On numbers and classes *†

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Abstract. *The concept of class is analyzed and it is concluded that a number is the number of a pure founded class. Some well-known definitions of numbers are analyzed and it is concluded that this analysis supports the thesis that numbers are conventional. Nevertheless, an argument is offered supporting the thesis that von Neumann's numbers are the numbers.*

Key words: *class, pure founded class, number, von Neumann's number*

Sažetak. **O brojevima i klasama.** *Analizom pojma klase zaključeno je da je broj uvijek broj čiste utemeljene klase. Analizirane su neke dobro poznate definicije broja, i zaključeno je da analiza podupire tezu da su brojevi konvencije. Ipak, ponuđen je dodatni argument koji podupire tezu da su von Neumannovi brojevi pravi brojevi.*

Ključne riječi: *klasa, čista utemeljena klasa, broj, von Neumannov broj*

1. On numbers and classes

What are numbers? A number is the number of something. Hence, we could start answering our question, as Frege did, by asking another one. What is a number the number of? Frege's answer is that number is the number of a concept. Namely, we can say with equal truth both "Here are four books" and "Here are 500 book leafs". Now what changes here from one judgement to the other is neither any individual object, nor the whole, the agglomeration of them, but rather our terminology. But that is itself only a sign that one concept has been substituted for another (Frege, 1978, p. 59).

But a concept is not something subjective like an idea. It is as objective as any object is. We assert something of a concept as truly or as falsely as we assert something of any object. If, for example, we bring the concept of whale under that of mammal, we are asserting something objective; but if the concepts themselves were subjective, then the subordination of one to the other, being a relation between

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them, would be subjective too, just as a relation between ideas is (Frege, 1978, p. 60).

The objectivity of concepts does not imply their actuality (i.e. their spatio-temporal causal efficiency). We distinguish what we call objective from what is actual. The axis of the earth is objective, so is the centre of mass of the solar system, but we should not call them actual in the way the earth itself is (Frege, 1978, p. 35). If something is objective but not actual it is called an abstract object. Hence, a concept is an abstract object. A number is the number of an abstract object.

Now, what counts in ascribing a number to a concept is the extension of the concept. We could say, a step further from Frege, that a number is the number of the *extension* of a concept. A number is the number of a class.

Classes are abstract objects too. Whatever we asserted about objectivity and nonactuality of concepts could be asserted, equally true, about classes. But what are classes and what classes are there? The best answer to this question is the standard first-order theory of classes (i.e. sets and proper classes).[1] Of course we may restrict ourselves to pure founded classes (see [3]), because to every unfounded class there corresponds a founded one of the same size and to every founded class there corresponds a pure one of the same size. As far as numbers are concerned and, as far as mathematics is concerned, individuals (i.e. objects which are not classes) are completely unnecessary. Mathematics does not need them. It is founded on the empty set of individuals. Numbers are the numbers of pure founded classes.

Now that we have learned what the objects are, of which numbers are the numbers of, we may get back to the main question. What are numbers? A tempting strategy is to explain numbers away. The meaningfulness of a statement containing a singular term does not necessarily presuppose an object named by it. For example, we could say that:

(S) $2 + 3 = 5$

is just a manner of speaking. The real meaning of the statement is given by:

(MS) If X and Y are any two disjoint classes, and if the number 2 belongs to the class X and the number 3 belongs to the class Y, then the number 5 belongs to their union $X \cup Y$.

Of course, the real meaning of the statements "the number 2 belongs to the class X", "the number 3 belongs to the class Y" and "the number 5 belongs to the class $X \cup Y$" is given by:

(M2) "There is a member z of the class X and such that there is a member y of the class $X - \{z\}$ such that there is no member of the class $X - \{z\} - \{y\}$."

and by analogous **(M3)** and **(M5)**. Nevertheless, there are many other mathematical contexts from which it is quite impossible to explain numbers away. The most

[1] Mathematicians struggled for it for more than seventy years. Those who were logically inclined wanted to base everything on the presupposedly logical principles of comprehension and extensionality and that was not possible. Those who were more mathematically minded did not care for that. But they were not successful in explaining (with no use of the independent notion of ordinal number) that the universe of all classes is just the cumulative hierarchy. Von Neumann was the first one to succeed.

common contexts are those which mention classes of numbers. Such contexts are in constant use: integers are defined as classes of pairs of natural numbers; rationals as classes of pairs of integers; real numbers as classes of rationals, etc. But, if a number is to be the member of a class, then it has to be an object and there is no way to explain it away. There is a possibilitiy of explaining *classes* away, but mathematics, i.e. the standard classical mathematics, is founded on classes and does not explain them away. (After all, we are interested in what numbers are, and not interested in what numbers could be in a could-be-reconstructed-mathematics.)

If numbers are not to be explained away, then what are they? We know that they belong to pure founded classes. If a and b are numbers, then they are numbers of some pure founded classes A and B: $a = \mathrm{num}(A)$, $b = \mathrm{num}(B)$. Hence, to specify what numbers are, is to define the function num on the universe of all pure founded classes. Numbers are to be the values of this function. Of course, this should be done in accordance with the criterion of number identity: $\mathrm{num}(A) = \mathrm{num}(B)$ iff A and B are of the same size. "A and B are of the same size" is to be defined as "there is a one-to-one correspondence between A and B; i.e. $A \approx B$". Hence the function num is to be defined in accordance with the following:

(NI) *Number identity* $\mathrm{num}(A) = \mathrm{num}(B) \leftrightarrow A \approx B$

The difficulty is that there is no unique function num, defined on the universe of pure founded classes, which satisfies the criterion of number identity **(NI)**. In one word, there is no unique solution of **(NI)**.

The uniqueness of the num function could be forced by imposing further conditions on the function. Needless to say the conditions should be appropriate, just as **(NI)** is. The Fregean condition:

(FC) $\quad x \in \mathrm{num}(x)$,

which postulates that each set is a member of the number ascribed to it, is one such condition.[2] There is exactly one function num which satisfies **(NI)** and **(FC)**. This is a Fregean number function:

(FN) $\quad \mathrm{num}(x) = [x]$, where $[x] = \{y : x \approx y\}$.

The difficulty of Fregean definition **(FN)** is that, according to the definition, numbers are proper classes. It means that numbers could not be members of further classes. It renders the definition mathematically useless and, therefore, inadmissible (cf. above).

Scott's definition of numbers (see [2]):

(SN) $\quad \mathrm{num}(x) = [x]_{\min}$,

where $[x]_{\min}$, is the subset of $[x]$ consisting of elements of the least rank, also satisfies **(NI)**. It dispenses with Frege's difficulty, because $[x]_{\min}$ is a set contained in $[x]$. Nevertheless, it has to be abandoned, because the further condition it satisfies (that $\mathrm{num}(x)$ should consist of elements of $[x]$ of the least rank) is an ad hoc condition. The only motivation for using Scott's condition is to eliminate Frege's difficulty.

[2]Incidentally, **(FC)** precludes the proper classes to have a number, because a proper class could not be a member of any class.

Another, rather natural condition is that mutually equinumerous members of $[x]$ are represented by one of them. It means that num associates to any given x its representative from $[x]$:

(RC) $\quad \mathrm{num}(x) \in [x]$.

The problem is that there are many num-functions which satisfy **(NI)** and the representative condition **(RC)**. A further condition is needed to separate the unique num-function from all of them. Without such condition $\mathrm{num}(x)$ is as conventional as the Paris metre is. (The Paris metre represents all lengths of its span, just as $\mathrm{num}(x)$ represents all sets of its numerosity.) Hence, we could not talk about numbers, but rather about conventional numbers. Nevertheless, to whatever use numbers may be put, conventional numbers may be put equally well. For example, using von Neumann's conventional numbers $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$, etc. we could interpret:

0) "0 belongs to x", 1) "1 belongs to y", 2) "2 belongs to z", etc.

into the following way:

c0) "$x \approx \emptyset$", c1) "$y \approx \{\emptyset\}$", c2) "$z \approx \{\emptyset, \{\emptyset\}\}$", etc.

Any other conventional numbers would do just as well. Similarly, conventional integers are easily defined as classes of pairs of conventional natural numbers; conventional rationals as classes of pairs of conventional integers; conventional real numbers as classes of conventional rationals, etc. In a sense, conventional numbers explain numbers away. But conventional numbers are objects, which are elements of further classes, and the main problem of explaining numbers away is not present now. To whatever use numbers may be put, conventional numbers may be put equally well. Mathematics has no need of numbers. Conventional numbers suffice.

Nevertheless, we think that von Neumann's conventional numbers are not as conventional as they seem to be. First of all, von Neumann's conventional numbers are conventional ordinal numbers. As far as natural numbers are concerned there is no difference. Natural numbers are cardinal numbers of finite sets and there is always unique well-ordering of a finite set. On the other hand, there are many well-orderings of an infinite set. Hence, to the cardinal number of an infinite set there correspond many ordinal numbers. The first of them is unique and it is quite appropriate to identify the cardinal number with this unique ordinal number. A cardinal number is the first ordinal number of its cardinality. Cardinal numbers, defined in this way, are as conventional as their corresponding ordinal numbers are. We will show that von Neumann's ordinal numbers are not as conventional as they seem to be.

We should start with the question: What is an ordinal number the ordinal number of? We have seen that a cardinal number is the cardinal number of a pure founded class. In the same way we could show that an ordinal number is the ordinal number of a pure founded well-ordered class. If α and β are ordinal numbers, then they are ordinal numbers of some pure founded well ordered classes $(A, <_A)$ and $(B, <_B)$:

$$\alpha = \mathrm{Ord}(A, <_A), \qquad \beta = \mathrm{Ord}(B, <_B).$$

Hence, to specify what ordinal numbers are is to define the function Ord on the universe of all pure founded well-ordered classes. Ordinal numbers are to be the values of this function. This should be done in accordance with the criterion of ordinal number identity:

(ONI) *Ordinal number identity* $\mathrm{Ord}(A, <_A) = \mathrm{Ord}(B, <_B) \leftrightarrow (A, <_A) \sim (B, <_B)$.

Equiorderedness relation $\sim$ holds between $(A, <_A)$ and $(B, <_B)$, if there is a one-one correspondence between A and B which is order preserving. In this case it is also said that A and B are of the same length. The difficulty is that there is no unique function Ord which satisfies **(ONI)**, even if we agree that values of Ord should be pure founded well-ordered classes. The uniqueness of the Ord-function could be forced by imposing further conditions on the function. A rather natural one is that mutually equiordered sets[3] are represented by one of them:

(ORC) $\mathrm{Ord}(A, <_A) \in [(A, <_A)]$.

The problem is that there are still many Ord-functions which satisfy **(ONI)** and **(ORC)**. Further conditions are needed to separate the unique Ord-function from all of them. Without such conditions $\mathrm{Ord}(A, <_A)$ is completely conventional.

There are such conditions which are quite appropriate. An ordinal number $\mathrm{Ord}(A)$ is the ordinal number of a well-ordered class A, i.e. $\mathrm{Ord}(A)$ is the length of A[4]. But, the primary meaning of ordinal numbers is that they are ordinal numbers of the members of a well-ordered class. The well-ordered class A has its 5th member, its 157th member and, perhaps, its ωth member and its ϵ_0th member. The ordinal number of a in A is denoted by $\mathrm{ord}(a)$[5] and it is the distance of a in A. The connection between "small" ord and "big" Ord is straightforward. The distance of a in A is the length of $\{x : x < a\}$, i.e.

(oO) $\mathrm{ord}(a) = \mathrm{Ord}\{x : x < a\}$.

On the other hand, the length of $\{x : x < a\}$ is completely determined by the set of distances $\{\mathrm{ord}\, x : x < a\}$, i.e.

(Oo) $\mathrm{Ord}\{x : x < a\} = \{\mathrm{ord}(x) : x < a\}$

is an appropriate condition. From **(oO)** and **(Oo)** it follows:

$$\mathrm{ord}(a) = \mathrm{Ord}\{x : x < a\} = \{\mathrm{ord}(x) : x < a\}.$$

From the principle of transfinite recursion on well-ordered classes it follows that for any well-ordered class $(A, <)$ there is a unique function ord which satisfies $\mathrm{ord}(a) = \{\mathrm{ord}(x) : x < a\}$ and values of this function are von Neumann's numbers. Therefore, there is a unique function Ord which satisfies **(ONI)**, **(ORC)**, **(oO)** and **(Oo)**.

We may conclude that Ord is not conventional and, therefore, von Neumann's numbers are not just conventional numbers. They are the numbers.

[3]We may restrict ourselves to well-ordered sets, because all proper well-ordered classes are of the same ordinal type. Hence, one and the same ordinal number belongs to all of them and it could be dealt with separately.

[4]"Well-ordered class A" is short for "well-ordered class $(A, <_A)$".

[5]The function ord is always associated with some well-ordered class A, on which it is defined. Strictly, we should write $\mathrm{ord}_A(a)$. If it is clear which A it is, we usually omit A.

References

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