A remarkable Osijeker and his graph*

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Abstract. *This is an excerpt of The Colloquium Lecture devoted to late Professor Danilo Blanuša (born in Osijek), his results in [3], and later impact of [3] to graph theory.*

Key words: *map coloring, graph coloring, cubic graphs, snarks*

Sažetak.**O jednom znamenitom Osječaninu i njegovom grafu.** *Ovo je kratak prikaz predavanja koje je bilo posvećeno pokojnom profesoru Danilu Blanuši (rođenom Osječaninu), njegovim rezultatima koje je objavio u [3] i kasnijem odjeku tih rezultata u teoriji grafova.*

Ključne riječi: *bojanje karata, bojanje grafova, trivalentni grafovi*

The title of this talk conceals the name of Danilo Blanuša, who was a longtime professor at the University of Zagreb. He died in 1987, having taught mathematics to many generations of engineers and mathematicians. This talk presents the results of his paper published in 1946 that left a lasting mark on graph theory. The paper in question is [3], which was published in Croatian with a longer summary in French. It is worth pointing out that the paper appeared in the first issue of the first Croatian journal devoted to mathematics, physics and astronomy. Its mathematical successor is the present-day journal "Glasnik Matematički".

The English title of [3] is "The four color problem". At that time the four color conjecture was in vogue and was attacked by amateurs and professional mathematicians alike. The statement of the conjecture is that every map drawn on a sheet of paper can be colored using only four colors in such a way that countries sharing a border have different colors (where "country" means a connected region of the sheet). If a country is allowed to consist of more than one region and these regions are separated, then one can construct an example of a map with five countries each of which is adjacent to each of the other four, consequently, five colors are required. In 1976, this conjecture was solved by K. Appel and W. Haken ([1] and [2]). This was about 100 years after the first "solutions" to the problem were offered by A. B. Kempe [8] in 1879 and P.G. Tait [11] in 1880. The flaw in Kempe's analysis was corrected in the final solution by Appel and Haken by examining some 1,500 arrangements of countries by means of computers. Tait considered a conjecture equivalent to the four-color conjecture. Given a map, construct a graph whose

*The lecture presented at the Mathematical Colloquium in Osijek organized by Croatian Mathematical Society - Division Osijek, April 24, 1998.

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edges are boundaries between countries and vertices are any points where more than two countries meet. Assuming that a map is four-colored, the graph given by borders has the following properties: it is connected; it is planar (i.e. it can be drawn in a plane with no edge intersections); and it has no bridges (a bridge is an edge whose removal renders the graph disconnected - the existence of a bridge implies that a border has the same color on both sides). Tait showed how, essentially, any map can be transformed into a map whose borders form trivalent graphs (i.e. graphs where exactly three different edges meet at every vertex, also known as cubic or 3-graphs). Moreover, he showed that a four-coloring of countries is equivalent to a three-coloring of the graphs formed by their borders, where a three-coloring of a graph means a coloring edges so that all edges are colored by three colors and all edges with a common vertex are colored by different colors. Thus, Tait showed that the four-color problem is equivalent to the three-coloring problem of edges of planar, connected, bridgeless, trivalent graphs. Tait's proof was short and elegant, but he was intuitively convinced that any trivalent graph is three-colorable.

However, soon after Tait's paper, J. Petersen [9] pointed out that the graph in *Fig. 1* is not three-colorable. This graph bears his name today. It is a fact that this graph is not planar, but this does not help Tait's proof, since it was not known whether a planar graph was 3-colorable. At the time Blanuša's paper was written, many believed that Petersen's graph is essentially the only one that is not three-colorable. (It was possible to obtain examples by some trivial modifications. An example of a trivial modification is the following. Cut an edge at its midpoint, thereby obtaining two new vertices, and make a graph by adding two edges between those two new vertices. If the original graph was not three-colorable, the modified one is not, either.) It was not easy to find another nontrivial example.

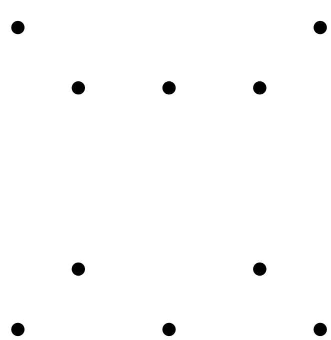

Figure 1.

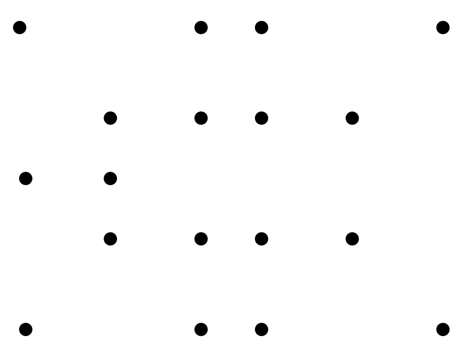

Figure 2.

In his paper Blanuša first treats again the above mentioned equivalence between four-colorings of maps and three-colorings of cubic graphs and gives another proof of that equivalence. In order to prove that the three-coloring of cubic graphs implies the four-coloring of maps he needed an auxiliary result that turned out to be a useful and powerful lemma, the so-called edge cut lemma (e.g. see [6], p.82). Let us call by an edge cut a set of edges Z whose removal disconnects a graph. The edge cut lemma says: if Z is any edge cut of a cubic 3-colorable graph G and k, l and m are the numbers of edges of Z colored by colors 1, 2 and 3, then k, l and m have the same parity, in other words, all are either even or odd.

Then Blanuša discusses the four-color conjecture in terms of three-colorings of cubic graphs, saying that in order to answer the four-color conjecture it suffices to find either all trivalent graphs that are not three-colorable or characterize them in some constructive way in order to see if there is a planar one among them (see also [4] and [7]). This task was not easy and he did not try to carry it out, but he did offer a new non-three-colorable cubic graph which today bears his name. It is depicted in *Fig. 2.* Thus, after almost 50 years, another nontrivial example was discovered. In his paper, Blanuša briefly says that his graph was obtained by connecting two Petersen's graphs after simultaneously deleting a vertex in each. This operation was later formalized by Isaacs [7] and is today called the "dot product" (see [6]).

W.T. Tutte saw Blanuša's graph soon after it appeared (through private correspondence) and then published the third example of a non-three-colorable trivalent graph under the alias Blanche Descartes [4]. The fourth one was discovered in 1973 by Szekeres [10] and in 1975 Isaacs constructed two nontrivial infinite families of such graphs. A year later, M. Gardner [5] proposed the name "snark" for a nontrivial non-three-colorable trivalent graph, which is today widely accepted (see [6], [12]). (The term "snark" was coined by Lewis Carroll: "The hunting of the snark".) Today there exists a series of research articles studying snarks. For example, it is known that there are no snarks with fewer than ten vertices. The Petersen graph is the unique snark with ten vertices. There are no nontrivial snarks with 12, 14 or 16 vertices and there are exactly two with 18 vertices, both named after Blanuša.

D. Blanuša was one of the founders of The Croatian Mathematical Society and its very devoted member. To honor him, The Croatian Mathematical Society uses Blanuša's snark in its logo, which is presented in black and white in *Fig. 3.* (In the color version, the squares (vertices) are red and the edges are golden.)

Figure 3.

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