On absolute matrix summability methods

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Abstract. *In this paper a theorem on* $\mid A, p_n \mid_k$ *summability methods, which generalizes a theorem of Bor [2] on* $\mid \bar{N}, p_n \mid_k$ *summability methods, has been proved.*

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1. Introduction

Let $\sum a_n$ be a given infinite series with the partial sums (s_n), and let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of non-zero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \;\; n = 0, 1, \ldots \tag{1}$$

The series $\sum a_n$ is said to be summable $\mid A \mid_k, k \geq 1$, if (see [5])

$$\sum_{n=1}^{\infty} n^{k-1} \mid \overline{\Delta} A_n(s) \mid^k < \infty, \tag{2}$$

where

$$\overline{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{3}$$

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The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{4}$$

defines the sequence (t_n) of the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [3]). The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \mid t_n - t_{n-1} \mid^k < \infty, \tag{5}$$

and it is said to be summable $\mid A, p_n \mid_k$, $k \geq 1$, if (see [4])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \mid \overline{\Delta} A_n(s) \mid^k < \infty. \tag{6}$$

In the special case when $p_n = 1$ for all n, $\mid A, p_n \mid_k$ summability is the same as $\mid A \mid_k$ summability. Also if we take $a_{nv} = \frac{p_v}{P_n}$, then $\mid A, p_n \mid_k$ summability is the same as $\mid \bar{N}, p_n \mid_k$ summability.

Bor [2] has proved the following theorem for $\mid \bar{N}, p_n \mid_k$ summability of infinite series.

Theorem A. *Let (p_n) be a sequence of positive numbers such that*

$$P_n = O(np_n) \ \ as \ \ n \to \infty. \tag{7}$$

If (X_n) is a positive monotonic non-decreasing sequence such that

$$\mid \lambda_n \mid X_n = O(1) \ \ as \ \ n \to \infty, \tag{8}$$

$$\sum_{n=1}^{\infty} n X_n \mid \Delta^2 \lambda_n \mid = O(1), \tag{9}$$

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n} \mid t_n \mid^k = O(X_m), \tag{10}$$

where

$$t_n = \frac{1}{n+1} \sum_{v=1}^{n} v a_v,$$

then the series $\sum a_n \lambda_n$ is summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$.

2. The main result

The aim of this paper is to generalize *Theorem A* for absolute matrix summability. Before stating the main theorem we must first introduce some further notations. Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\overline{A} = (\overline{a}_{nv})$ and $\widehat{A} = (\widehat{a}_{nv})$ as follows:

$$\overline{a}_{nv} = \sum_{i=v}^{n} a_{ni},\ n, v = 0, 1, \ldots \tag{11}$$

and

$$\widehat{a}_{00} = \overline{a}_{00} = a_{00},\ \widehat{a}_{nv} = \overline{a}_{nv} - \overline{a}_{n-1,v},\ n = 1, 2, \ldots \tag{12}$$

It may be noted that $\overline{A}$ and $\widehat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$\begin{aligned} A_n(s) &= \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} a_{nv} \sum_{i=0}^{v} a_i \\ &= \sum_{i=0}^{n} a_i \sum_{v=i}^{n} a_{nv} \sum_{i=0}^{n} \overline{a}_{ni} a_i \end{aligned} \tag{13}$$

and

$$\begin{aligned} \overline{\Delta} A_n(s) &= \sum_{i=0}^{n} \overline{a}_{ni} a_i - \sum_{i=0}^{n-1} \overline{a}_{n-1,i} a_i \\ &= \overline{a}_{nn} + \sum_{i=0}^{n-1} (\overline{a}_{ni} - \overline{a}_{n-1,i}) a_i \\ &= \widehat{a}_{nn} + \sum_{i=0}^{n-1} \widehat{a}_{ni} a_i = \sum_{i=0}^{n} \widehat{a}_{ni} a_i. \end{aligned} \tag{14}$$

Now we shall prove the following theorem.

Theorem. *Let $A = (a_{nv})$ is a positive normal matrix such that*

$$\overline{a}_{no} = 1,\ n = 0, 1, \ldots, \tag{15}$$

$$a_{n-1,v} \geq a_{nv},\ for\ n \geq v + 1, \tag{16}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{17}$$

$$\widehat{a}_{n,v+1} = O(v \mid \Delta_v \widehat{a}_{nv} \mid). \tag{18}$$

If (X_n) *is a positive monotonic non-decreasing sequence such that conditions (8)-(10) of* Theorem A *are satisfied, then the series* $\sum a_n\lambda_n$ *is summable* $\mid A, p_n \mid_k$, $k \geq 1$.

It should be noted that if we take $a_{nv} = \frac{p_v}{P_n}$, then we get *Theorem A*. Furthermore, in this case condition (18) reduces to condition (7).

We need the following lemma for the proof of our theorem.

Lemma ([2]). *Under the conditions of* Theorem A, *we have that*

$$\sum_{n=1}^{\infty} X_n \mid \Delta\lambda_n \mid < \infty, \tag{19}$$

$$nX_n \mid \Delta\lambda_n \mid = O(1) \ as \ \ n \to \infty. \tag{20}$$

3. Proof of the *Theorem*

Let (T_n) denotes an A-transform of the series $\sum a_n\lambda_n$. By (13) and (14) then we have

$$\overline{\Delta} T_n = \sum_{v=0}^{n} \widehat{a}_{nv} a_v \lambda_v.$$

Applying Abel's transformation, we get that

$$\begin{aligned}
\overline{\Delta} T_n &= \sum_{v=1}^{n} \frac{\widehat{a}_{nv}\lambda_v}{v} v a_v \\
&= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\widehat{a}_{nv}\lambda_v}{v}\right)(v+1)t_v + \frac{n+1}{n}\widehat{a}_{nn}\lambda_n t_n \\
&= \sum_{v=1}^{n-1} \frac{v+1}{v}\Delta_v(\widehat{a}_{nv})\lambda_v t_v + \sum_{v=1}^{n-1} \frac{v+1}{v}\widehat{a}_{n,v+1}\Delta\lambda_v t_v \\
&+ \sum_{v=1}^{n-1} \frac{\widehat{a}_{n,v+1}\lambda_{v+1}t_v}{v} + \frac{n+1}{n}a_{nn}\lambda_n t_n \\
&= T_n(1) + T_n(2) + T_n(3) + T_n(4), \ say.
\end{aligned}$$

Since

$$\mid T_n(1) + T_n(2) + T_n(3) + T_n(4) \mid^k \leq 4^k \left(\mid T_n(1) \mid^k + \mid T_n(2) \mid^k + \mid T_n(3) \mid^k + \mid T_n(4) \mid^k \right),$$

to complete the proof of the *Theorem*, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \mid T_n(r) \mid^k < \infty, \ for \ r = 1, 2, 3, 4.$$

Applying Hölder's inequality with indices k and k', where $k>1$ and $\frac{1}{k}+\frac{1}{k'}=1$, we have that

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} \mid T_n(1)\mid^k &\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}\mid \Delta_v(\widehat{a}_{nv})\mid\mid \lambda_v\mid\mid t_v\mid\right)^k\\ &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}\mid \Delta_v(\widehat{a}_{nv})\mid\mid \lambda_v\mid^k\mid t_v\mid^k\right)\\ &\times\left(\sum_{v=0}^{n-1}\mid \Delta_v(\widehat{a}_{nv})\mid\right)^{k-1}\end{aligned}$$

Since

$$\begin{aligned}\Delta_v(\widehat{a}_{nv}) &= \widehat{a}_{nv}-\widehat{a}_{n,v+1}\\ &= \overline{a}_{nv}-\overline{a}_{n-1,v}-\overline{a}_{n,v+1}+\overline{a}_{n-1,v+1}\\ &= a_{nv}-a_{n-1,v},\end{aligned} \tag{21}$$

by using (15) and (16), we get that

$$\sum_{v=0}^{n-1}\mid \Delta_v(\widehat{a}_{nv})\mid=\sum_{v=0}^{n-1}(a_{n-1,v}-a_{nv})=1-1+a_{nn}=a_{nn}. \tag{22}$$

Hence we get

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\mid T_n(1)\mid^k &= O(1)\sum_{v=1}^{m}\mid \lambda_v\mid^{k-1}\mid \lambda_v\mid\mid t_v\mid^k\\ &\cdot\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^{k-1}\mid \Delta_v(\widehat{a}_{nv})\mid\\ &= O(1)\sum_{v=1}^{m}\mid \lambda_v\mid\mid t_v\mid^k\sum_{n=v+1}^{m+1}\mid \Delta_v\widehat{a}_{nv}\mid.\end{aligned}$$

By (21), we have that

$$\sum_{n=v+1}^{m+1}\mid \Delta_v\widehat{a}_{nv}\mid \sum_{n=v+1}^{m+1}(a_{n-1,v}-a_{nv})\sum_{n=v}^{m}a_{nv}-\sum_{n=v+1}^{m+1}a_{nv}a_{vv}-a_{m+1,v}\leq a_{vv}.$$

Thus, we obtain

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\mid T_n(1)\mid^k &= O(1)\sum_{v=1}^{m}\mid \lambda_v\mid\mid t_v\mid^k a_{vv}\\ &= O(1)\sum_{v=1}^{m}\frac{p_v}{P_v}\mid \lambda_v\mid\mid t_v\mid^k\end{aligned}$$

$$= O(1)\sum_{v=1}^{m-1}\Delta(\mid\lambda_v\mid)\sum_{r=1}^{v}\frac{p_r}{P_r}\mid t_r\mid^k + O(1)\mid\lambda_m\mid\sum_{v=1}^{m}\frac{p_v}{P_v}\mid t_v\mid^k$$

$$= O(1)\sum_{v=1}^{m-1}\mid\Delta\lambda_v\mid X_v + O(1)\mid\lambda_m\mid X_m = O(1) \quad as\ m\to\infty,$$

by virtue of the hypothesis of the *Theorem* and *Lemma*. By using (18) and (22), we have that

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\mid T_n(2)\mid^k = O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}\widehat{a}_{n,v+1}\mid\Delta\lambda_v\mid\mid t_v\mid\right)^k$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}v\mid\Delta_v\widehat{a}_{nv}\mid\mid\Delta\lambda_v\mid\mid t_v\mid\right)^k$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}(v\mid\Delta\lambda_v\mid)^k\mid t_v\mid^k\mid\Delta_v\widehat{a}_{nv}\mid\right)$$

$$\times\left(\sum_{v=1}^{n-1}\mid\Delta_v\widehat{a}_{nv}\mid\right)^{k-1}$$

$$= O(1)\sum_{v=1}^{m}(v\mid\Delta\lambda_v\mid)^{k-1}(v\mid\Delta\lambda_v\mid)\mid t_v\mid^k\sum_{n=v+1}^{m+1}\mid\Delta_v\widehat{a}_{nv}\mid$$

$$= O(1)\sum_{v=1}^{m}v\mid\Delta\lambda_v\mid\mid t_v\mid^k a_{vv}$$

$$= O(1)\sum_{1}^{m}v\mid\Delta\lambda_v\mid\mid t_v\mid^k\frac{p_v}{P_v}$$

$$= O(1)\sum_{v=1}^{m-1}\Delta(v\mid\Delta\lambda_v\mid)\sum_{r=1}^{v}\frac{p_r}{P_r}\mid t_r\mid^k$$

$$+O(1)m\mid\Delta\lambda_m\mid\sum_{v=1}^{m}\frac{p_v}{P_v}\mid t_v\mid^k$$

$$= O(1)\sum_{v=1}^{m-1}\mid\Delta(v\mid\Delta\lambda_v\mid)\mid X_v + O(1)m\mid\Delta\lambda_m\mid X_m$$

$$= O(1)\sum_{v=1}^{m-1}vX_v\mid\Delta^2\lambda_v\mid + O(1)\sum_{v=1}^{m-1}\mid\Delta\lambda_{v+1}\mid X_v$$

$$+O(1)m\mid\Delta\lambda_v\mid X_m$$

$$= O(1) \quad as\ m\to\infty,$$

by virtue of the hypothesis of the *Theorem* and *Lemma*.

Again using Hölder's inequality, as in $T_n(1)$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} \mid T_n(3) \mid^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}\widehat{a}_{n,v+1}\frac{\mid \lambda_{v+1}\mid}{v}\mid t_v \mid\right)^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}\mid \Delta_v\widehat{a}_{nv}\mid\mid \lambda_{v+1}\mid\mid t_v\mid\right)^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}\mid \Delta_v\widehat{a}_{nv}\mid\mid \lambda_{v+1}\mid^k\mid t_v\mid^k\right) \\
&\times\left(\sum_{v=1}^{n-1}\mid \Delta_v\widehat{a}_{nv}\mid\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^{k-1}\sum_{v=1}^{n-1}\mid \Delta_v\widehat{a}_{nv}\mid\mid \lambda_{v+1}\mid^k\mid t_v\mid^k \\
&= O(1)\sum_{v=1}^{m}\mid \lambda_{v+1}\mid^k\mid t_v\mid^k\sum_{n=v+1}^{m+1}\mid \Delta_v\widehat{a}_{nv}\mid \\
&= O(1)\sum_{v=1}^{m} a_{vv}\mid \lambda_{v+1}\mid^{k-1}\mid \lambda_{v+1}\mid\mid t_v\mid^k \\
&= O(1)\sum_{v=1}^{m}\frac{p_v}{P_v}\mid \lambda_{v+1}\mid\mid t_v\mid^k = O(1) \quad as\ m\to\infty.
\end{aligned}$$

Finally, again as in $T_n(1)$, we get

$$\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}\mid T_n(4)\mid^k &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^{k-1}\mid \lambda_n\mid^k\mid t_n\mid^k \\
&= O(1)\sum_{n=1}^{m}\frac{p_n}{P_n}\mid \lambda_n\mid\mid t_n\mid^k = O(1) \quad as\ m\to\infty,
\end{aligned}$$

by virtue of the hypothesis of the *Theorem* and *Lemma.* Therefore, we have that

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}\mid T_n(r)^k = O(1),\ as\ m\to\infty,\ for\ r=1,2,3,4.$$

This completes the proof of the *Theorem.*

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