Micropolar fluid flow with rapidly variable initial conditions

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Abstract. *In this paper we consider nonstationary 1D-flow of a micropolar viscous compressible fluid, which is in a thermodynamic sense perfect and polytropic. Assuming that the initial data for the specific volume, velocity, microvorticity and temperature are rapidly variable functions and making use of the method of two-scale asymptotic expansion, we find out the homogenized model of the considered flow.*

Key words: *micropolar fluid, homogenization*

Sažetak. **Homogenizacija 1D-toka mikropolarnog kompresibilnog fluida.** *U ovom radu razmatramo nestacionarni 1D-tok mikropolarnog viskoznog kompresibilnog fluida, koji je u termodinamičkom smislu perfektan i politropan. Pretpostavljajući da su početni podaci za specifični volumen, brzinu, mikrovrtložnost i temperaturu brzo varijabilne funkcije i koristeći metodu dvoskalnog asimptotičkog razvoja, nalazimo homogenizirani model razmatranog toka.*

Ključne riječi: *mikropolarni fluid, homogenizacija*

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1. Introduction

In this paper we consider nonstationary 1D-flow of a micropolar viscous compressible fluid, which is in a thermodynamic sense perfect and polytropic. Initial-boundary value problem for such flow is well posed ([9]). Assuming that the initial data for the specific volume, velocity, microvorticity and temperature are locally ϵ - periodic functions, where $\epsilon > 0$ is a small parameter, and making use of the method of two-scale asymptotic expansion ([5], [10]), we find out the homogenized model of the considered flow. It turns out that the homogenized model has the same structure as the original one. The analogous problem for the classical viscous

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compressible fluid was considered in [7]. Homogenization problems for incompressible flows were treated in [1]-[3] (see also [4] and [6], where lubrication problems close to homogenization were considered).

Let

$$\rho, \nu, \omega, \theta :]0,1[\times \mathbb{R}^+ \;\to\; \mathbb{R} \tag{1}$$

denote the mass density, velocity, microvorticity and temperature of the fluid in the Lagrangean description, respectively. Governing equations of the flow under consideration are as follows ([8]):

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial \nu}{\partial x} = 0, \tag{2}$$

$$\frac{\partial \nu}{\partial t} = \frac{\partial}{\partial x}\left(\rho \frac{\partial \nu}{\partial x}\right) - K \frac{\partial}{\partial x}(\rho\theta), \tag{3}$$

$$\rho \frac{\partial \omega}{\partial t} = A\left[\rho \frac{\partial}{\partial x}\left(\rho \frac{\partial \omega}{\partial x}\right) - \omega\right], \tag{4}$$

$$\rho \frac{\partial \theta}{\partial t} = -K\rho^2\theta \frac{\partial \nu}{\partial x} + \rho^2 \left(\frac{\partial \nu}{\partial x}\right)^2 + \rho^2 \left(\frac{\partial \omega}{\partial x}\right)^2 + \omega^2 + D\rho \frac{\partial}{\partial x}\left(\rho \frac{\partial \theta}{\partial x}\right), \tag{5}$$

where K, A and D are positive constants. It is convenient to introduce (instead of the mass density ρ) the specific volume $V = \frac{1}{\rho}$. Then we obtain the system

$$\frac{\partial V}{\partial t} - \frac{\partial \nu}{\partial x} = 0, \tag{6}$$

$$\frac{\partial \nu}{\partial t} - \frac{\partial}{\partial x}\left(\frac{1}{V}\left(\frac{\partial \nu}{\partial x} - K\theta\right)\right) = 0, \tag{7}$$

$$\frac{1}{V}\frac{\partial \omega}{\partial t} - A\left(\frac{1}{V}\frac{\partial}{\partial x}\left(\frac{1}{V}\frac{\partial \omega}{\partial x}\right) - \omega\right) = 0, \tag{8}$$

$$\frac{1}{V}\frac{\partial \theta}{\partial t} + K\frac{\theta}{V^2}\frac{\partial \nu}{\partial x} - \frac{1}{V^2}\left(\frac{\partial \nu}{\partial x}\right)^2 - \frac{1}{V^2}\left(\frac{\partial \omega}{\partial x}\right)^2 - \omega^2 - D\frac{1}{V}\frac{\partial}{\partial x}\left(\frac{1}{V}\frac{\partial \theta}{\partial x}\right) = 0 \tag{9}$$

in $Q =]0,1[\times \mathbb{R}^+$. We take the following homogeneous boundary conditions:

$$\nu(0,t) = \nu(1,t) = 0, \tag{10}$$

$$\omega(0,t) = \omega(1,t) = 0, \tag{11}$$

$$\frac{\partial \theta}{\partial x}(0,t) = \frac{\partial \theta}{\partial x}(1,t) = 0 \tag{12}$$

for $t > 0$. Let the functions

$$V_0, \nu_0, \omega_0, \theta_0 : [0,1] \times \mathbb{R} \;\to\; \mathbb{R} \tag{13}$$

be 1-periodic in the second, so-called fast variable y. Then, for sufficiently small $\epsilon > 0$ the functions

$$V_0^\epsilon(x) = V_0(x, y), \tag{14}$$

$$\nu_0^\epsilon(x) = \nu_0(x, y), \tag{15}$$

$$\omega_0^\epsilon(x) = \omega_0(x, y), \tag{16}$$

$$\theta_0^\epsilon(x) = \theta_0(x, y), \tag{17}$$

where

$$y = \frac{x}{\epsilon}, \tag{18}$$

are locally ϵ-periodic on $]0,1[$. For the system (6)–(9) we take the initial conditions

$$V(x, 0) = V_0^\epsilon(x), \tag{19}$$

$$\nu(x, 0) = \nu_0^\epsilon(x), \tag{20}$$

$$\omega(x, 0) = \omega_0^\epsilon(x), \tag{21}$$

$$\theta(x, 0) = \theta_0^\epsilon(x) \tag{22}$$

for $x \in]0,1[$.

Let the functions (13) be sufficiently smooth and let the compatibility conditions

$$\nu_0(0, .) = \nu_0(1, .) = 0, \tag{23}$$

$$\omega_0(0, .) = \omega_0(1, .) = 0 \tag{24}$$

on $]0,1[$ and the conditions

$$V_0 > 0,\ \theta_0 > 0 \ \text{ on } \ [0,1]^2 \tag{25}$$

hold true. Then for each $\epsilon > 0$ the initial-boundary value problem (6)–(12), (19)–(22) has a unique strong solution

$$(V^\epsilon, \nu^\epsilon, \omega^\epsilon, \theta^\epsilon) \tag{26}$$

in Q, having the properties

$$V^\epsilon > 0, \quad \theta^\epsilon > 0, \quad \text{on} \quad [0,1] \times [0, \infty) \tag{27}$$

([9]). In order to describe a propagation of the initial rapidly variable heterogenities (19)–(22), we have to identify a limit of solutions (26), as ϵ tends to zero.

2. Two-scale asymptotic expansion

According to the homogenization method ([5], [10]), we shall look for a solution to the problem (6)–(12), (19)–(22) in the form of two-scale asymptotic expansion:

$$(V^\epsilon, \nu^\epsilon, \omega^\epsilon, \theta^\epsilon)(x,t) = (V^0, \nu^0, \omega^0, \theta^0)(x,y,t) + \epsilon(V^1, \nu^1, \omega^1, \theta^1)(x,y,t) + \cdots, \tag{28}$$

$$y = \frac{x}{\epsilon},$$

where functions

$$(V^k, \nu^k, \omega^k, \theta^k) \ :]0,1[\times \mathbb{R} \times \mathbb{R}^+ \ \to \ \mathbb{R}, \ \ k = 0, 1, ... \tag{29}$$

are 1-periodic in the fast variable $y \in \mathbb{R}$. The idea of the method is to insert the expansion (28) into the system (6)–(9) and to identify the powers of ϵ. In this way we obtain a sequence of equations for the functions (29). Our purpose is to identify the function $(V^0, \nu^0, \omega^0, \theta^0)$ that is, at least in a formal sense, a limit of the solutions (26), as ϵ tends to zero.

In order to present computations in a simple form, it is useful to consider first x and y as independent variables and next to replace y by $\frac{x}{\epsilon}$. Note that the operator $\frac{\partial}{\partial x}$, applied to the function $\varphi(x, \frac{x}{\epsilon})$ becomes

$$\frac{\partial}{\partial x} + \frac{1}{\epsilon}\frac{\partial}{\partial y}. \tag{30}$$

In that what follows we use the notation

$$\bar{\varphi}(x) = \int_0^1 \varphi(x, y)dy. \tag{31}$$

For simplicity we shall assume that the functions ν^0, ω^0 and θ^0 do not depend on the fast variable:

$$\nu^0 = \nu^0(x, t), \quad \omega^0 = \omega^0(x, t), \quad \theta^0 = \theta^0(x, t); \tag{32}$$

in fact one can easily see that (32) holds true.

The equation (6). Identifying the terms of the order ϵ^0, we get

$$\frac{\partial V^0}{\partial t} - \frac{\partial \nu^0}{\partial x} - \frac{\partial \nu^1}{\partial y} = 0. \tag{33}$$

Integrating this equation over $]0, 1[$ with respect to variable y and taking into account (32) and the periodicity of the function ν^1, we obtain

$$\frac{\partial \bar{V}^0}{\partial t} - \frac{\partial \nu^0}{\partial x} = 0. \tag{34}$$

The equation (7). Identifying the terms of the order ϵ^{-1}, one gets

$$\frac{\partial}{\partial y}\left(\frac{1}{V^0}\left(\frac{\partial \nu^0}{\partial x} + \frac{\partial \nu^1}{\partial y} - K\theta^0\right)\right) = 0. \tag{35}$$

Therefore, there exists a function $\alpha(x, t)$, such that

$$\frac{\partial \nu^0}{\partial x} + \frac{\partial \nu^1}{\partial y} - K\theta^0 = \alpha V^0. \tag{36}$$

After integration with respect to y we have

$$\frac{\partial \nu^0}{\partial x} - K\theta^0 = \alpha \bar{V}^0, \tag{37}$$

or

$$\alpha = \frac{1}{\bar{V}^0}\left(\frac{\partial \nu^0}{\partial x} - K\theta^0\right). \tag{38}$$

From (36) and (38), it follows

$$\frac{\partial \nu^1}{\partial y} = \frac{V^0 - \bar{V}^0}{\bar{V}^0}\left(\frac{\partial \nu^0}{\partial x} - K\theta^0\right). \tag{39}$$

Identifying the terms of the order ϵ^0 and taking into account (39), we obtain

$$\frac{\partial \nu^0}{\partial t} - \frac{\partial}{\partial x}\left(\frac{1}{\bar{V}^0}\left(\frac{\partial \nu^0}{\partial x} - K\theta^0\right)\right) - \frac{\partial}{\partial y}\left(\frac{1}{V^0}\left(\frac{\partial \nu^1}{\partial x} + \frac{\partial \nu^2}{\partial y} - K\theta^1\right)\right) +$$
$$+\frac{\partial}{\partial y}\left(\frac{V^1}{V^0\bar{V}^0}\left(\frac{\partial \nu^0}{\partial x} - K\theta^0\right)\right) = 0. \tag{40}$$

Integrating with respect to y, we get

$$\frac{\partial \nu^0}{\partial t} - \frac{\partial}{\partial x}\left(\frac{1}{\bar{V}^0}\left(\frac{\partial \nu^0}{\partial x} - K\theta^0\right)\right) = 0. \tag{41}$$

The equation (8). Identifying the terms of the order ϵ^{-1}, we have

$$\frac{\partial}{\partial y}\left(\frac{1}{V^0}\left(\frac{\partial \omega^0}{\partial x} + \frac{\partial \omega^1}{\partial y}\right)\right) = 0. \tag{42}$$

Therefore, ther exists a function $\beta(x,t)$, such that

$$\frac{\partial \omega^0}{\partial x} + \frac{\partial \omega^1}{\partial y} = \beta V^0. \tag{43}$$

Integrating with respect to y, we obtain

$$\frac{\partial \omega^0}{\partial x} = \beta \bar{V}^0, \tag{44}$$

or

$$\beta = \frac{1}{\bar{V}^0}\frac{\partial \omega^0}{\partial x}. \tag{45}$$

From (43) and (45), it follows

$$\frac{\partial \omega^1}{\partial y} = \frac{V^0 - \bar{V}^0}{\bar{V}^0}\frac{\partial \omega^0}{\partial x}. \tag{46}$$

The terms of the order ϵ^0 (together with (46)) lead to the equation

$$\frac{1}{A}\frac{\partial \omega^0}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{\bar{V}^0}\frac{\partial \omega^0}{\partial x}\right) - \omega^0 V^0 + \frac{\partial}{\partial y}\left(\frac{1}{V^0}\left(\frac{\partial \omega^1}{\partial x} + \frac{\partial \omega^2}{\partial y} - \frac{V^1}{\bar{V}^0}\frac{\partial \omega^0}{\partial x}\right)\right). \tag{47}$$

After integration with respect to y, we obtain

$$\frac{1}{\bar{V}^0}\frac{\partial \omega^0}{\partial t} - A\left(\frac{1}{\bar{V}^0}\frac{\partial}{\partial x}\left(\frac{1}{\bar{V}^0}\frac{\partial \omega^0}{\partial x}\right) - \omega^0\right) = 0. \tag{48}$$

The equation (9). The terms of the order ϵ^{-1} give

$$\frac{\partial}{\partial y}\left(\frac{1}{V^0}\left(\frac{\partial\theta^0}{\partial x}+\frac{\partial\theta^1}{\partial y}\right)\right)=0. \tag{49}$$

Therefore, ther exists a function $\gamma(x,t)$, such that

$$\frac{\partial\theta^0}{\partial x}+\frac{\partial\theta^1}{\partial y}=\gamma V^0. \tag{50}$$

Integrating with respect to y, we have

$$\frac{\partial\theta^0}{\partial x}=\gamma\bar{V}^0, \tag{51}$$

or

$$\gamma=\frac{1}{\bar{V}^0}\frac{\partial\theta^0}{\partial x}. \tag{52}$$

Inserting it in (50), we obtain

$$\frac{\partial\theta^1}{\partial y}=\frac{V^0-\bar{V}^0}{\bar{V}^0}\frac{\partial\theta^0}{\partial x}. \tag{53}$$

The terms of the order ϵ^0 together with (53) give

$$\begin{aligned}\frac{\partial\theta^0}{\partial t} \quad = \quad & -K\frac{\theta^0}{V^0}\frac{\partial\nu^0}{\partial x}-\frac{K^2}{V^0}(\theta^0)^2+\frac{V^0}{(V^0)^2}\left(\frac{\partial\nu^0}{\partial x}\right)^2-\frac{2KV^0}{(V^0)^2}\frac{\partial\nu^0}{\partial x}\theta^0\\ & +\frac{K^2V^0}{(V^0)^2}(\theta^0)^2+\frac{V^0}{(V^0)^2}\left(\frac{\partial\omega^0}{\partial x}\right)^2+V^0(\omega^0)^2+D\frac{\partial}{\partial x}\left(\frac{1}{V^0}\frac{\partial\theta^0}{\partial x}\right)\\ & +D\frac{\partial}{\partial y}\left(\frac{1}{V^0}\left(\frac{\partial\theta^1}{\partial x}+\frac{\partial\theta^2}{\partial y}-\frac{V^1}{V^0}\frac{\partial\theta^0}{\partial x}\right)\right).\end{aligned} \tag{54}$$

After integration with respect to y, we have

$$\begin{aligned}&\frac{1}{\bar{V}^0}\frac{\partial\theta^0}{\partial t}+K\frac{\theta^0}{(\bar{V}^0)^2}\frac{\partial\nu^0}{\partial x}-\frac{1}{(\bar{V}^0)^2}\left(\frac{\partial\nu^0}{\partial x}\right)^2-\frac{1}{(\bar{V}^0)^2}\left(\frac{\partial\omega^0}{\partial x}\right)^2-\\ &-(\omega^0)^2-\frac{D}{\bar{V}^0}\frac{\partial}{\partial x}\left(\frac{1}{\bar{V}^0}\frac{\partial\theta^0}{\partial x}\right)=0.\end{aligned} \tag{55}$$

3. The homogenized problem

For the functions V^0, ν^0, ω^0 and θ^0, describing a macroscopic behaviour of the flow, we get the system (34), (41), (48), (55). From (10)–(12) it follows

$$\nu^0(0,t)=\nu^0(1,t)=0, \tag{56}$$

$$\omega^0(0,t)=\omega^0(1,t)=0, \tag{57}$$

$$\frac{\partial\theta^0}{\partial x}(0,t)=\frac{\partial\theta^0}{\partial x}(1,t)=0 \tag{58}$$

for $t > 0$. Taking into account that the functions V_0^ϵ, ν_0^ϵ, ω_0^ϵ and θ_0^ϵ tend in some sense to the functions $\bar{V}_0$, $\bar{\nu}_0$, $\bar{\omega}_0$ and $\bar{\theta}_0$, respectively ([5]), from (19)–(22) we conclude that the following initial conditions hold true:

$$\bar{V}^0(x,0) = \bar{V}_0(x), \tag{59}$$

$$v^0(x,0) = \bar{\nu}_0(x), \tag{60}$$

$$\omega^0(x,0) = \bar{\omega}_0(x), \tag{61}$$

$$\theta^0(x,0) = \bar{\theta}_0(x) \tag{62}$$

for $x \in]0,1[$. Because of (23)–(25) the conditions

$$\bar{\nu}_0(0) = \bar{\nu}_0(1) = 0, \tag{63}$$

$$\bar{\omega}_0(0) = \bar{\omega}_0(1) = 0, \tag{64}$$

$$\bar{V}_0 > 0, \quad \bar{\theta}_0 > 0 \quad \text{on} \quad [0,1] \tag{65}$$

are satisfied. Therefore, we have the following result:

Theorem 1. *The homogenized initial-boundary value problem (34), (41), (48), (55), (56)–(62) has a unique strong solution , having the properties*

$$V^0 > 0, \;\; \theta^0 > 0 \quad on \quad [0,1] \times [0,\infty). \tag{66}$$

Remark 1. *If the functions (13) do not depend on the first variable x, then the functions (14)–(17) are ϵ-periodic and, consequently, $\bar{V}_0$, $\bar{\nu}_0$, $\bar{\omega}_0$ and $\bar{\theta}_0$ are constants; it means that the initial state of the macroscopic flow in this case is a homogeneous one.*

Remark 2. *A rigorous proof of a convergence*
$(V^\epsilon, \nu^\epsilon, \omega^\epsilon, \theta^\epsilon) \to (V^0, \nu^0, \omega^0, \theta^0)$ *as $\epsilon \to 0$ is an open problem.*

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