Picard iteration converges faster than Mann iteration for a class of quasi-contractive operators

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Abstract. *The purpose of this paper is to introduce a new class of quasi-contractive operators and to show that the most used fixed point iterative methods, that is, the Picard and Mann iterations, are convergent to the unique fixed point. The comparison of these methods with respect to their convergence rate is obtained.*

Key words: *Picard iteration, Mann iteration, quasi-contractive operators, comparison function*

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1. Introduction

In the last four decades many papers have been published on the iterative approximation of fixed points for certains classes of operators, using the Picard, Krasnoselskij, Mann and Ishikawa iteration methods, see [2] for a recent survey. These papers were motivated by the fact that, under weaker contractive type conditions, the Picard iteration (or the method of succesive approximations) need not converge to the fixed point of the operators in question.

However, there exist large classes of operators, as for example that of quasi-contractive type operators introduced in [2], [7], [11], [12], for which not only the Picard iteration, but also the Krasnoselskij, Mann and Ishikawa iterations can be used to approximate the fixed points. In such situations, it is of theoretical and practical importance to compare these methods in order to establish, if possible, which one converges faster.

As far as we know, there are only a few papers devoted to this very important numerical problem, see [1] - [4], [12]. It is the main purpose of this paper to compare the Picard and Mann iterations over a class of quasi-contractive mappings which included the class of Zamfirescu operators and the class of φ- contractions. This new class is different from the class of quasi-contractions introduced by Ćirić [7].

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2. Quasi-φ-contractions

Let us recall a few basic definitions and results concerning a class of quasi-contractive mappings, see [5], [13].

Definition 2.1 *A function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a* comparison function *if* φ *satisfies the following conditions:*

(i) φ *is monotone increasing, i.e.,* $t_1 \leq t_2$ *implies* $\varphi(t_1) \leq \varphi(t_2)$*,*

(ii) $\{\varphi^n(t)\}_{n=0}^{\infty}$ *converges to* 0 *for all* $t \geq 0$*.*

Lemma 2.2 *If* φ *is a comparison function, then* φ *also satisfies*

(iii) $\varphi(t) < t$ *for all* $t > 0$*,*

(iv) $\varphi(0) = 0$*.*

Definition 2.3 *Let* (X, d) *be a metric space. A mapping* $T : X \to X$ *is a* φ-contraction *if* φ *is a comparison function and*

$$d(Tx, Ty) \leq \varphi(d(x, y)) \tag{2.1}$$

for all $x, y \in X$*.*

Definition 2.4 *Let* (X, d) *be a metric space. A mapping* $T : X \to X$ *is a* Picard mapping *if there exists* $x^* \in X$ *such that* $F(T) = \{x^*\}$ *and* $\{T^n x_0\}_{n=0}^{\infty}$ *converges to* x^* *for all* $x_0 \in X$*, where* $F(T) := \{x \in X;\, Tx = x\}$*.*

Lemma 2.5 *Let* (X, d) *be a complete metric space and* $T : X \to X$ *a mapping. We suppose that:*

(i) for each $\epsilon > 0$ *there exists a* $\delta(\epsilon) > 0$ *such that*

$$d(x, Tx) < \delta(\epsilon) \Rightarrow \overline{B}(x; \epsilon) \in I(T),$$

where

$$\overline{B}(x; \epsilon) := \{y \in X;\, d(y, x) \leq \epsilon\}, \quad I(T) := \{A \in P(X);\, A \neq \phi\ ,\, T(A) \subset A\},$$

(ii) there exists an element $x_0 \in X$ *asymptotic regular under* T*, i.e.,* $\left\{d(T^n x, T^{n+1} x\right\}$ *converges to zero as* $n \to \infty$*.*

Then the sequence $\{T^n x_0\}_{n=0}^{\infty}$ *converges to a fixed point of* T*.*

The following theorem is a generalization of Banach's contraction principle.

Theorem 2.6 *Let* (X, d) *be a complete metric space and* $T : X \to X$ *a* φ*-contraction. Then* T *is a Picard mapping.*

Theorem 2.7 *Let* (X, d) *be a nonempty set,* d *and* ρ *two metrics on* X *and* $T : X \to X$ *a mapping. Suppose that:*

(i) there exists $c > 0$ *such that* $d(Tx, Ty) \leq c\rho(x, y)$ *for all* $x, y \in X$*,*

(ii) (X, d) *is a complete metric space,*

(iii) $T : (X, d) \to (X, d)$ *is continuous,*

(iv) $T : (X, \rho) \to (X, \rho)$ *is a* φ*-contraction.*

Then the mapping $T : (X, d) \to (X, d)$ *is a Picard mapping.*

The previous theorem is a generalization of Maia's theorem, see [10].

Now we consider a class more generally than the class of φ-contractions and extend *Theorems 2.6* and *2.7*.

Definition 2.8 *Let* (X, d) *be a metric space. A mapping* $T : X \to X$ *is a* quasi-φ-contraction *if* φ *is a comparison function and*

$$d(Tx, Ty) \leq \varphi(d(x, y)) + L\, m(x, y) \tag{2.2}$$

for some $L \geq 0$ *and for all* $x, y \in X$*, where*

$$m(x, y) := \min\{d(x, Tx),\ d(y, Ty),\ d(x, Ty),\ d(y, Tx)\}.$$

Theorem 2.9 *Let* (X, d) *be a complete metric space and* $T : X \to X$ *a quasi-*φ*-contraction. Then* T *is a Picard mapping.*

Proof. First we remark that $cardF(T) \leq 1$. Indeed, assuming $x^*,\ y^* \in F(T)$, $x^* \neq y^*$, since $m(x^*, y^*) = 0$, we get by the property (iii) from *Lemma 2.2* that $d(x^*, y^*) = d(Tx^*, Ty^*) \leq \varphi(d(x^*, y^*)) < d(x^*, y^*)$, which is a contradiction.
Let x_0 be an element of X. Let $x_n := T^n x$. By using (2.2) we get

$$d(x_n, x_{n+1}) \leq \varphi(d(x_{n-1}, x_n)) \tag{2.3}$$

because $m(x_{n-1}, x_n) = 0$, and therefore $d(x_n, x_{n+1}) \leq \varphi^n(d(x_0, x_1)) \to 0$ as $n \to \infty$.
So every $x_0 \in X$ is asymptotic regular under T.
Let $\epsilon > 0$ be given and $\delta(\epsilon) := (\epsilon - \varphi(\epsilon))/(L + 1)$. Let $y \in \overline{B}(x_0; \epsilon)$. We have

$$d(Ty, x_0) \leq d(Ty, Tx_0) + d(Tx_0, x_0) \leq \varphi(d(y, x_0)) + Ld(x_0, Tx_0) + d(x_0, Tx_0) \tag{2.4}$$

and thus

$$d(Ty, x_0) \leq \varphi(d(y, x_0)) + (L + 1)d(x_0, Tx_0) \leq \varphi(\epsilon) + (L + 1)d(x_0, Tx_0). \tag{2.5}$$

Hence $d(x_0, Tx_0) < \delta(\epsilon) \Rightarrow d(Ty, x_0) \leq \varphi(\epsilon) + \epsilon - \varphi(\epsilon) = \epsilon$, so $\overline{B}(x_0; \epsilon) \in \dot{I}(T)$.
Now the theorem follows from *Lemma 2.5*. □

Theorem 2.10 *Let* X *be a nonempty set,* d *and* ρ *two metrics on* X *and* $T : X \to X$ *a mapping. Suppose that:*

(i) there exists $c > 0$ *such that* $d(Tx, Ty) \leq c\rho(x, y)$ *for all* $x, y \in X$*,*

(ii) (X, d) *is a complete metric space,*

(iii) $T : (X, d) \to (X, d)$ *is continuous,*

(iv) $T : (X, \rho) \to (X, \rho)$ *is a quasi-*φ*-contraction.*

Then the mapping $T : (X, d) \to (X, d)$ *is a Picard mapping.*

Proof. Let $x_0 \in X$. From (iv) and the previous theorem the sequence $\{T^n x_0\}_{n=0}^{\infty}$ is a Cauchy in (X, ρ). By (i), $\{T^n x_0\}_{n=0}^{\infty}$ is a Cauchy sequence in (X, d) and by (ii) it converges. Let $x^* := \lim\limits_{n\to\infty} T^n x_0$. By (iii) $x^* \in F(T)$ and by (iv), $F(T) = \{x^*\}$.
□

3. The Picard and Mann iterations

Let E be a normed linear space, $T : E \to E$ a given operator. Let $x_0 \in E$ be arbitrary and $\{\alpha_n\} \subset [0,1]$ a sequence of real numbers. The sequence $\{x_n\}_{n=0}^{\infty} \subset E$ defined by

$$x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n\, Tx_n,\ \ n = 0, 1, 2, ..., \tag{3.1}$$

is called the Mann iteration or Mann iterative procedure.

If we take $\alpha_n \equiv \lambda$ (const.), $\lambda \in (0,1]$, we obtain

$$x_{n+1} = (1-\lambda)\, x_n + \lambda\, Tx_n,\ \ n \geq 0 \tag{3.2}$$

i.e., the Krasnoselskij iteration, which gives the Picard iteration

$$x_{n+1} = Tx_n \tag{3.3}$$

for $\lambda \equiv 1$.

Let $y_0 \in E$ be arbitrary and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of real numbers in $[0,1]$. The sequence $\{y_n\}_{n=0}^{\infty} \subset E$ defined by

$$y_{n+1} = (1-\alpha_n)\, y_n + \alpha_n\, Tz_n,\ \ n = 0, 1, 2, ..., \tag{3.4}$$

$$z_n = (1-\beta_n) y_n + \beta_n\, Ty_n,\ n = 0, 1, 2, ...,$$

is called the Ishikawa iteration or Ishikawa iteration procedure.

Zamfirescu proved the following theorem.

Theorem 3.1 ([14]) *Let (X,d) be a complete metric space, and $T : X \to X$ a map for which there exist real numbers a, b and c satisfying $0 < a < 1,\ 0 < b, c < 1/2$ such that for each pair x, y in X at least one of the following is true:*

(z_1) $d(Tx, Ty) \leq a\, d(x,y)$;

(z_2) $d(Tx, Ty) \leq b\,[d(x, Tx) + d(y, Ty)]$;

(z_3) $d(Tx, Ty) \leq c\,[d(x, Ty) + d(y, Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by (3.3) converges to p, for any $x_0 \in X$.

An operator T which satisfies the contraction conditions $((z_1) - (z_3))$ of *Theorem 3.1* will be called a Zamfirescu operator [2].

Definition 3.2 ([4]) *Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be two sequences of real numbers that converge to a and b, respectively, and assume that there exists*

$$l = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} \tag{3.5}$$

If $l = 0$, then we say that $\{a_n\}_{n=0}^{\infty}$ converges faster *to a than $\{b_n\}_{n=0}^{\infty}$ to b.*

Definition 3.3 ([4]) *Suppose that for two fixed point iteration procedures $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ both converging to the same fixed point p with the error estimates*

$$\|u_n - p\| \leq a_n,\ \ n = 0, 1, 2, ..., \tag{3.6}$$

$$\|v_n - p\| \le b_n, \ \ n = 0, 1, 2, ..., \tag{3.7}$$

where $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ *are two sequences of positive numbers (converging to zero). If* $\{a_n\}_{n=0}^{\infty}$ *converges faster than* $\{b_n\}_{n=0}^{\infty}$*, then we say that* $\{u_n\}_{n=0}^{\infty}$ converges faster *than* $\{v_n\}_{n=0}^{\infty}$ *to* p*.*

Based on *Definition 3.3*, Berinde [4] compared the Picard and Mann iterations of the class of Zamfirescu operators defined on a closed convex subset of a Banach space and concluded that the Picard iteration always converges faster than the Mann iteration.

Using *Definition 3.3* Babu and Vara Prasad [1] showed that the Mann iteration converges faster than the Ishikawa iteration.

Example 3.4 *If we have* $a_n = 1/n^2$*,* $b_n = 1/n$*, then* $\{u_n\}$ *converges faster than* $\{v_n\}$*. But if we take* $b_n = 1/n^3$ *(supposing that (3.7) is still available) we obtain that* $\{v_n\}$ *converges faster than* $\{u_n\}$*.*

The previous example shows that *Definition 3.3* is not consistent. We will adopt the following concept.

Definition 3.5 *Suppose that two fixed point iteration procedures* $\{u_n\}_{n=0}^{\infty}$ *and* $\{v_n\}_{n=0}^{\infty}$ *converge to the same fixed point* p*. If*

$$\|u_n - p\| = a_n, \ \ n = 0, 1, 2, \dots,$$

$$\|v_n - p\| = b_n, \ \ n = 0, 1, 2, \dots, \tag{3.8}$$

and $\{a_n\}_{n=0}^{\infty}$ *converges faster than* $\{b_n\}_{n=0}^{\infty}$*, then we say that* $\{u_n\}_{n=0}^{\infty}$ converges faster *than* $\{v_n\}_{n=0}^{\infty}$ *to* p*.*

We use *Definition 3.5* to prove the following results. We replace the class of Zamfirescu operators with the class of quasi-δ-contractions.

Definition 3.6 *Let* (X, d)*, be a metric space. A mapping* $T : X \to X$ *is a* quasi-δ-contraction *if there exist* δ*,* $0 \le \delta < 1$ *and* $L > 0$ *such that*

$$d(Tx, Ty) \le \delta \ d(x, y) + L \ m(x, y) \tag{3.9}$$

for all $x, y \in X$*.*

Obviously, every quasi-δ-contraction is a quasi-φ-contraction. We remark also that a Zamfirescu operator is a quasi-δ-contraction. Indeed, a Zamfirescu operator satisfying the following inequalities (see [2]) :

$$d(Tx, Ty) \le 2\delta \ d(x, Tx) + \delta \ d(x, y), \tag{3.10}$$

$$d(Tx, Ty) \le 2\delta \ d(x, Ty) + \delta \ d(x, y). \tag{3.11}$$

If we take $L = 2\delta$, then inequality (3.9) holds.

Theorem 3.7 *Let* E *be an arbitrary Banach space,* K *a closed convex subset of* E*, and* $T : K \to K$ *a quasi-δ-contraction. Let* $\{y_n\}_{n=0}^{\infty}$ *be defined by (3.1) and* $y_0 \in K$*,* $y_0 \notin F(T)$ *with* $\{\alpha_n\} \subset [0, 1]$ *satisfying* *(i)* $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then $\{y_n\}_{n=0}^{\infty}$ *converges strongly to the fixed point of* T *and, moreover, the Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *defined by (3.3) and* $x_0 \in K$ *converges faster than the Mann iteration if*

(ii) $\alpha_n < \frac{1}{1+\delta}, n = 0, 1, 2, ...,$

(iii) $\lim\limits_{n\to\infty} \prod\limits_{k=0}^{n} \left[\frac{\delta}{1-(1+\delta)\alpha_k}\right] = 0.$

Proof. Using (3.1) we get

$$\|y_{n+1} - p\| \le (1 - \alpha_n) \|y_n - p\| + \alpha_n \|Ty_n - p\| . \tag{3.12}$$

Take $x := p$ and $y := y_n$ in (3.9) we obtain

$$\|Ty_n - p\| \le \delta \cdot \|y_n - p\| , \tag{3.13}$$

and then

$$\|y_{n+1} - p\| \le [1 - (1 - \delta)\alpha_n] \cdot \|y_n - p\| , \quad n = 0, 1, 2, ... \tag{3.14}$$

By induction, we get

$$\|y_{n+1} - p\| \le \prod_{k=0}^{n} [1 - (1 - \delta)\alpha_k] \cdot \|y_o - p\| , \quad n = 0, 1, 2, ... \tag{3.15}$$

As $\delta < 1$, $\alpha_k \in [0, 1]$ and $\sum\limits_{k=0}^{\infty} \alpha_k = \infty$, it follows that

$$\lim_{n\to\infty} \prod_{k=0}^{n} [1 - (1 - \delta)\alpha_k] = 0, \tag{3.16}$$

which by (3.15) implies that

$$\lim_{n\to\infty} \|y_{n+1} - p\| = 0, \tag{3.17}$$

that is, $\{y_n\}_{n=0}^{\infty}$ converges strongly to p.
Taking $y := x_n, x := p$ in (3.9), we obtain

$$\|x_{n+1} - p\| < \delta \cdot \|x_n - p\| \tag{3.18}$$

which inductively yields

$$\|x_{n+1} - p\| < \delta^{n+1} \cdot \|x_0 - p\| , \ n \ge 0. \tag{3.19}$$

Now, by (3.1) we have

$$\begin{aligned} \|y_{n+1} - p\| &= \|(1 - \alpha_n)y_n + \alpha_n Ty_n - [(1 - \alpha_n) + \alpha_n] p\| \\ &\ge (1 - \alpha_n) \|y_n - p\| - \alpha_n \|Ty_n - p\| . \end{aligned} \tag{3.20}$$

Using (3.13) we get

$$\|y_{n+1} - p\| \geq [1 - \alpha_n(1+\delta)] \ \|y_n - p\|, \tag{3.21}$$

which implies that

$$\|y_{n+1} - p\| \geq \prod_{k=0}^{n} [1 - \alpha_k(1+\delta)] \ \|y_0 - p\|, \ n = 0, 1, 2, ... \tag{3.22}$$

In order to compare $\{x_n\}$ and $\{y_n\}$, we must compare δ^{n+1} and $\prod\limits_{k=0}^{n} [1 - (1+\delta)\alpha_n]$. First, note that $1 - (1+\delta)\alpha_k > 0$, for all $\delta \in [0, 1)$ and $\{\alpha_k\}_{k=0}^{\infty}$ satisfying (ii). Moreover, by (iii) we have

$$\lim_{n\to\infty} \frac{\delta^{n+1}}{\prod_{k=0}^{n} [1 - \alpha_k(1+\delta)]} = 0. \tag{3.23}$$

By (3.23) and *Definition 3.5* we obtain that the Picard iteration converges faster than the Mann iteration. □

Remark 3.8 Theorem 3.7 *will remain true if conditions (ii) and (iii) are replaced by*

(iv) there exists c satisfying $\delta < c < 1$ and

$$\alpha_n < \frac{c - \delta}{c(1+\delta)} \tag{3.24}$$

for all $n = 0, 1, 2, ...$

Corollary 3.9 *Let E be an arbitrary Banach space, K a closed convex subset of E, and $T : K \to K$ a quasi-δ-contraction. Let $\{y_n\}_{n=0}^{\infty}$ be defined by (3.2) and $y_0 \in K$, with $\lambda < \frac{c-\delta}{c(1+\delta)}$ for some c such that $\delta < c < 1$. Then $\{y_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T and, moreover, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by (3.3) and $x_0 \in K$ converges faster than the Krasnoselskij iteration $\{y_n\}_{n=0}^{\infty}$ if $y_0 \notin F(T)$.*

Remark 3.10 *Since strict contractions, Kannan mappings [9], Hardy and Rogers generalized contractions [8], as well as Chatterjea mappings [6] belong to the class of Zamfirescu operators, so also to the class of quasi-L-contractions, by* Theorem 3.7 *we obtain similar results for these classes of contractive mappings.*

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