xi$.*

Definition 2.8 ([12]) *A double sequence $x = (x_{ij})$ is said to be Cauchy sequence if for each $\epsilon > 0$ there exists a positive integer m such that $|x_{ij} - x_{pq}| < \epsilon$ for every $i \geq p \geq m$ and $j \geq q \geq m$.*

Definition 2.9 ([12]) *A double sequence $x = (x_{ij})$ is said to be bounded if there exists a real number $M > 0$ such that $|x_{ij}| < M$ for each i and j, i.e., if $\|x\|_{(\infty,2)} = sup_{ij}|x_{ij}| < \infty$. We shall denote the set of all bounded double sequences by ℓ^2_∞. Note that in contrast to the case for single sequences a convergent double sequence need not be bounded.*

Mursaleen and Osama [11] introduced the two dimensional analogue of natural density; however the same concept was also introduced by F. Morciz [10]. Before starting the main results, we also recall the following definitions of [10] and [11].

Definition 2.10 *Let $K \subset N^2$ and K (m,n) denotes the number of (i,j) in K such that $i \leq m$ and $j \leq n$. Then the lower asymptotic density of K is defined by $\underline{\delta}_2(K) = lim\,inf_{m,n\to\infty}\frac{K(m,n)}{mn}$. In case the sequence $(\frac{K(m,n)}{mn})$ has a limit in Pringsheim's sense then we say that K has a double natural density and is defined* *by* $lim_{m,n\to\infty}\frac{K(m,n)}{mn} = \delta_2(K)$.

Definition 2.11 *A real double sequence* $x = (x_{ij})$ *is said to be statistically convergent to a number* ξ *if for each* $\epsilon > 0$*, the set*

$$A(\epsilon) = \{(i,j), i \le m, j \le n : |x_{ij} - \xi| \ge \epsilon\}$$

has double natural density zero. In this case we write, $st - lim_{i,j\to\infty}x_{ij} = \xi$ *. Let* st_2 *denote the set of all double sequences which are statistical convergent.*

Definition 2.12 *A real double sequence* $x = (x_{ij})$ *is said to be statistically Cauchy if for each* $\epsilon > 0$*, there exist positive integers* $m(\epsilon)$ *and* $n(\epsilon)$ *such that for every* $i, p \ge m$ *and* $j, q \ge n$*, the set* $\{(i,j), i \le m, j \le n : |x_{ij} - x_{pq}| \ge \epsilon\}$ *has double natural density zero.*

3. I-convergence

For further study we shall take X = N^2 and I will denote the ideal of subsets of N^2. As earlier, the following proposition express a relation between the notions of an ideal and a filter.

Proposition 3.1 $I \subset P(N^2)$ *is a non-trivial ideal if and only if the class* $F = F(I) = \{N^2 - A : A \in I\}$ *is a filter on* N^2.

Definition 3.1 *Let* $I \subset P(N^2)$ *be a non-trivial ideal in* N^2*. A double sequence* $x = (x_{ij})$ *of numbers is said to be I-convergent to a number* ξ *if for each* $\epsilon > 0$ *the set* $A(\epsilon) = \{(i,j) \in N^2 : |x_{ij} - \xi| \ge \epsilon\}$ *belongs to I. The number* ξ *is called the I-limit of the sequence* (x_{ij}) *and we write* $I - lim_{i,j\to\infty}x_{ij} = \xi$*. Let* I_2 *denotes the set of all double sequences which are I convergent.*

Remark 3.1 *If we take* $I = \{E \subset N^2$*: E is of the form* $(N \times A) \cup A \times N)$ *where A is a finite subset of* $N\}$*. Then I-convergence is equivalent to the usual Pringsheim's convergence.*

Remark 3.2 *Let* $I = I_{\delta_2} = \{$ *A: A is subset of* N^2 *such that* $\delta_2(A) = 0\}$*. Then I-convergence coincides with statistical convergence.*

Proposition 3.2 *I-limit of any double sequence if exist is unique.*

Proof. Let $x = (x_{ij})$ be any double sequence and suppose that $I - lim_{i,j\to\infty}x_{ij} = \xi$, $I - lim_{i,j\to\infty}x_{ij} = \eta$ where $\xi \ne \eta$. Since $\xi \ne \eta$, we may suppose that $\xi > \eta$. Select $\epsilon = \frac{\xi-\eta}{3}$, so that the neighborhoods $(\eta - \epsilon, \eta + \epsilon))$ and $(\xi - \epsilon, \xi + \epsilon)$ of η and ξ respectively are disjoints. Since ξ and η both are I-limit of the sequence $x = (x_{ij})$, therefore both the sets A= $\{(i,j) \in N^2 : |x_{ij} - \xi| \ge \epsilon\}$ and B = $\{(i,j) \in N^2 : |x_{ij} - \eta| \ge \epsilon\}$ belongs to I. This implies that the sets $A^C = \{(i,j) \in N^2 : |x_{ij} - \xi| < \epsilon\}$ and $B^C = \{(i,j) \in N^2 : |x_{ij} - \eta| < \epsilon\}$ belongs to F(I). Since F (I) is a filter on N^2 therefore $A^C \cap B^C$ is a non empty set in F(I). In this way we obtain a contradiction to the fact that the neighborhoods $(\eta - \epsilon, \eta + \epsilon))$ and $(\xi - \epsilon, \xi + \epsilon)$ of η and ξ respectively are disjoints. Hence we have $\xi = \eta$. □

Proposition 3.3 *If* $x = (x_{ij})$ *and* $y = (y_{ij})$ *are two double sequences, then*

(i) If I contains all sets of the form $N \times \{n\}, \{n\} \times N$*, for* $n \in N$ *then* $P - lim_{i,j\to\infty}x_{ij} = \xi$ *implies* $I - lim_{i,j\to\infty}x_{ij} = \xi$ *.*

(ii) If $I - lim_{i,j\to\infty}x_{ij} = \xi$ *and* $I - lim_{i,j\to\infty}y_{ij} = \eta$*, then* $I - lim_{i,j\to\infty}(x_{ij} + y_{ij}) = \xi + \eta$*.*

(iii) *If* $I-lim_{i,j\to\infty}x_{ij} = \xi$ *and* $I-lim_{i,j\to\infty}y_{ij} = \eta$*, then* $I-lim_{i,j\to\infty}(x_{ij}y_{ij}) = \xi\eta$*, where* $x_{ij}y_{ij}$ *means usual multiplication of the corresponding entries of the sequences x and y.*

Proof. (i) Let $\epsilon > 0$ be given. Since $x = (x_{ij})$ is P-convergent to ξ, therefore there exists a positive integer m such that $|x_{ij} - \xi| < \epsilon$ whenever i, j $\geq m$. This implies that the set A $= \{(i,j) \in N^2 : |x_{ij} - \xi| \geq \epsilon\} \subset N \times \{1,2,3...m-1\} \cup \{1,2,3...m-1\} \times N$. Since I contains all sets of the form $N \times \{n\}, \{n\} \times N$, for $n \in N$ therefore the set on the right side belongs to I. As I is an ideal therefore A belongs to I. This shows that, $I-lim_{i,j\to\infty}x_{ij} = \xi$.
(ii) Let $\epsilon > 0$ be given. Since $I-lim_{i,j\to\infty}x_{ij} = \xi$ and $I-lim_{i,j\to\infty}y_{ij} = \eta$, therefore the sets A $= \{(i,j) \in N^2 : |x_{ij} - \xi| \geq \frac{\epsilon}{2}\}$ and B $= \{(i,j) \in N^2 : |y_{ij} - \eta| \geq \frac{\epsilon}{2}\}$ belongs to I. Let C $= \{(i,j) \in N^2 : |(x_{ij} + y_{ij}) - (\xi + \eta)| \geq \epsilon\}$. Since I is an ideal therefore to prove the result it is sufficient to prove that C $\subset A \cup B$. For this let, $(i,j) \in C$, then we have $\epsilon \leq |(x_{ij} + y_{ij}) - (\xi + \eta)| \leq |x_{ij} - \xi| + |y_{ij} - \eta|$. As both of $\{|x_{ij} - \xi|, |y_{ij} - \eta|\}$ can not be (together) strictly less than $\frac{\epsilon}{2}$, and therefore we have either $|x_{ij} - \xi| \geq \frac{\epsilon}{2}$ or $|y_{ij} - \eta| \geq \frac{\epsilon}{2}$. This shows that (i, j) belongs to A or B i. e, (i, j) $\in A \cup B$. Hence C $\subset$ A $\cup$ B and therefore the result follows.
(iii) Since $I-lim_{i,j\to\infty}x_{ij} = \xi$, therefore the set $\{(i,j) \in N^2 : |x_{ij} - \xi| \geq 1\}$ belongs to I, which implies that the set A $= \{(i,j) \in N^2 : |x_{ij} - \xi| < 1$ belongs to F(I). Also for any (i,j) in A we have $|x_{ij}| < |\xi| + 1$. Let $\epsilon > 0$ be given. Choose $\delta > 0$ such that $0 < 2\delta < \frac{\varepsilon}{|\xi|+|\eta|+1}$. It follows from the assumption that the sets B $= \{(i,j) \in N^2 : |x_{ij} - \xi| < \delta\}$, C $= \{(i,j) \in N^2 : |y_{ij} - \eta| < \delta\}$ belongs to F(I). Since F(I) is a filter therefore A $\cap$ B $\cap$ C $\in$ F(I). Also for each (i,j) $\in A \cap B \cap C$, we have $|x_{ij}y_{ij} - \xi\eta| = |x_{ij}y_{ij} - x_{ij}\eta + x_{ij}\eta - \xi\eta| \leq |x_{ij}||y_{ij} - \eta| + |\eta||x_{ij} - \xi| < (|\xi|+1)\delta + |\eta|\delta = (|\xi|+|\eta|+1)\delta < \epsilon$. Hence $\{(i,j) \in N^2 : |x_{ij}y_{ij} - \xi\eta| \geq \epsilon\}$ belongs to I, and therefore (iii) holds. □

Proposition 3.4 *Let* $x = (x_{ij})$ *and* $y = (y_{ij})$ *be two real double sequences. Then:*

(i) *If* $I-lim_{i,j\to\infty}x_{ij} = \xi$ *and* $x_{ij} \geq 0$ *for every (i,j) in K, where* $K \in F(I)$*, then* $\xi \geq 0$.

(ii) *If* $x = (x_{ij})$ *and* $y = (y_{ij})$ *be two double sequences such that* $x_{ij} \leq y_{ij}$ *for every (i,j) in K, where* $K \in F(I)$ *and if* $I-lim_{i,j\to\infty}x_{ij} = \xi$, $I-lim_{i,j\to\infty}y_{ij} = \eta$ *then* $\xi \leq \eta$.

Proof. (i) If possible, let $\xi < 0$. Select $\epsilon = -\frac{\xi}{2}$. Since $I-lim_{i,j\to\infty}x_{ij} = \xi$, therefore the set A$= \{(i,j) \in N^2 : |x_{ij} - \xi| < \epsilon\}$ belongs to F(I). Since F(I) is a filter on N^2 and the sets $A, K \in F(I)$, therefore $A \cap K$ is a non empty set in F(I). So we can find out a pair (i_0, j_0) in K such that $|x_{i_0j_0} - \xi| < \epsilon$. This implies that $x_{i_0j_0} < 0$. In this way we obtain a contradiction to the fact that $x_{ij} \geq 0$ for every (i, j) in K. Hence we have $\xi \geq 0$.
(ii). If possible, let $\xi > \eta$. Select $\epsilon = \frac{\xi-\eta}{3}$, so that the neighborhoods $(\eta - \epsilon, \eta + \epsilon)$, $(\xi - \epsilon, \xi + \epsilon)$ of η and ξ respectively are disjoints. Since $I-lim_{i,j\to\infty}x_{ij} = \xi$, $I-lim_{i,j\to\infty}y_{ij} = \eta$, therefore both the sets A $= \{(i,j) \in N^2 : |x_{ij} - \xi| < \epsilon\}$ and B $= \{(i,j) \in N^2 : |y_{ij} - \eta| < \epsilon\}$ belongs to F(I). This implies that $\emptyset \neq A \cap B \cap K \in$ F (I), and therefore there exists a pair (i_0, j_0) in K such that $|x_{i_0j_0} - \xi| < \epsilon$ and

$|y_{i_0j_0} - \eta| < \epsilon$. This shows that $y_{i_0j_0} < x_{i_0j_0}$. In this way we obtain a contradiction to the fact that $x_{ij} \le y_{ij}$ for every (i, j) in K. Hence we have $\xi \le \eta$. □

Theorem 3.1 (Sandwich theorem) *If $x = (x_{ij})$, $y = (y_{ij})$ and $z = (z_{ij})$ be three double sequences such that*

(i) $x_{ij} \le y_{ij} \le z_{ij}$, for every (i, j) in K, where $K \in F(I)$, and

(ii) $I - lim_{i,j\to\infty} x_{ij} = \xi$, $I - lim_{i,j\to\infty} z_{ij} = \xi$,

then $I - lim_{i,j\to\infty} y_{ij} = \xi$.

Proof. Let $\epsilon > 0$ be given. By condition (ii) the sets $\{(i,j) \in N^2 : |x_{ij} - \xi| \ge \epsilon\}$ and $\{(i,j) \in N^2 : |z_{ij} - \xi| \ge \epsilon\}$ belongs to I. This implies that the sets

$$A = \{(i,j) \in N^2 : |x_{ij} - \xi| < \epsilon\}, \quad C = \{(i,j) \in N^2 : |z_{ij} - \xi| < \epsilon\}$$

belongs to F(I). Let B = $\{(i,j) \in N^2 : |y_{ij} - \xi| < \epsilon\}$. It is clear that, the set $A \cap C \cap K$ is contained in B. Since F(I) is a filter on N^2 and $A \cap C \cap K$ belongs to F(I) therefore $B \in F(I)$. Hence the set $\{(i,j) \in N^2 : |y_{ij} - \xi| \ge \epsilon\}$ belongs to I and therefore the theorem is proved. □

Theorem 3.2 *Let $I \subset P(N^2)$ be an admissible ideal in N^2. Then $I_2 \cap \ell^2_\infty$ is a closed linear sub space of the normed linear space ℓ^2_∞.*

Proof. By *Proposition 3.3*, it is obvious that $I_2 \cap \ell_\infty$ is a linear subspace of the normed linear space ℓ^2_∞. So to prove the result it is sufficient to prove that $I_2 \cap \ell_\infty$ is closed. Let $= x^{(mn)} = (x^{(mn)}_{ij})$ be a convergent sequence in $I_2 \cap \ell^2_\infty$. Suppose that $x^{(mn)}$ converges to x. It is clear that $x \in \ell^2_\infty$. Since $x^{(mn)} \in I_2$, therefore by definition of I-convergence there exist real numbers a_{mn} such that $I - lim_{i,j\to\infty} x^{(mn)}_{ij}$ $= a_{mn}$ (m, n = 1, 2, 3 . . .). As $x^{(mn)} \to x$, this implies that $x^{(mn)}$ is a Cauchy sequence. So for each $\epsilon > 0$, there exists a positive integer n_0 such that

$$|x^{(pq)} - x^{(mn)}| < \frac{\epsilon}{3} \quad for \quad every \quad p \ge m \ge n_0, q \ge n \ge n_0 \tag{1}$$

where |.| denotes the norm in the linear space. Since $I - lim_{i,j\to\infty} x^{(pq)}_{ij} = a_{pq}$ and $I - lim_{i,j\to\infty} x^{(mn)}_{ij} = a_{mn}$, therefore by definition of I-convergence both the sets $\{(i,j) \in N^2 : |x^{(pq)}_{ij} - a_{pq}| \ge \frac{\epsilon}{3}\}$ and $\{(i,j) \in N^2 : |x^{(mn)}_{ij} - a_{mn}| \ge \frac{\epsilon}{3}\}$ belongs to I.

$$Let, K_1 = \{(i,j) \in N^2 : |x^{(pq)}_{ij} - a_{pq}| < \frac{\epsilon}{3}\}, K_2 = \{(i,j) \in N^2 : |x^{(mn)}_{ij} - a_{mn}| < \frac{\epsilon}{3}\}. \tag{2}$$

Then both sets K_1 and K_2 belongs to F(I). Since F(I) is a filter on N^2 therefore$K_1 \cap K_2$ is a non empty set in F(I). Choose $(k_1, k_2) \in K_1 \cap K_2$, then we have from (2) that

$$|x^{(mn)}_{k1k2} - a_{mn}| < \frac{\epsilon}{3} \quad and \quad |x^{(pq)}_{k1k2} - a_{pq}| < \frac{\epsilon}{3} \tag{3}$$

Therefore for each $p \ge m \ge n_0$ and $q \ge n \ge n_0$, we have from (1) to (3) $|a_{pq} - a_{mn}| = |a_{pq} - x^{(pq)}_{k1k2} + x^{(pq)}_{k1k2} - x^{(mn)}_{k1k2} + x^{(mn)}_{k1k2} - a_{mn}| \le |a_{pq} - x^{(pq)}_{k1k2}| + |x^{(pq)}_{k1k2} - x^{(mn)}_{k1k2}| + |x^{(mn)}_{k1k2} - a_{mn}| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$. This implies that (a_{mn}) is a Cauchy sequence and consequently convergent. Let,

$$lim_{mn\to\infty} a_{mn} = a \tag{4}$$

Now to prove the theorem it is enough to show that the sequence x is I-convergent to a. Since $x^{(mn)}$ is convergent to x in ℓ^2_∞ , so by the structure of ℓ^2_∞ it is also coordinate wise convergent. Therefore for each $\epsilon > 0$, there exist a positive integer $n_1(\epsilon)$ such that

$$|x_{ij}^{(mn)} - x_{ij}| < \frac{\epsilon}{3} \quad for \quad every \quad m,n \quad \geq n_1(\epsilon) \tag{5}$$

By (4) for each $\epsilon > 0$, there exist a positive integer $n_2(\epsilon)$ such that

$$|a_{mn} - a| < \frac{\epsilon}{3} \quad for \quad every \quad m,n \geq n_2(\epsilon) \tag{6}$$

Let $n_3(\epsilon) = max\{n_1(\epsilon), n_2(\epsilon)\}$ and chose $m_0, n_0 \geq n_3(\epsilon)$. Then for any (i,j)$\in N^2$

$$\begin{aligned} |x_{ij} - a| &= |x_{ij} - x_{ij}^{(m_0n_0)} + x_{ij}^{(m_0n_0)} - a_{m_0n_0} + a_{m_0n_0} - a| \\ &\leq |x_{ij} - x_{ij}^{(m_0n_0)}| + |x_{ij}^{(m_0n_0)} - a_{m_0n_0}| + |a_{m_0n_0} - a| \end{aligned}$$

$$< \quad \frac{\epsilon}{3} + |x_{ij}^{(m_0n_0)} - a_{m_0n_0}| + \frac{\epsilon}{3} \quad \text{(by using (5) and (6))} \tag{7}$$

$$\begin{aligned} Let, \quad A_{m_0n_0}(\frac{\epsilon}{3}) &= \{(i,j) \in N^2 : |x_{ij}^{(m_0n_0)} - a_{m_0n_0}| \geq \frac{\epsilon}{3}\} \\ A(\epsilon) &= \{(i,j) \in N^2 : |x_{ij} - a| \geq \epsilon\} \\ A^C_{m_0n_0}(\frac{\epsilon}{3}) &= \{(i,j) \in N^2 : |x_{ij}^{(m_0n_0)} - a_{m_0n_0}| < \frac{\epsilon}{3}\} and \\ A^C(\epsilon)&\{(i,j) \in N^2 : |x_{ij} - a| < \epsilon\} \end{aligned}$$

So for any (i,j) $\in A^C_{m_0n_0}(\frac{\epsilon}{3})$ we have by (7), $|x_{ij} - a| < \epsilon$ and therefore $A^C_{m_0n_0}(\frac{\epsilon}{3}) \subset A^C(\epsilon)$. This implies that $A(\epsilon) \subset A_{m_0n_0}(\frac{\epsilon}{3})$. Since $A_{m_0n_0}(\frac{\epsilon}{3}) \in$ I, therefore we have $A(\epsilon) \in$ I. Hence x is I-convergent to a and therefore x $\in I_2$. This proves that $I_2 \cap \ell^2_\infty$ is a closed linear subspace of ℓ^2_∞. □

Let $\Im$ denote the class of all admissible ideals in N^2 then $\Im$ is a partially ordered set with respect to the usual inclusion. If $I_0 \subset \Im$ is a non-void linearly ordered subset of $\Im$, then it is clear that $\cup\, I_0$ is an admissible ideal in N^2 which is an upper bound of I_0. So by Zorn's lemma $\Im$ has a maximal ideal. The following lemma gives a characterization of a maximal admissible ideal.

Lemma 3.1 *Let I_0 be an admissible ideal ideal in N^2, then I_0 is maximal if and only if $A \in I_0$ or $N^2 - A \in I_0$ holds for every $A \subset N^2$.*

Theorem 3.3 *Let $I \subset P\ (N^2)$ be an admissible ideal inN^2. Then $I_2 \cap \ell^2_\infty = \ell^2_\infty$ if and only if I is maximal ideal.*

Proof. First assume that I is maximal ideal and let x $= (x_{ij}) \in \ell^2_\infty$, then there exist a positive real number M such that $|x_{ij}| \leq M$ for every i and j. Let A_1={ $(i,j) \in N^2$: $-M \leq x_{ij} \leq 0$} and B_1= $\{(i,j) \in N^2$: $0 \leq x_{ij} \leq M\}$. Then it is clear that $N^2 = A_1 \cup B_1$. Since I is an admissible ideal therefore we have either $A_1 \notin$I or $B_1 \notin$I i.e., at least one of them does not belongs to I. Let K_1 denote the set which does not belongs to I and J_1 be the corresponding interval then we have $K_1 = \{(i,j) \in N^2 : x_{ij} \in J_1\} \notin$ I. We can therefore inductively construct a sequence $J_1 \supset J_2 \supset J_3 \supset ...J_i \supset J_{i+1} \supset ...$ of closed intervals such that $j_n \to 0$ as $n \to \infty$ and the sets K_p =$\{ (i,j) \in N^2$: $x_{ij} \in J_p\} \notin$ I for p = 1, 2, 3, ... By nested interval property we have $\cap_{p=1}^{\infty} J_p \neq \emptyset$. Let $\xi \in \cap_{p=1}^{\infty} J_p$. We shall prove that $I - lim_{i,j\to\infty} x_{ij} = \xi$. Let $\epsilon > 0$ be given . Since J_n decreasing to zero and $\xi \in \cap_{p=1}^{\infty} J_p$ therefore we can choose a positive integer m such that $J_n \subset (\xi - \epsilon, \xi + \epsilon)$ for every $n \geq m$. Now $K_m = \{(i,j) \in N^2 : x_{ij} \in J_m\} \notin$ I implies that the set $\{(i,j) \in N^2 : |x_{ij} - \xi| < \epsilon\} \notin$I. The maximality of I implies that $\{(i,j) \in N^2 : |x_{ij} - \xi| \geq \epsilon\}$ belongs to I. Hence $I - lim_{i,j\to\infty} x_{ij} = \xi$. Conversely-Assume that $I_2 \cap \ell_\infty^2 = \ell_\infty^2$. We prove that I is maximal. Suppose that I is not maximal. Then by *Lemma 3.1*, there exists a subset A = $\{(i,j)\}$ $i,j = 1,2,3,\ldots$ of N^2 such that $A \notin I$ and $A^C \notin I$. Define the sequence x = (x_{ij}) as follow:

$$x_{ij} = \begin{cases} 1, \text{ if (i,j)} \in A \\ 0, \text{ otherwise.} \end{cases}$$

We claim that x is not I-convergent. Suppose that there exist a real number ξ such that $I - lim_{i,j\to\infty} x_{ij} = \xi$. Since for sufficient small $\epsilon > 0$, the set $A(\epsilon) = \{(i,j) \in N^2 : |x_{ij} - \xi| \geq \epsilon\}$ is equal to A or A^C or N^2 and none of these sets belongs to I. Hence x is not I-convergent. Also it is obvious that $x \in \ell_\infty^2$. Thus we have a bounded sequence (x_{ij}) which is not I-convergent. This contradicts the assumption $I_2 \cap \ell_\infty^2 = \ell_\infty^2$. Hence I is maximal ideal. □

4. I^*-convergence of double sequences

In [13] Salat proved that a sequence x = (x_n) of real numbers is statistically convergent to ξ if and only if there exists a subset $K = \{m_1 < m_2 < m_3 ... < m_k ...\} \subset N$ with $\delta(K)$ =1 such that $lim_{k\to\infty} x_{m_k} = \xi$. Kostyrko, Salat and Wilczynski [7] used this result to introduce the concept of I^*-convergence for single sequences. Mursaleen and Osama [11] extend the above result of Salat analogously to double sequences as follow: A real double sequence x = (x_{ij}) is statistically convergent to a number ξ if and only if there exist a subset K = $\{(i,j)\} \subset N^2$, $i,j = 1,2,3,\ldots$ such that $\delta_2(K) = 1$ and $lim_{(i,j)\in K, i,j\to\infty} x_{ij} = \xi$. Analogous to [7], we use this result to introduce the concept of I^*-convergence for real double sequences as follow:

Definition 4.1 *A real double sequence $x = (x_{ij})$ is said to be I^*-convergent to a number ξ if and only if there exist a set $K = \{(i,j)\}$, i, j = 1, 2, 3,...in F (I) such that $lim_{(i,j)\in K, i,j\to\infty} x_{ij} = \xi$. Let I_2^* denotes the set of all double real sequences which are I^*-convergent.*

Proposition 4.1 *Let I be an admissible ideal such that I contain all sets of the form $H \times N$, $N \times H$ where H is a finite subset of N. If I^*-$lim_{i,j\to\infty} x_{ij} = \xi$, then $I - lim_{i,j\to\infty} x_{ij} = \xi$.*

Proof. Let I^*-$lim_{i,j\to\infty} x_{ij} = \xi$, therefore there exist a set K = {(i, j)} i, j = 1, 2, 3. . . In F (I) such that

$$lim_{(i,j)\in K, i,j\to\infty} x_{ij} = \xi \tag{8}$$

Let $\epsilon > 0$ be given. By virtue of (8) there exists a positive integer n_1 such that $|x_{ij} - \xi| < \epsilon$ for every $(i,j) \in K$ with $i,j \geq n_1$. Let A = {1, 2 ... n_1-1}; B =

$\{(i,j) \in K : |x_{ij} - \xi| \geq \epsilon\}$. Then it is clear that $B \subset (A \times N) \cup (N \times A)$ and therefore belongs to I. Also K $\in$ F (I), therefore K $=N^2$- H for some $H \in$I. Obviously the set $= \{(i,j) \in N^2 : |x_{ij} - \xi| \geq \epsilon\} \subset$ B $\cup$ H and therefore the proposition follows.□

The following example shows that the converse of the above proposition is not true.

Example 4.1 *Let N $=\cup_{i=1}^{\infty} N_i$ be a disjoint decomposition of N such that each N_i is an infinite set. Then it is obvious that $N^2 = \cup_{i=1}^{\infty} \cup_{j=1}^{\infty} (N_i \times N_j)$ is a disjoint decomposition of N^2. Let $I = \{A \subset N^2$:A is contained in $(N \times (\cup_{i=1}^{p} N_i) \cup (\cup_{j=1}^{q} N_j) \times N)$ for some positive integer p and q $\}$. Then it is clear that I is an admissible ideal in N^2 such that I contains all sets of the form $H \times N$, $N \times H$ where H is a finite subset of N. We define the sequence $x = (x_{mn})$ as follow: For (m, n) $\in N_i \times N_j$, define $x_{mn} = \frac{1}{i} + \frac{1}{j}$ where i,j= 1, 2, 3,... Obviously $lim_{m,n \to \infty} x_{mn}$= 0 and therefore by* Proposition 3.3, *$I-lim_{m,n \to \infty} x_{mn}$= 0. Next we prove that I^* - $lim_{m,n \to \infty} x_{mn}$ = 0 does not hold. Suppose that I^*-$lim_{m,n \to \infty} x_{mn}$ = 0, then by definition there exists a set K = $\{(m,n)\}$, m, n = 1, 2, 3,...in F (I) such that $lim_{(m,n) \in K, m,n \to \infty} x_{mn}$= 0. Since K $\in$F(I), therefore there is a set $B \in I$ such that $K = N^2 - B$. By definition of the ideal I there exist positive integers p and q such that B is contained in $(N \times (\cup_{i=1}^{p} N_i) \cup (\cup_{j=1}^{q} N_j) \times N)$. But then K contains the set $N_{p+1} \times N_{q+1}$ and therefore $x_{mn} = \frac{1}{p+1} + \frac{1}{q+1}$ for infinitely many $(m,n) \in N_{p+1} \times N_{q+1} \subset K$. This shows that $lim_{(m,n) \in K, m,n \to \infty} x_{mn}$ does not exist and therefore we obtain a contradiction to the fact that $lim_{(m,n) \in K, m,n \to \infty} x_{mn} = 0$.*

Definition 4.2 ([7]) *An admissible ideal $I \subset P(N^2)$ is said to be satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, \ldots\}$ belonging to I there exists a countable family $\{B_1, B_2, \ldots\}$ in I such that $A_i \triangle B_i$ is a finite set for each $i \in N$ and $B = \cup_{i=1}^{\infty} B_i \in I$.*

Proposition 4.2 *If the ideal I has the property (AP), then I-convergence implies $I^* -$ convergence for real double sequence.*

Proof. Suppose that the ideal I satisfies the condition (AP). Let x = (x_{ij}) be a real double sequence such that $I - lim_{i,j \to \infty} x_{ij} = \xi$. Then for each $\epsilon > 0$, the set $A(\epsilon) = \{(i,j) \in N^2 : |x_{ij} - \xi| \geq \epsilon\}$ belongs to I.

For $n \in N$, we define the set A_n as follow: Put $A_1 = \{(i,j) \in N^2 : |x_{ij} - \xi| \geq 1\}$ and $A_n = \{(i,j) \in N^2 : \frac{1}{n} \leq |x_{ij} - \xi| < \frac{1}{n-1}\}$ for $n \geq 2$, $n \in N$. Now it is clear that $\{A_1, A_2 ...\}$ is a countable family of mutually disjoint sets belonging to I and therefore by the condition (AP) there is a countable family of sets $\{B_1, B_2 ...\}$ in I such that $A_i \triangle B_i$ is a finite set for each $i \in N$ and $B = \cup_{i=1}^{\infty} B_i \in I$. Since B$\in$ I so there is set K in F (I) such that K = N^2 - B. Now to prove the result it is sufficient to prove that $lim_{(i,j) \in K, i,j \to \infty} x_{ij} = \xi$. Let $\eta > 0$ be given. Chose a positive integer q such that $\eta > \frac{1}{q+1}$. Then we have

$$\{(i,j) \in N^2 : |x_{ij} - \xi| \geq \eta\} \subset \{(i,j) \in N^2 : |x_{ij} - \xi| \geq \frac{1}{q+1}\} = \cup_{i=1}^{q+1} A_i \qquad (9)$$

Since $A_i \triangle B_j$ is a finite set for each i = 1, 2, 3 ...q +1, therefore there exist a positive integer n_0 such that $\{\{\cup_{i=1}^{q+1} B_i\} \cap \{(i,j) \in N^2 : i,j > n_0\}\} = \{\{\cup_{i=1}^{q+1} A_i\} \cap \{(i,j) \in N^2 : i,j > n_0\}\}$. If $i,j > n_0$ and $(i,j) \in K$, then (i,j) $\notin$ B. This implies that (i,j) $\notin \cup_{i=1}^{q+1} B_i$ and therefore (i,j) $\notin \cup_{i=1}^{q+1} A_i$. Hence for every $i,j > n_0$ and (i,j)$\in$ K we have by (9) $|x_{ij} - \xi| < \eta$. This completes the proof of the proposition. $\square$

Theorem 4.1 *For an admissible ideal I in N^2, closure $(I_2^* \cap \ell_\infty^2) = I_2 \cap \ell_\infty^2$.*

Proof. Since $(I_2^* \cap \ell_\infty^2) \subset I_2 \cap \ell_\infty^2$ and $I_2 \cap \ell_\infty^2$ is a closed linear subspace of ℓ_∞^2 , we get closure $(I_2^* \cap \ell_\infty^2) \subset I_2 \cap \ell_\infty^2$. Next we prove that $I_2 \cap \ell_\infty^2 \subset$ closure $(I_2^* \cap \ell_\infty^2)$. For $z \in \ell_\infty^2$ and $\delta > 0$, let $B(z, \delta) = \{x \in \ell_\infty^2 : \|x - z\|_{(\infty,2)} < \delta\}$ denote the open ball in ℓ_∞^2. So to prove the result it is sufficient to prove that for each $(x_{ij}) \in I_2 \cap \ell_\infty^2$ and $0 < \delta < 1$ we have $B(x, \delta) \cap I_2^* \cap \ell_\infty^2 \neq \emptyset$. Take $0 < \delta < 1$ and let $(x_{ij}) \in I_2 \cap \ell_\infty^2$ with $I - lim_{i,j\to\infty} x_{ij} = \xi$. Choose $\eta \in (0, \delta)$, then I-convergence of (x_{ij}) implies that the set A $= \{(i, j) \in N^2 : |x_{ij} - \xi| \geq \eta\}$ belongs to I. Let K $= N^2$ - A then K $\in$ F (I). We define a sequence (y_{ij}) as follow:

$$y_{ij} = \begin{cases} \xi, & \text{if } (i, j) \in K \\ x_{ij,} & \text{otherwise.} \end{cases}$$

Thus we have a set K $\in$ F(I) such that $lim_{(i,j)\in K, i,j\to\infty} y_{ij} = \xi$. This shows that $I-^* lim_{i,j\to\infty} y_{ij} = \xi$. As $(y_{ij}) \in \ell_\infty^2$, therefore $(y_{ij}) \in (I_2^* \cap \ell_\infty^2)$. Also it is obvious that $(y_{ij}) \in B(x, \eta)$. $\square$

5. I - Cauchy sequence

K. Dems [3] proved that, in a complete metric space (X, ρ); I-Cauchy sequence is necessary and sufficient for the I-convergence of a sequence. He also extended this result for double sequences. The same result was proved by Tripathy and Tripathy [15]. The proof given by the authors is very short and interesting however we give its different proof.

Definition 5.1 ([15]) *A real double sequence $x = (x_{ij})$ is said to be I- Cauchy sequence if for each $\epsilon > 0$, there exists (m, n) in N^2 such that the set*

$$\{(i, j) \in N^2 : |x_{ij} - x_{mn}| \geq \epsilon\} \text{ belongs to } I.$$

Theorem 5.1 *Let I$\subset$P(N^2) be an admissible ideal. A double sequence $x = (x_{ij})$ is I-convergent if and only if it is I- Cauchy.*

Proof. Necessity: Suppose that (x_{ij}) is I-convergent to ξ. Let $\epsilon > 0$ be given. Since $I - lim_{i,j\to\infty} x_{ij} = \xi$, therefore the set $A(\frac{\epsilon}{2}) = \{(i, j) \in N^2 : |x_{ij} - \xi| \geq \frac{\epsilon}{2}\}$ belongs to I. This implies that the set $A^C(\frac{\epsilon}{2}) = \{(i, j) \in N^2 : |x_{ij} - \xi| < \frac{\epsilon}{2}\}$ belongs to F(I) and therefore is non empty. So we can choose positive integers m and n such that $(m, n) \notin A(\frac{\epsilon}{2})$, but then we have $|x_{mn} - \xi| < \frac{\epsilon}{2}$. Let B $= \{(i, j) \in N^2 : |x_{ij} - x_{mn}| \geq \epsilon\}$. We prove that $B \subset A(\frac{\epsilon}{2})$. Let $(i, j) \in B$ then we have $\epsilon \leq |x_{ij} - x_{mn}| \leq |x_{ij} - \xi| + |x_{mn} - \xi| < |x_{ij} - \xi| + \frac{\epsilon}{2}$. This implies that $\frac{\epsilon}{2} < |x_{ij} - \xi|$ and therefore $(i, j) \in A(\frac{\epsilon}{2})$. Since $B \subset A(\frac{\epsilon}{2})$ and $A(\frac{\epsilon}{2})$ belongs to I, therefore B $\in I$. This shows that x=(x_{ij}) is I- Cauchy sequence.

Sufficiency- Assume that x=(x_{ij}) is I- Cauchy sequence. We shall prove that x is I-convergent.To this effect, let (ϵ_p) be a strictly decreasing sequence of numbers converging to zero. Since x is I- Cauchy, therefore there exist two strictly increasing

sequences (m_p) and (n_p) of positive integers such that
$A_p = \{(i,j) \in N^2 : |x_{ij} - x_{m_p n_p}| \geq \epsilon_p\} \in I$, p=1, 2, 3 . This implies that

$$\emptyset \neq \{(i,j) \in N^2 : |x_{ij} - x_{m_p n_p}| < \epsilon_p\} \quad \text{belongs to } F(I),\ p = 1,2,3\ldots \tag{10}$$

Let p and q be two positive integers such that p $\neq$ q. By (10), both the sets $\{(i,j) \in N^2 : |x_{ij} - x_{m_p n_p}| < \epsilon_p\}$ and $\{(i,j) \in N^2 : |x_{ij} - x_{m_q n_q}| < \epsilon_q\}$ are non empty sets in F (I). Since F (I) is a filter on N^2, therefore

$$\emptyset \neq \{(i,j) \in N^2 : |x_{ij} - x_{m_p n_p}| < \epsilon_p\} \cap \{(i,j) \in N^2 : |x_{ij} - x_{m_q n_q}| < \epsilon_q\}$$

belongs to F(I). Thus for each pair p and q of positive integers with p $\neq$ q, we can select a pair $(i_{pq}, j_{pq}) \in N^2$ such that $|x_{i_{pq},j_{pq}} - x_{m_p n_p}| < \epsilon_p$ and $|x_{i_{pq},j_{pq}} - x_{m_q n_q}| < \epsilon_q$. It follows that $|x_{m_p n_p} - x_{m_q n_q}| \leq |x_{i_{pq} j_{pq}} - x_{m_p n_p}| + |x_{i_{pq} j_{pq}} - x_{m_q n_q}| \leq \epsilon_p + \epsilon_q \to 0$ as $p, q \to \infty$. This implies that $(x_{m_p n_p})$ p = 1, 2, 3 ...is an ordinary single Cauchy sequence and therefore it satisfies the Cauchy convergence criterion. Thus the sequence in the usual sense goes to a finite limit ξ (say).i.e., $lim_{p\to\infty} x_{m_p n_p} = \xi$. Also we have $\epsilon_p \to 0$ as $p \to \infty$, so for each $\epsilon > 0$ we can choose a positive integer p_0 such that

$$\epsilon_{p_0} < \frac{\epsilon}{2} \quad and |x_{m_p n_p} - \xi| < \frac{\epsilon}{2} \quad for p \geq p_0 \tag{11}$$

Next we prove that the set A=$\{(i,j) \in N^2 : |x_{ij} - \xi| \geq \epsilon\}$ is contained in A_{p_0}. Let $(i,j) \in$ A, then we have $\epsilon \leq |x_{ij} - \xi| \leq |x_{ij} - x_{m_{p_0} n_{p_0}}| + |x_{m_{p_0} n_{p_0}} - \xi| < |x_{ij} - x_{m_{p_0} n_{p_0}}| + \frac{\epsilon}{2}$ (by 11) This implies that $\frac{\epsilon}{2} < |x_{ij} - x_{m_{p_0} n_{p_0}}|$ and therefore by first half of (11) we have $\epsilon_{p_0} < |x_{ij} - x_{m_{p_0} n_{p_0}}|$. This implies that (i,j) $\in A_{p_0}$ and therefore A is contained in A_{p_0} . Since A_{p_0} belongs to I therefore A belongs to I. This proves that x=(x_{ij}) is I-convergent to ξ. □

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