Constructive problems and the method of similarity*†

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Abstract. *We study three types of constructive problems which can be solved by the method of similarity. By solving typical problems of each group, we are going to note the ways of further generalization and creation of new problems.*

Key words: *method of similarity, the Thales theorem, triangle*

Sažetak.**Konstruktivni problemi i metoda sličnosti.** *Opisana su tri tipa konstruktivnih zadataka koji se rješavaju metodom sličnosti i dane su ideje kako generirati niz novih zadataka.*

Ključne riječi: *metoda sličnosti, Talesov teorem, trokut*

In teaching mathematics in the seventh class of primary school and the first class in grammar school and not only there, we meet with constructive problems which may be solved by the method of similarity. It is interesting to note the rules which help us solve these problems and create the new ones.

We are going to study three types of problems and by solving typical problems of each group, we are going to note the ways of further generalization and creation of new problems.

First, let us pay attention to the basic constructive problems, i.e. to the problems in which the construction of a triangle is required. We shall use the standard indications for the elements of the triangle, i.e. we shall denote the sides of the triangle by letters a, b and c, the angles by α, β, γ, the altitudes by v_a, v_b, and v_c, the medians by letters t_a, t_b, t_c, etc.

Let us solve in detail a typical problem of this group.

1. Construct a triangle ABC, if the following is given: α, β, $3a + 2b$.

Solution. Analysis. According to the theorem about similarity of triangles (K-K), the angles α and β determine the family of the triangles which are mutually similar. Besides this great number of triangles, it is necessary to define exactly

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that one which has the given combination $3a + 2b$. So if the triangle $A'B'C'$ is constructed by the angles α and β, then the triangle requested ABC is similar to the triangle $A'B'C'$ and the ratio of lengths $3a' + 2b'$ and $3a + 2b$ coresponds to the coefficient of similarity, i.e. there holds $(3a' + 2b') : (3a + 2b) = a' : a$.

Well, the problem led to constructing the fourth proportional which is simply performed by the Thales theorem with known elements $3a + 2b$, $3a' + 2b'$ and a'. The side b is constructed in the same way.

Construction. Let us construct any triangle $A'B'C'$ with the angles $\angle A' = \alpha$, $\angle B' = \beta$. For example, let the side c' be 1.8 cm. On a semiline Op put the segment $\overline{B'C'}$ three times and the segment $\overline{A'C'}$ two times. In such a way we have got points M', N', P', Q', R' (figure). On another semiline Oq let's put the segment $\overline{OR}$, $|OR| = 3a + 2b$. Let us connect the points R and R'. The parallels with the line RR' through points M', N', P', Q' intersect the semiline Oq at the points M, N, P, Q. At the same time the length of the segments $\overline{OM}, \overline{MN}$ and $\overline{NP}$ is equal to a and the length of the segments $\overline{PQ}$ and $\overline{QR}$ is equal to b.

Now, it is easy to construct a triangle with angles α, β and with sides a and b.

The proof is obvious from the analysis that has been performed, because it is evident that the triangle that has been obtained has got angles α, β and that its combination $3a + 2b$ is equivalent to the given segment.

Discussion. The problem has got a unique solution.

How to generalize this problem? In the analysis we have mentioned that the triangle we are searching for is defined with the third element, i.e. with linear combination of sides. This fact is an exhaustible source of new problems. In other words, if we change $3a + 2b$ with any other linear combination of sides and not only them, but any other linear elements of the triangle: altitude, median, radius, etc., we will get a series of problems which are solved analogously. Here are some suggestions of that type and it is absolutely clear how to generate nuberous new problems.

2. Construct a triangle if there are given α, β and
1) v_c; 2) t_b; 3) R; 4) s_α; 4) $2s$; 5) $v_a + t_b$.

3. Construct a right-angled triangle if an acute–angle is set and
1) $a+2b$; 2) $a+c$; 3) $c-a$; 4) v_c; 5) v_c+c.

4. Construct an isosceles triangle if are given an angle between sides and 1) perimeter; 2) $a+b$; 3) sum of the three altitudes.

The other way of generalization is based on the substitution of the similarity theorem on the basis of which the analysis is performed. In analysing the first problem we have used the similarity theorem popularly named K-K. What is necessary to b e changed in the given elements, in order to use one of the other three theorems? The answer is evident: instead of two angles of the triangle it is necessary to set those elements which generate the families of mutually similar triangles. Particulary it means that if we want to apply the theorem S-S-S, the problem will have the form: *Construct a triangle if are given the ratio* $a:b:c$ *and the linear combination of the linear elements of the triangle.* The use of the theorem S-K-S and S-S $-^{>}$ K leads us to the problem of the form: *Construct a triangle if are given the ratio of the two sides, an angle and a linear combination of linear elements of the triangle.*

Let us illustrate what we have said before.

5. Let's construct a triangle if $a:b:c=3:4:6$ and $t_b=2.5$ cm.

Solution. Construction. We will construct any triangle $A'B'C'$ where the ratio of the sides is given. For example, let $a'=3$ cm, $b'=4$ cm and $c'=6$ cm. In it we draw a median $\overline{B'D}$ from the vertex B'. On the line $B'D'$ we determine the point D so that $|B'D|=t$. The parallel through the point D intersects the lines $A'B'$ and $B'C'$ successively at the points A and C.

6. Construct a triangle if the following is given: 1) $a:b:c=3:2:4$ and $s_\alpha=6$ cm; 2) $a:b:c=5:7:9$ and the perimeter; 3) $a:b:c=4:7:10$ and $2c-b$.

7. Construct a triangle if the following is given $a:b=3:4$, angle α and altitude $v_c=4$.

Solution. The triangle ABC whose we are looking for is similar to $A'B'C'$ whose ratio of the sides $a':b'$ is equal to $3:4$ and the angle at the vertex A' is equal to α. First, let's construct the triangle $A'B_1'C'$. We construct the angle a with the vertex A' and sides p and q. On the side q we put the point C' such that $|A'C'|=4$ cm. We draw a circle with radius 3 cm and center C' and which intersects the side p at two points B' and B'. In the triangle $A'B_1'C'$ we make a perpendicular line $C'D'$ from the point C' and on the line $C'D'$ we determine the point D such that $|C'D|=v_c=4$. The parallel with $A'B_1'$ through the point D intersects the lines $C'A'$ and $B'C'$ in the points A and B. The triangle AB_1C' is one solution of the problem. We get the other solution if we apply the same procedure on the triangle $A'B'_2C'$. Generally, when the ratio of the sides and the angle opposite to the smaller side are given, there are going to appear two solutions.

8. Construct a triangle if the following is given:

1) $a : b = 3 : 5$, $\alpha = 30^\circ$, $t_c = 4$; 2) $a : b = 2 : 5$, $\alpha = 30^\circ$, perimeter; 3) $a : b = 3 : 5$, $\beta = 30^\circ$, s_β; 4) $a : b = 2 : 5$, β, $a + b$; 5) $a : b = 4 : 7$, $\gamma = 60^\circ$, $v_a = 5$; 6) $a : b = 4 : 9$, γ, $a + 2b + 3c$.

Let us point out once more the idea on which solving of given problems is based so as creation of the new onces. The two of given elements of the triangle generate a class of mutually similar triangles and the rest, the third given element determines the triangle being searched.

The natural generalization of these problems is passing to constructions in the set of squares and polygons. When solving these problems, the problem is usually deducted to the construction of some auxiliary triangle.

9. The ratio of the side a and the diagonal d of the rectangle is $5 : 8$. Draw this rectangle, if $a + d = 20$ cm.

Solution: We draw any right triangle with the ratio of the short side and hypotenuse $a' : d' = 5 : 8$ and complement it up to a rectangle. Then, by using the Thales theorem from segments $a + d$, $a' + d'$ and a' we determine a and construct the rectangle which is similar to the rectangle already drawn, but with the side a.

10. Construct a trapezoid if a : b : c : d and the altitude is given.

11. Construct the parallelogram, the sides of which relate as $4 : 7$, if the angle between them is 45° and the length of one of the diagonals is 8.

And finally, let us describe another type of constructive problems whose solution is connected to the homothety. These problems are characterized by the formulation: "In the given figure draw...", or "Around the given figure draw..." The typical example of this kind of problems is the following.

12. In the given triangle ABC draw the triangle PRS ($P \in \overline{AB}$, $R \in \overline{BC}$, $S \in \overline{AC}$) , in such a way that its sides are parallel with the given three lines p, r, s of which no one is parallel to any of others.

Solution. The triangle we are looking for must satisfy six conditions: $RS \| p$, $PR \| s$, $PS \| r$, $R \in \overline{BC}$, $S \in \overline{AC}$, $P \in \overline{AB}$. Let us ask ourselves if the problem can be solved by using only first five conditions.

We construct any triangle $P'R'S'$ with property that its sides are successively parallel with the given lines and that $R' \in \overline{BC}$ and $S' \in \overline{AC}$. We can see that there are many solutions of this type depending on the choice of the point R'. By this construction the point P' generally will not belong to the side $\overline{AB}$. So, the triangle $P'R'S'$ must be transformed in such a way that the parallels and the incidentia remain maintained and that the point P' maps into the point on the side $\overline{AB}$. If we remember the basic property of homothety that the segment and its homothetic picture are parallel, it is natural to think that this kind of transformation should be homothety. Well, from this point we can conclude that the center of the homothety is a point C. For complete determination of homothety one pair of joint points must be set. Let P be the intersection of line CP' and the side AB. Now, by making parallels through the point P with $P'R'$ and $P'S'$ we get points R and S.

Here are some more problems the solution of which is based on using of the homothety which retains the requests which we have chosen for constructing a similar figure and by means of which in this class of similar figures we find out the solution we are looking for.

13. In the given circle draw a rectangle whose sides relate as $3 : 5$.

14. In the given semi-circle draw a square in such a way that its one side lies on the diametar.

15. In the given triangle draw a rectangle whose lengths of sides are related as $1 : 3$.

It is evident how by changing the given elements we can easily generate new problems solving of which is analogue to the previous ones