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Three-point singular boundary-value problem for a system of three differential equations*

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Abstract. *A singular Cauchy-Nicoletti problem for a system of three ordinary differential equations is considered. An approach which combines topological method of T. Ważewski and Schauder's principle is used. Theorem concerning the existence of a solution of this problem (a graph of which lies in a given domain) is proved. Moreover, an estimation of its coordinates is obtained.*

Key words: *Three-point boundary value problem, singular Cauchy-Nicoletti problem, Ważewski's principle*

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1. Introduction

In the presented paper the following Cauchy-Nicoletti problem

$$y_i'(x) = \omega_i(x)y_i + f_i(x, y_1, y_2, y_3),\ i = 1, 2, 3, \tag{1}$$

$$y_1(x_1^+) = A_1,\ y_2(x_2^\pm) = A_2,\ y_3(x_3^-) = A_3, \tag{2}$$

where $x \in I = [a, b], a = x_1 < x_2 < x_3 = b$ and $A_i, i = 1, 2, 3$ are real constants, is considered. Denote $I_i = I \setminus \{x_i\},\ i = 1, 2, 3$ and $J = I_1 \cap I_2 \cap I_3$. We shall suppose that $\omega_i \in C(I_i, \mathbb{R}),\ i = 1, 2, 3$ and $f_i \in C(\Omega_i, \mathbb{R}),\ i = 1, 2, 3$ where $\Omega_i \subset I_i \times \mathbb{R}^3,\ \Omega_i \cap \{x = x^*\} \neq \emptyset$ for $x^* \in I_i$. Note that continuity of the functions ω_i and f_i is not required at point $x_i,\ i = 1, 2, 3$. A solution of the problem (1), (2) is defined in the following sense:

Definition 1. *A vector-function $y(x) = (y_1(x), y_2(x), y_3(x)) \in C(I)$ where $y_i(x) \in C^1(I_i),\ i = 1, 2, 3,$ is said to be a solution of the problem (1), (2) if it satisfies the system (1) on J and, moreover, $y_1(x_1^+) = A_1,\ y_2(x_2^\pm) = A_2,\ y_3(x_3^-) = A_3$.*

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Although singular boundary value problems were widely considered by using of various methods (see e.g. [1] – [3], [5] – [7]) the method used here is based on a new approach – a combination of topological method of T. Ważewski and Schauder's principle. (Note that method of T. Ważewski was used to investigation various asymptotic and singular problems, e.g., in [2], [3], [8] — [10].) Each equation of the system (1) is considered separately (as scalar equation) under supposition that nondiagonal variables are changed by functions taken from a given set of functions M. For every scalar equation (together with corresponding Cauchy initial condition which follows from (2)) it is shown, with the aid of Ważewski's principle, that there is its solution with the same properties which are supposed for coordinates of corresponding functions from M. By this way an operator T is defined. Stationary point of operator T is a solution of the problem (1), (2).

2. Existence of solutuins of problem (1), (2)

Let us suppose that $\omega_i \in C(I_i, \mathbb{R})$, $i = 1, 2, 3$, function $f_i \in C(\Omega_i, \mathbb{R})$, $i = 1, 2, 3$ where

$$\Omega_i = \{(x, y_1, y_2, y_3) : x \in I_i, (x, y_1, y_2, y_3) \in \Omega\},$$

$$\Omega = \{(x, y_1, y_2, y_3) : x \in I, \alpha_i(x) - \varepsilon^* \le y_i \le \beta_i(x) + \varepsilon^*, i = 1, 2, 3\},$$

ε^* is a small positive number and $\alpha_i(x), \beta_i(x), i = 1, 2, 3$ are real functions such that $\alpha_i(x)$, $\beta_i(x) \in C^1(I)$, $\alpha_i(x) < \beta_i(x)$, for $x \in I_i$, $\alpha_1(x_1) = \beta_1(x_1) = A_1$, $\alpha_2(x_2) = \beta_2(x_2) = A_2$, $\alpha_3(x_3) = \beta_3(x_3) = A_3$. Involve (for $i, j = 1, 2, 3$ and $i < j$) the set

$$\Omega_{ij} = \{(x, y_i, y_j) : x \in I_k, k \in \{1, 2, 3\}, k \ne i, j, \ \alpha_s(x) \le y_s \le \beta_s(x), s = i, j\}.$$

Let us define auxiliary functions

$$F_i(x, y_1, y_2, y_3) \equiv \omega_i(x) y_i - y_i' + f_i(x, y_1, y_2, y_3), \ i = 1, 2, 3.$$

The result of the paper is given in the following theorem.

Theorem 1. *Assume that* $F_1(x, \alpha_1(x), y_2, y_3) \cdot F_1(x, \beta_1(x), y_2, y_3) < 0$ *if* $(x, y_2, y_3) \in \Omega_{23}$,
$F_2(x, y_1, \alpha_2(x), y_3) \cdot F_2(x, y_1, \beta_2(x), y_3) < 0$ *if* $(x, y_1, y_3) \in \Omega_{13}$ *and*

$$F_3(x, y_1, y_2, \alpha_3(x)) \cdot F_3(x, y_1, y_2, \beta_3(x)) < 0 \ \text{ if } \ (x, y_1, y_2) \in \Omega_{12}. \tag{3}$$

Let, moreover,

$$|f_i(x, y_1, y_2, y_3) - f_i(x, z_1, z_2, z_3)| \le M_i(x)|y_1 - z_1| + N_i(x)|y_2 - z_2| + P_i(x)|y_3 - z_3| \tag{4}$$

for any $(x, y_1, y_2, y_3), (x, z_1, z_2, z_3) \in \Omega_i$ *where* $M_i(x), N_i(x), P_i(x)$ *are functions, continuous on* $I_i, i = 1, 2, 3$, *such that*

$$|\omega_i(x)| > M_i(x) + N_i(x) + P_i(x), \ i = 1, 2, 3, \ x \in I_i \tag{5}$$

and

$$\begin{aligned} &\omega_1(x) F_1(x, \beta_1(x), y_2, y_3) > 0 \quad \text{if} \quad (x, y_2, y_3) \in \Omega_{23}, \\ &\omega_2(x) F_2(x, y_1, \beta_2(x), y_3) > 0 \quad \text{if} \quad (x, y_1, y_3) \in \Omega_{13}, \\ &\omega_3(x) F_3(x, y_1, y_2, \beta_3(x)) > 0 \quad \text{if} \quad (x, y_1, y_2) \in \Omega_{12}. \end{aligned} \tag{6}$$

Then there is at least one solution $y(x) = (y_1(x), y_2(x), y_3(x))$ *of the problem* (1), (2) *such that* $\alpha_i(x) < y_i(x) < \beta_i(x)$ *where* $x \in I_i$, $i = 1, 2, 3$.

Proof. The idea of the proof is as follows. At first, with the aid of (1), (2), an operator T is constructed, such that $T(M) \subset M$ where

$$M = \{(\varphi_1(x), \varphi_2(x), \varphi_3(x)) : x \in I, \varphi_i(x) \in C(I), \alpha_i(x) \leq \varphi_i(x) \leq \beta_i(x), i = 1, 2, 3\}.$$

The next step is verification of conditions of Schauder principle for this operator. The stationary point $\varphi(x) \in M$ of T will be a solution of the problem (1), (2). The construction of operator T uses the topological principle of T. Ważewski. (The details of the application of this principle can be found, e.g., in [3], [4], [8] — [10] and therefore will be omitted.)
The proof is divided into two parts (construction of the operator T and verification of Schauder's principle).

I. *Construction of operator* T. Let us consider the system of three equations

$$y_1' = \omega_1(x)y_1 + f_1(x, y_1, \varphi_2(x), \varphi_3(x)), \tag{7}$$
$$y_2' = \omega_2(x)y_2 + f_2(x, \varphi_1(x), y_2, \varphi_3(x)), \tag{8}$$
$$y_3' = \omega_3(x)y_3 + f_3(x, \varphi_1(x), \varphi_2(x), y_3), \tag{9}$$

where $(\varphi_1(x), \varphi_2(x), \varphi_3(x)) \in M$. This system consists of separated scalar equations. Therefore we shall consider equations of this system separately in the sequel. Define (for $i = 1, 2, 3$): $\mathrm{w}_i(x, y_i) \equiv (y_i - \alpha_i(x))(y_i - \beta_i(x))$,
$\mathcal{N}_i = \{(x, y_i) : x \in I_i, w_i(x, y_i) = 0\}$,
$\mathcal{D}_i = \{(x, y_i) : x \in \text{ int } I_i, w_i(x, y_i) < 0\}$. **a)** Let us investigate the equation (7). Next we prove that there is at least one solution $y_1 = y_1(x) \in C(I) \cap C^1(I_1)$ satisfying the following properties:

$$y_1(x_1^+) = A_1;\ \alpha_1(x) < y_1(x) < \beta_1(x),\ x \in I_1. \tag{10}$$

Let us evaluate the derivative of $w_1(x, y_1)$ along the trajectories of the equation (7) if $(x, y_1) \in \mathcal{N}_1$. Then either $y_1 = \beta_1(x)$ or $y_1 = \alpha_1(x)$. In the first case we have

$$\left.\frac{dw_1(x, y_1)}{dx}\right|_{y_1 = \beta_1(x)} = (\beta_1(x) - \alpha_1(x)) \cdot F_1(x, \beta_1(x), \varphi_2(x), \varphi_3(x))$$

and in the second one

$$\left.\frac{dw_1(x, y_1)}{dx}\right|_{y_1 = \alpha_1(x)} = F_1(x, \alpha_1(x), \varphi_2(x), \varphi_3(x)) \cdot (\alpha_1(x) - \beta_1(x)).$$

In view of condition (1.) we have either

$$F_1(x, \alpha_1(x), \varphi_2(x), \varphi_3(x)) < 0 \tag{11}$$

or

$$F_1(x, \alpha_1(x), \varphi_2(x), \varphi_3(x)) > 0. \tag{12}$$

If (11) holds, then

$$\left.\frac{dw_1(x, y_1)}{dx}\right|_{(x, y_1) \in \mathcal{N}_1} > 0$$

and all points of the set $\mathcal{N}_1$ for $x \in (x_1, x_3)$ are the points of strict egress for $\mathcal{D}_1$with respect to (7). Then each point (x_3, y_1^*) where $y_1^* \in (\alpha_1(x_3), \beta_1(x_3))$ defines a solution of equation (7) such that (10) holds. In the sequel we will take the unique solution $y_1(x)$ which satisfies condition $y_1(x_3^-) = y_1^* = \frac{1}{2}(\alpha_1(x_3) + \beta_1(x_3))$.
If (12) holds, then

$$\left.\frac{dw_1(x, y_1)}{dx}\right|_{(x,y_1)\in\mathcal{N}_1} < 0$$

and all points of the set $\mathcal{N}_1$ for $x \in (x_1, x_3)$ are the points of strict ingress for $\mathcal{D}_1$ with respect to (7). Then from the Ważewski principle follows that there is at least one solution $y_1 = y_1(x)$ such that $w_1(x, y_1(x)) < 0$ if $x \in I_1$, i.e. (10) holds. Suppose that a set Y_1 consists of all such solutions and denote $y_1^{**} = \min\{y_1(x_3) : y_1(x) \in Y_1\}$. The value y_1^{**} exists in view of elementary properties of solutions of differential equations. In the sequel we will take the unique solution $y_1(x)$ of equation (7) which satisfies condition $y_1(x_3^-) = y_1^{**}$. Therefore for both cases (11) or (12), we have defined, by an unique manner, a solution $y_1(x)$ of equation (7) with property (10).
b) Now consider the equation (9). We prove (similarly as in the part a)) that there is at least one solution $y_3 = y_3(x) \in C(I) \cap C^1(I_3)$ satisfying the conditions

$$y_3(x_3^-) = A_3;\ \alpha_3(x) < y_3(x) < \beta_3(x),\ x \in I_3. \tag{13}$$

We evaluate the derivative of $w_3(x, y_3)$ along the trajectories (9) if $(x, y_3) \in \mathcal{N}_3$. By analogy with the previous computations we (in view of (3)) get

$$\left.\frac{dw_3(x, y_3)}{dx}\right|_{(x,y_3)\in\mathcal{N}_3} > 0 \tag{14}$$

if

$$F_3(x, \varphi_1(x), \varphi_2(x), \alpha_3(x)) < 0 \tag{15}$$

and

$$\left.\frac{dw_3(x, y_3)}{dx}\right|_{(x,y_3)\in\mathcal{N}_3} < 0 \tag{16}$$

if

$$F_3(x, \varphi_1(x), \varphi_2(x), \alpha_3(x)) > 0. \tag{17}$$

Therefore, as in the part a), for both cases (14) or (16) (with the aid of the sets $\mathcal{N}_3, \mathcal{D}_3$), we can define, by an unique manner, a solution $y_3(x)$ of eguation (9) with property (13).
c) Let us consider the equation (8). We prove that there is at least one solution $y_2 = y_2(x) \in C(I) \cap C^1(I_2)$ such that

$$y_2(x_2^{\pm}) = A_2;\ \alpha_2(x) < y_2(x) < \beta_2(x),\ x \in I_2. \tag{18}$$

By condition (1.) there are possible following four cases:

$$\left.\frac{dw_2(x, y_2)}{dx}\right|_{(x,y_2)\in\mathcal{N}_\in} > 0, \quad \text{or} \quad \left.\frac{dw_2(x, y_2)}{dx}\right|_{(x,y_2)\in\mathcal{N}_\in} < 0,$$

or

$$\left.\frac{dw_2(x,y_2)}{dx}\right|_{\substack{(x,y_2)\in\mathcal{N}_\in \\ x\in[x_1,x_2)}} > 0, \qquad \left.\frac{dw_2(x,y_2)}{dx}\right|_{\substack{(x,y_2)\in\mathcal{N}_\in \\ x\in(x_2,x_3]}} < 0,$$

or

$$\left.\frac{dw_2(x,y_2)}{dx}\right|_{\substack{(x,y_2)\in\mathcal{N}_\in \\ x\in[x_1,x_2)}} < 0, \qquad \left.\frac{dw_2(x,y_2)}{dx}\right|_{\substack{(x,y_2)\in\mathcal{N}_\in \\ x\in(x_2,x_3]}} > 0.$$

By analogy, as in part a) above we define (with the aid of the sets $\mathcal{N}_2, \mathcal{D}_2$), by an unique manner, a solution $y_2(x)$ of equation (8) which satisfies conditions (18).
d) From parts a) – c) above it follows that for each function $\varphi(x) = (\varphi_1(x), \varphi_2(x), \varphi_3(x)) \in M$ there can be chosen, by indicated rule, a unique function $y(x) = (y_1(x), y_2(x), y_3(x)) \in M$ (here we put $y_1(x_1) = y_1(x_1^+), y_2(x_2) = y_2(x_2^\pm)$ and $y_3(x_3) = y_3(x_3^-)$). This correspondence defines mentioned operator T on M, i.e. for each $\varphi(x) \in M$ we have $T\varphi(x) \in M$ and, therefore, $T(M) \subset M$.

II. *Verification of Schauder's assumptions.* Let us consider the Banach space Λ of functions $\lambda(x) = (\lambda_1(x), \lambda_2(x), \lambda_3(x))$, continuous on I, with the norm $\|\lambda(x)\| = \max_{i=1,2,3}\left\{\max_I |\lambda_i(x)|\right\}$. Obviously $M \subset \Lambda$ and, as it follows from the properties of the functions $\alpha_i(x), \beta_i(x), i = 1,2,3,$ M is a closed, bounded and convex set.

It remains to prove that T is a continuous mapping such that $T(M)$ is a relatively compact subset. With respect to relatively compactness of $T(M)$ it is sufficient to prove by Arczela–Ascoli Theorem that $T(M)$ is uniformly bounded and equicontinuous on I.
α) The uniform boudnedness follows from inequality $\|\varphi\| \le L$ where $L = \max_I\{|\alpha_i(x)|, |\beta_i(x)|, i = 1,2,3\}$ which holds for every $\varphi \in M$.
β) Let us prove the equicontinuity of each function $\varphi(x) \in T(M)$. On I_1 the first coordinate $\varphi_1(x)$ of $\varphi(x)$ satisfies an equation of the type

$$\varphi_1'(x) = \omega_1(x)\varphi_1(x) + f_1(x, \varphi_1(x), \nu_2(x), \nu_3(x)) \tag{19}$$

where $(\varphi_1(x), \nu_2(x), \nu_3(x)) \in M$. Since $\omega_1(x) \in C(I_1, \mathbb{R})$ and $f_1 \in C(\Omega_1, \mathbb{R})$, from (19) we get $|\varphi_1'(x)| < K_\delta,\ x \in [x_1+\delta, x_3],\ x_1+\delta < x_3,\ 0 < \delta =$ const, where the constant K_δ exists and depends on δ. Let us put $\delta_1 = \min(\delta/2, \varepsilon/K_{\delta/2})$ where ε is an arbitrary positive number and δ is so small that

$$\max_{[x_1,x_1+\delta]} |\beta_1(x) - A_1| < \varepsilon/2, \quad \max_{[x_1,x_1+\delta]} |\alpha_1(x) - A_1| < \varepsilon/2.$$

Let us suppose that $|z_1 - z_2| < \delta_1, z_1, z_2 \in [x_1, x_3]$. Then either $z_1, z_2 \in [x_1, x_1+\delta]$ or $z_1, z_2 \in [x_1+\delta/2, x_3]$. In the first case

$$|\varphi_1(z_1) - \varphi_1(z_2)| \le |\varphi_1(z_1) - A_1| + |\varphi_1(z_2) - A_1| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

and in the second case (by Lagrange's theorem) $|\varphi_1(z_1) - \varphi_1(z_2)| \le K_{\delta/2}|z_1 - z_2| < \varepsilon$. So, for each positive ε there is a $\delta_1 > 0$ such that $|\varphi_1(z_1) - \varphi_1(z_2)| < \varepsilon$ for $|z_1 - z_2| < \delta_1$ and each function of the type of $\varphi_1(x)$ is equicontinuous. By analogy we can show that the functions of the type $\varphi_2(x)$ or $\varphi_3(x)$ are equicontinuous too. Finally, for $|z_1 - z_2| < \delta_1$, we get $\|\varphi(z_1) - \varphi(z_2)\| < \varepsilon$ and the equicontinuity of the set $T(M)$ is proved.

γ) Continuity of the operator T. Let us suppose that $y^0(x) \in M,\ \tilde{y}(x) \in M$ and

$$Y^0(x) = Ty^0(x),\ \tilde{Y}(x) = T\tilde{y}(x).$$

In the sequel we prove that the operator T is continuous, i.e. that

$$\|Y^0(x) - \tilde{Y}(x)\| < \varepsilon \quad \text{if} \quad \|y^0(x) - \tilde{y}(x)\| < \delta \leq \varepsilon. \tag{20}$$

The last inequality (in which ε is an arbitrary sufficiently small positive number) will be supposed in the sequel. Consider the identity

$$Y_i^{0\prime}(x) \equiv \omega_i(x)Y_i^0(x) + f_i(x, \eta_1^0(x), \eta_2^0(x), \eta_3^0(x)),$$

where $i = 1, 2, 3,\ \eta_i^0(x) = Y_i^0(x),\ \eta_j^0(x) \equiv y_j^0(x), j \neq i,\ (x, \eta_1^0(x), \eta_2^0(x), \eta_3^0(x)) \in \Omega_i$ and the equation (which has a solution $\tilde{Y}_i = Y_i(x)$)

$$\tilde{Y}_i' = \omega_i(x)\tilde{Y}_i + f_i(x, \tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3), \tag{21}$$

where $i = 1, 2, 3,\ \tilde{\eta}_i = \tilde{Y}_i, \tilde{\eta}_j = \tilde{\eta}_j(x) \equiv \tilde{y}_j(x), j \neq i,\ (x, \tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3) \in \Omega_i$. Define (for $i = 1, 2, 3$): $\mathrm{W}_i(x, \tilde{Y}_i) = (\tilde{Y}_i - Y_i^0(x))^2 - \varepsilon^2,\ \varepsilon = \text{const},\ 0 < \varepsilon < \varepsilon^*$,
$\mathcal{P}_i = \{(x, \tilde{Y}_i) : x \in I_i, W_i(x, \tilde{Y}_i) = 0\}$. γ_1) Let us evaluate the derivative of $W_1(x, \tilde{Y}_1)$ along the trajektories of equation (21) for $i = 1$ if $(x, \tilde{Y}_1) \in \mathcal{P}_1$. Then either $\tilde{Y}_1 = Y_1^0(x) + \varepsilon$ or $\tilde{Y}_1 = Y_1^0(x) - \varepsilon$. Therefore

$$\left.\frac{dW_1(x, \tilde{Y}_1)}{dx}\right|_{\tilde{Y}_1 = Y_1^0 \pm \varepsilon} =$$

$$= \pm 2\varepsilon \left[\omega_1(x)(\pm\varepsilon) + f_1(x, Y_1^0(x) \pm \varepsilon, \tilde{y}_2(x), \tilde{y}_3(x)) - f_1(x, Y_1^0(x), y_2^0(x), y_3^0(x))\right].$$

According to (4) and (5)

$$|f_1(x, Y_1^0(x) \pm \varepsilon, \tilde{y}_2(x), \tilde{y}_3(x)) - f_1(x, Y_1^0(x), y_2^0(x), y_3^0(x))| \leq$$

$$\leq (M_1(x) + N_1(x) + P_1(x))\,\varepsilon < |\omega_1(x)|\varepsilon.$$

Therefore

$$\left.\frac{dW_1(x, \tilde{Y}_1)}{dx}\right|_{(x, \tilde{Y}_1) \in \mathcal{P}_1} > 0 \text{ if } \omega_1(x) > 0 \text{ on } I_1 \tag{22}$$

and

$$\left.\frac{dW_1(x, \tilde{Y}_1)}{dx}\right|_{(x, \tilde{Y}_1) \in \mathcal{P}_1} < 0 \text{ if } \omega_1(x) < 0 \text{ on } I_1. \tag{23}$$

If (22) and (11) hold simultaneously, then all points of the set $\partial\mathcal{Q}_1$ where $\mathcal{Q}_1 = \{(x, \tilde{Y}_1) : x \in (x_1, x_3), w_1(x, \tilde{Y}_1) < 0, W_1(x, \tilde{Y}_1) < 0\}$ are, for $x \in (x_1, x_3)$, the points of strict egress for $\mathcal{Q}_\infty$ with respect to (21) where $i = 1$ (since this equation is at the same time an equation of the type (7)). Since $Y_1^0(x_1^+) = \tilde{Y}_1(x_1^+)$ and (in view of construction of operator T) $Y_1^0(x_3^-) = \tilde{Y}_1(x_3^-)$, then $|Y_1^0(x) - \tilde{Y}_1(x)| < \varepsilon$. Indeed, if this inequality does not hold, then there is a $x^* \in I_1$ such that $|Y_1^0(x^*) - \tilde{Y}_1(x^*)| = \varepsilon$ and by (22) $|Y_1^0(x) - \tilde{Y}_1(x)| > \varepsilon$ on $(x^*, x_3]$. This is impossible. If (23) and (12) hold, then all points of the set $\partial\mathcal{Q}_1$ are, for $x \in (x_1, x_3)$, the points of strict ingress for $\mathcal{Q}_\infty$ with respect to (21) where $i = 1$. If inequality $|Y_1^0(x) - \tilde{Y}_1(x)| < \varepsilon$ does not hold, then there is a $x^* \in I_1$ such that $|Y_1^0(x^*) - \tilde{Y}_1(x^*)| = \varepsilon$ and $|Y_1^0(x) - \tilde{Y}_1(x)| < \varepsilon$ on (x_1, x^*). This is impossible. In both considered cases $|Y_1^0(x) - \tilde{Y}_1(x)| < \varepsilon$ on I_1 and, consequently, on I too. We conclude that in cases (22), (11) and (23), (12)

$$|\tilde{Y}_1(x) - Y_1^0(x)| < \varepsilon \text{ on } I \text{ if } \|\tilde{y}(x) - y^0(x)\| < \delta.$$

Cases (22), (12) and (23), (11) are impossible according to (6).
γ_2) Let us evaluate the derivative of $W_3(x, \tilde{Y}_3)$ along the trajectories of equation (21) for $i = 3$ if $(x, \tilde{Y}_3) \in \mathcal{P}_\ni$. We get

$$\left.\frac{dW_3(x, \tilde{Y}_3)}{dx}\right|_{(x,\tilde{Y}_3)\in\mathcal{P}_3} > 0 \text{ if } \omega_3(x) > 0 \text{ on } I_3, \tag{24}$$

or

$$\left.\frac{dW_3(x, \tilde{Y}_3)}{dx}\right|_{(x,\tilde{Y}_3)\in\mathcal{P}_3} < 0 \text{ if } \omega_3 < 0 \text{ on } I_3. \tag{25}$$

In both of these cases we can prove, as in the part $\gamma_{\mathbf{1}}$), that $|Y_3^0(x) - \tilde{Y}_3^0(x)| < \varepsilon$ on I if $\|\tilde{y}(x) - y^0(x)\| < \delta$. Cases (24), (17) and (25), (15) are impossible in view of (6).
γ_3) Let us evaluate the derivative of W_2 along the trajectories of equation (21) for $i = 2$ if $(x, \tilde{Y}_2) \in \mathcal{P}_\in$. We get

$$\left.\frac{dW_2(x, \tilde{Y}_2)}{dx}\right|_{(x,\tilde{Y}_2)\in\mathcal{P}_2} > 0 \ \text{ if } \omega_2 > 0 \text{ on } I_2, \quad \text{or} \quad \left.\frac{dW_2(x, \tilde{Y}_2)}{dx}\right|_{(x,\tilde{Y}_2)\in\mathcal{P}_2} < 0 \ \text{ if } \omega_2(x) < 0 \text{ on } I_2,$$

or

$$\begin{cases} \left.\dfrac{dW_2(x, \tilde{Y}_2)}{dx}\right|_{\substack{(x,\tilde{Y}_2)\,\in\,\mathcal{P}_\in \\ x\,\in\,[x_1,x_2)}} > 0 \ \text{ if } \omega_2(x) > 0 \text{ on } [x_1, x_2), \\ \left.\dfrac{dW_2(x, \tilde{Y}_2)}{dx}\right|_{\substack{(x,\tilde{Y}_2)\,\in\,\mathcal{P}_\in \\ x\,\in\,(x_2,x_3]}} < 0 \ \text{ if } \omega_2(x) < 0 \text{ on } (x_2, x_3], \end{cases}$$

or

$$\begin{cases} \left.\dfrac{dW_2(x, \tilde{Y}_2)}{dx}\right|_{\substack{(x,\tilde{Y}_2)\,\in\,\mathcal{P}_\in \\ x\,\in\,[x_1,x_2)}} < 0 \text{ if } \omega_2(x) < 0 \text{ on } [x_1, x_2), \\ \left.\dfrac{dW_2(x, \tilde{Y}_2)}{dx}\right|_{\substack{(x,\tilde{Y}_2)\,\in\,\mathcal{P}_\in \\ x\,\in\,(x_2,x_3]}} > 0 \ \text{ if } \omega_2(x) > 0 \text{ on } (x_2, x_3]. \end{cases}$$

Each of the these cases can be considered as above in the parts $\gamma_{\mathbf{1}}$) and $\gamma_{\mathbf{2}}$) and, therefore, $|Y_3^0(x) - \tilde{Y}_3(x)| < \varepsilon$ on I if $\|\tilde{y}(x) - y^0(x)\| < \delta$. Other cases are impossible in view of (6).

Connecting parts $\gamma_1)$ — $\gamma_3)$, we conclude that (20) holds and, consequently, operator T is continuous. All conditions of Schauder's principle are valid and, therefore, operator T has a fixed point, i.e. has a solution of problem (1), (2) with indicated properties which follow from the form of the set M. The proof is complete.

3. Examples

Example 1. *Let us consider singular problem:*

$$\begin{gathered}
xy_1' = y_1 + x(y_1 - x)(y_2 - y_3)\exp(-2/x), \\
(x - 1/2)y_2' = 2y_2 + \left(y_2 - (x - 1/2)^2\right)(y_1 - y_3)\exp\left(-(x - 1/2)^{-2}\right), \\
(x - 1)y_3' = y_3 + (x - 1)(x - 1 - y_3)(y_1 + y_2)\exp(2/(x - 1)), \\
y_1(0^+) = 0,\ y_2\left(0.5^{\pm}\right) = 0,\ y_3(1^-) = 0.
\end{gathered}$$

This problem has trivial solution. Moreover, if rewrite this system in the form (1), all conditions of Theorem 1 are valid for $\alpha_1(x) = x/2, \beta_1(x) = 2x, \alpha_2(x) = (x - 1/2)^2/2,\ \beta_2(x) = 2(x - 1/2)^2, \alpha_3(x) = (1 - x)/2$ and $\beta_3(x) = 2(1 - x)$. Consequently, there is at least one *nontrivial* solution of this problem $y(x) = (y_1(x), y_2(x), y_3(x))$ such that $x/2 < y_1(x) < 2x$ on $(0, 1]$, $(x - 1/2)^2/2 < y_2(x) < 2(x - 1/2)^2$ on $[0, 1/2) \cup (1/2, 1]$, $(1 - x)/2 < y_3(x) < 2(1 - x)$ on $[0, 1)$.

Example 2. *Let us consider singular problem:*

$$\begin{gathered}
y_1' = -\frac{y_1}{x^2} + \frac{y_2}{10} + 1, \\
y_2' = -\frac{5y_2}{(x - 1)^2} + \frac{y_3}{10} + 1, \\
y_3' = \frac{5y_3}{(x - 2)^2} - \frac{y_1}{10} - 1, \\
y_1(0^+) = y_2(1^{\pm}) = y_3(2^-) = 0.
\end{gathered}$$

All conditions of Theorem 1 are valid for $\alpha_1(x) = 0.1x^2, \beta_1(x) = 2x^2, \alpha_2(x) = 0.1(x - 1)^2,\ \beta_2(x) = 2(x - 1)^2, \alpha_3(x) = 0.1(x - 2)^2$ and $\beta_3(x) = 2(x - 2)^2$. Consequently, there is at least one solution of this problem $y(x) = (y_1(x), y_2(x), y_3(x))$ such that $\alpha_i(x) < y_i < \beta_i(x),\ x \in I_i,\ i = 1, 2, 3.$

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