Square–Gaussian random processes and estimators of covariance functions

YURIJ V. KOZACHENKO* OLEXANDER V. STUS*

Abstract. *In this paper inequalities for distributions of quadratic forms from square–Gaussian random variables and distributions of suprema of quadratic forms from square–Gaussian random processes are proved. These inequalities enable us to investigate the jointly distributions of estimators of covariance functions of Gaussian processes.*

Key words: *square–Gaussian random variables, random process, metric space, confidence ellipsoid*

Sažetak. **Kvadratno–Gaussovi slučajni procesi i procjenitelji kovarijacionih funkcija.** *U ovom članku dokazane su nejednakosti za distribucije kvadratnih formi kvadratno–Gaussovih slučajnih varijabli i za distribucije supremuma kvadratnih formi kvadratno-Gaussovih slučajnih procesa. Te nejednakosti omogućuju proučavanje distribucija procjenitelja kovarijacionih funkcija Gaussovih procesa.*

Ključne riječi: *kvadratno–Gaussove slučajne varijable, slučajni procesi, metrički prostori, elipsoid povjerenja*

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1. Introduction

In this paper we investigate spaces of square–Gaussian random variables $\mathrm{SG}_\Xi(\Omega)$, i.e. a closure in $L_2(\Omega)$ of quadratic forms from a family of jointly Gaussian random variables.

An inequality for the distributions of quadratic forms from random variables $\xi_k \in \mathrm{SG}_\Xi(\Omega)$ is proved. This inequality enables us to construct confidence ellipsoids for estimators of covariance functions of jointly Gaussian stationary random processes. Estimates of the distributions of the supremum of quadratic forms from random processes $\xi = \{\xi(t), t \in T\}$, $\xi \in \mathrm{SG}_\Xi(\Omega)$ are found, too. These estimates enable us to construct confidence ellipsoids for uniform (in some set) estimators of covariance functions of jointly Gaussian stationary random processes.

*Department of Mathematics, Taras Shevchenko University of Kyiv, Vladimirskaya 64, 252 601 Kyiv, Ukraine, e-mail: `mmp@mechmat.univ.kiev.ua`

2. Space of pre–Gaussian random variables

Let $\{\Omega, \Im, P\}\}$ be a standart probability space and let $U(x) = \exp\{|x|\} - 1$.

Definition 1. *A space of random variables $L_U(\Omega)$ will be called the Orlicz space generated by the function $U(x)$, if for any random variable $\xi \in L_U(\Omega)$ there exists a positive constant α such that $E\exp\{\alpha|\xi|\} < \infty$.*

The space $L_U(\Omega)$ is a Banach space with respect to the norm [10, 3]

$$\|\xi\| = \inf\left(r > 0 \colon E\exp\left\{\frac{|\xi|}{r}\right\} \le 2\right) \tag{1}$$

The norm $\|\xi\|$ is called the *Luxemburg norm.*

Definition 2. *The space of centered random variables ξ from $L_U(\Omega)$ will be called the space of pre–Gaussian random variables.*

This space will be denoted by $\mathrm{Prg}(\Omega)$. Pre–Gaussian random variables are introduced in [2].

3. Space of square–Gaussian random variables

In this section the notion of the space of square–Gaussian random variables is given. The notion of square–Gaussian random vectors and the notion of a family of square–Gaussian random variables were introduced and investigated in papers [5, 6, 7, 8, 9].

Definition 3. ([5]) *A random vector $\vec{\eta} \in \mathbb{R}^d$ is called square–Gaussian, if all its components η_i can be represented in the form*

$$\eta_i = \vec{\xi}_i^T A_i \vec{\xi}_i - E\vec{\xi}_i^T A_i \vec{\xi}_i \tag{2}$$

where $\vec{\xi}_i$, $E\vec{\xi}_i = 0$, are jointly Gaussian random vectors and A_i are symmetric matrices or mean-square limits of sequences of random variables of the form (2).

Remark 1. ([5]) *Let $\vec{\xi}_{ij}$, $\vec{\eta}_{ij}$, $E\vec{\xi}_{ij} = E\vec{\eta}_{ij} = 0$, $i = 1, \dots, d$, $j = 1, \dots, n$, be jointly Gaussian random vectors, and let A_i be symmetric matrices*

$$\theta_i = \sum_{j=1}^{n} \left(\vec{\xi}_{ij}^T A_i \vec{\eta}_{ij} - E\vec{\xi}_{ij}^T A_i \vec{\eta}_{ij}\right)$$

then the vector $\vec{\theta}^T = (\theta_1, \dots, \theta_d)$ is a square–Gaussian random vector (θ_i can be represented in the form (2)).

Definition 4. *Let $\Xi = \{\xi_t, t \in T\}$ be a family of jointly Gaussian random variables, $E\xi_t = 0$ (for example, ξ_t is a Gaussian random process). The space $\mathrm{SG}_\Xi(\Omega)$ is called the space of square–Gaussian random variables with respect to Ξ, if random variables from $\mathrm{SG}_\Xi(\Omega)$ can be presented in the form (2), where $\vec{\xi}_i^T = (\xi_{i1}, \xi_{i2}, \dots, \xi_{id})$, $\xi_{ik} \in \mathrm{SG}_\Xi(\Omega)$, $k = 1, \dots, d$, or if they are mean-square limits of sequences of such random variables.*

Lemma 1. *Let η_i, $i = 1, 2, \ldots, n$ be random variables from* $\mathrm{SG}_\Xi(\Omega)$*. Then, for all $|s| < 1$ and all $\lambda_i \in \mathbb{R}^1$, $i = 1, 2, \ldots, n$ the inequality*

$$E \exp\left\{ \frac{s}{\sqrt{2}} \frac{\eta}{(\operatorname{Var} \eta)^{\frac{1}{2}}} \right\} \le R(|s|) \tag{3}$$

holds, where $\eta = \sum_{i=1}^n \lambda_i \eta_i$, $R(s) = \exp\{\frac{1}{8}\} \exp\{-\frac{s}{2}\}(1 - s)^{-\frac{1}{2}}$.

Proof. Let $\vec{\xi}$ be a Gaussian random vector such that $E\vec{\xi} = 0$, and let A be an arbitrary symmetric matrix. It follows from [7], that there exist constants $\delta_1 > 0$, $\delta_2 > 0$, $\delta_1^2 + \delta_2^2 = 1$, such that for all $|s| < 1$ the inequality

$$E \exp\left\{ \frac{s}{\sqrt{2}} \frac{\vec{\xi}^T A \vec{\xi} - E\vec{\xi}^T A \vec{\xi}}{\left(\operatorname{Var} \vec{\xi}^T A \vec{\xi}\right)^{\frac{1}{2}}} \right\} \le L(s, \delta_1, \delta_2) R(s, \delta_1, \delta_2) \tag{4}$$

holds, where

$$L(s, \delta_1, \delta_2) = \exp\left\{-\frac{s}{2}(\delta_1 - \delta_2)\right\}(1 - s\delta_1)^{-\frac{1}{2}}(1 + s\delta_2)^{-\frac{1}{2}}$$

$$R(s, \delta_1, \delta_2) = \begin{cases} \exp\left\{\frac{(s\delta_2)^3}{6}\right\}, & s > 0 \\ \exp\left\{-\frac{(s\delta_1)^3}{6}\right\}, & s < 0. \end{cases}$$

It is easy to see that

$$L(s, \delta_1, \delta_2) R(s, \delta_1, \delta_2) \le R(s) \le R(|s|). \tag{5}$$

The assertion of *Lemma 1.* follows from (4), (5), *Remark1.* and the Fatou lemma. □

Lemma 2. *The space* $\mathrm{SG}_\Xi(\Omega)$ *is the subspace of the Orlicz space $L_U(\Omega)$ and of (*$\mathrm{Prg}(\Omega)$*), and for all $\eta_i \in$* $\mathrm{SG}_\Xi(\Omega)$*, $\lambda_i \in \mathbb{R}^1$, $i = 1, \ldots, n$, the inequality*

$$\left\| \sum_{i=1}^n \eta_i \lambda_i \right\| \le c_1 \left(\operatorname{Var} \left(\sum_{i=1}^n \eta_i \lambda_i \right) \right)^{\frac{1}{2}} \tag{6}$$

holds, where $c_1 = \frac{\sqrt{2}}{c_0}$, c_0 is the root of the equation $R(s) = 2$.

Lemma 2. follows from *Lemma 1.*

Lemma 3. *Let $\vec{\xi}^T = (\xi_1, \xi_2, \ldots, \xi_d)$ be a random vector, such that $\xi_i \in$* $\mathrm{SG}_\Xi(\Omega)$*, and let A be a symmetric positive definite matrix. Then for all $0 < s < \frac{1}{\sqrt{2}}$ the following inequality*

$$EG\left(\frac{s^2 \vec{\xi}^T A \vec{\xi}}{E\left(\vec{\xi}^T A \vec{\xi}\right)} \right) \le R(\sqrt{2}s), \tag{7}$$

holds, where $R(s)$ is defined in (3), $G(x) = \frac{\sinh(\sqrt{x})}{\sqrt{x}}$, $x > 0$.

Proof. Let us at first prove this lemma for $A = I$, where I is the identity matrix and for $\vec{\xi}$ such, that ξ_i are orthogonal $\big(\operatorname{Var}\eta = \operatorname{Var}\big(\sum_{i=1}^d \lambda_i\xi_i\big) = \sum_{i=1}^d \lambda_i^2 E\xi_i^2\big)$. Set $\sigma_i^2 = E\xi_i^2$, $i = 1, \dots, d$. In this case, from (3) (for $|s| < 1$) it follows that for all $\lambda_i \in \mathbb{R}$, $i = 1, 2, \dots, d$, the inequality

$$E\exp\left\{\frac{s\sum_{i=1}^d \lambda_i\xi_i}{\sqrt{2}\big(\sum_{i=1}^d \lambda_i^2\sigma_i^2\big)^{\frac{1}{2}}}\right\} \le R(|s|) \tag{8}$$

holds. Set $u = \frac{s}{\sqrt{2}\sqrt{\sum_{i=1}^d \lambda_i^2\sigma_i^2}}$. It follows from (8), that for $|u| < \big(2\sum_{i=1}^d \lambda_i^2\sigma_i^2\big)^{-\frac{1}{2}}$ the inequality

$$E\exp\left\{u\sum_{i=1}^d \lambda_i\xi_i\right\} \le R\left(\sqrt{2}|u|\sqrt{\sum_{i=1}^d \lambda_i^2\sigma_i^2}\right) \tag{9}$$

holds. Denote $s_i = u\lambda_i\sigma_i$. Then

$$\sum_{i=1}^d s_i^2 = u^2\sum_{i=1}^d \lambda_i^2\sigma_i^2 = \frac{s^2}{2},$$

therefore $\sum_{i=1}^d s_i^2 < \frac{1}{2}$.

It follows from (9), that for all s_i such that $\sum_{i=1}^d s_i^2 < \frac{1}{2}$ the next inequality holds:

$$E\exp\left\{\sum_{i=1}^d s_i\frac{\xi_i}{\sigma_i}\right\} \le R\left(\sqrt{2\sum_{i=1}^d s_i^2}\right). \tag{10}$$

From (10), we have that for all $\alpha_i > 0$, $\sum_{i=1}^d \alpha_i^2 < \frac{1}{2}$,

$$\begin{aligned} &E\int_{-\alpha_1}^{\alpha_1}\cdots\int_{-\alpha_d}^{\alpha_d}\exp\left\{\sum_{i=1}^d s_i\frac{\xi_i}{\sigma_i}\right\}ds_1\dots ds_d \\ &\le \int_{-\alpha_1}^{\alpha_1}\cdots\int_{-\alpha_d}^{\alpha_d} R\left(\sqrt{2\sum_{i=1}^d s_i^2}\right)ds_1\dots ds_d \le 2^d\prod_{i=1}^d \alpha_i R\left(\sqrt{2\sum_{i=1}^d \alpha_i^2}\right), \end{aligned} \tag{11}$$

$$\begin{aligned} &\int_{-\alpha_1}^{\alpha_1}\cdots\int_{-\alpha_d}^{\alpha_d}\exp\left\{\sum_{i=1}^d s_i\frac{\xi_i}{\sigma_i}\right\}ds_1\dots ds_d = \prod_{i=1}^d\int_{-\alpha_i}^{\alpha_i}\exp\left\{s_i\frac{\xi_i}{\sigma_i}\right\}ds_i \\ &= 2^d\prod_{i=1}^d\frac{\sinh\big(\frac{\alpha_i\xi_i}{\sigma_i}\big)}{\frac{|\xi_i|}{\sigma_i}} = 2^d\prod_{i=1}^d\alpha_i\cdot\prod_{i=1}^d\frac{\sinh\big(\frac{\alpha_i|\xi_i|}{\sigma_i}\big)}{\frac{\alpha_i|\xi_i|}{\sigma_i}}. \end{aligned} \tag{12}$$

It follows from (11) and (12), that

$$E\prod_{i=1}^d\frac{\sinh\big(\frac{\alpha_i|\xi_i|}{\sigma_i}\big)}{\frac{\alpha_i|\xi_i|}{\sigma_i}} \le R\left(\sqrt{2\sum_{i=1}^d \alpha_i^2}\right). \tag{13}$$

Denote $g(x)=\frac{\sinh\sqrt{x}}{\sqrt{x}}$. The function $g(x)$ is such that $g(0)=1$, and the function $f(x)=\ln g(x)$ is concave ($f(0)=0$). That is why for any $x_i>0$, $i=1,\dots,n$, it holds

$$\sum_{i=1}^{n} f(x_i)\geq f\biggl(\sum_{i=1}^{n} x_i\biggr).$$

That means that

$$\prod_{i=1}^{n} g(x_i)\geq g\biggl(\sum_{i=1}^{n} x_i\biggr).$$

That is why

$$E\prod_{i=1}^{d}\frac{\sinh\bigl(\frac{\alpha_i|\xi_i|}{\sigma_i}\bigr)}{\frac{\alpha_i|\xi_i|}{\sigma_i}}=E\prod_{i=1}^{d} g\biggl(\frac{\alpha_i^2\xi_i^2}{\sigma_i^2}\biggr)\geq Eg\biggl(\sum_{i=1}^{d}\frac{\alpha_i^2\xi_i^2}{\sigma_i^2}\biggr). \tag{14}$$

It follows from (13) and (14), that

$$Eg\biggl(\sum_{i=1}^{d}\frac{\alpha_i^2\xi_i^2}{\sigma_i^2}\biggr)\leq R\biggl(\sqrt{2\sum_{i=1}^{d}\alpha_i^2}\biggr). \tag{15}$$

If we set $\alpha_i^2=\frac{\sigma_i^2 s^2}{\sum_{i=1}^{d}\sigma_i^2}$, then from (15) we have

$$Eg\biggl(\frac{s^2\sum_{i=1}^{d}\xi_i^2}{\sum_{i=1}^{d}\sigma_i^2}\biggr)\leq R\bigl(\sqrt{2}s\bigr), \tag{16}$$

which is (7) for this case.

Consider now a general case. Let B be a matrix such that $BB=A$, $R=\operatorname{cov}\vec{\xi}$. Let O be the orthogonal matrix, that brings the matrix BRB in the diagonal form

$$OBRBO^T=D=\operatorname{diag}\bigl(d_k^2\bigr)_{k=1}^{d}.$$

Set $\vec{\eta}=OB\vec{\xi}$. Hence,

$$\vec{\eta}^T\vec{\eta}=\vec{\xi}^T BO^T OB\vec{\xi}=\vec{\xi}^T A\vec{\xi},$$

$\operatorname{cov}\vec{\eta}=OB\operatorname{cov}\xi BO^T=D$. It is easy to see that $\eta_i\in\mathrm{SG}_\Xi(\Omega)$, $\vec{\eta}^T=(\eta_1,\dots,\eta_d)$. Therefore, the inequality (16) holds for $\vec{\eta}$. So $\vec{\eta}^T\vec{\eta}=\sum_{i=1}^{d}\eta_i^2=\vec{\xi}A\vec{\xi}$, therefore

$$g\biggl(\frac{s^2\vec{\eta}^T\vec{\eta}}{E\vec{\eta}^T\vec{\eta}}\biggr)=g\biggl(\frac{s^2\vec{\xi}^T A\vec{\xi}}{E\vec{\xi}^T A\vec{\xi}}\biggr).$$

The lemma is proved. □

Lemma 4. *If the assumptions of* Lemma 3. *are satisfied, then for $x>2$ it holds*

$$\Pr\biggl\{\frac{\eta}{E\eta}>x\biggr\}\leq\biggl(\frac{e}{4}\biggr)^{\frac{1}{8}}\frac{x^{\frac{1}{4}}\bigl(\bigl(\frac{x}{2}\bigr)^{\frac{1}{2}}-1\bigr)}{\sinh\bigl(\bigl(\frac{x}{2}\bigr)^{\frac{1}{2}}-1\bigr)}=W_1(x), \tag{17}$$

where $\eta=\vec{\xi}^T A\vec{\xi}$.

Proof. It follows from the Chebyshev inequality and (7), that for $x > 0$, $0 < t < \frac{1}{\sqrt{2}}$,

$$\Pr\left\{\frac{\eta}{E\eta} > x\right\} \le \frac{Eg\left(t^2 \frac{\eta}{E\eta}\right)}{g(t^2 x)} \le \frac{R(\sqrt{2}t)}{g(t^2 x)},$$

set $t = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{x}}$, $x > 2$. *Lemma 4.* follows from the relations

$$g(t^2 x) = \frac{\sinh\left(\left(\frac{x}{2}\right)^{\frac{1}{2}} - 1\right)}{\left(\left(\frac{x}{2}\right)^{\frac{1}{2}} - 1\right)},$$

$$R(\sqrt{2}t) = \exp\left\{\frac{1}{8}\right\} \exp\left\{-\frac{1}{2} + \frac{1}{\sqrt{2x}}\right\} \left(\frac{x}{2}\right)^{\frac{1}{4}} \le \exp\left\{\frac{1}{8}\right\} \left(\frac{x}{2}\right)^{\frac{1}{4}}.$$

□

4. Examples of the application of inequality (17)

In this section we consider some examples of application of the inequality (17).

Example 1. *Let $\xi_k = \{\xi_k(t), t \in [0, 2T],\ k = 1, \dots, n\}$ be jointly Gaussian and jointly stationary stochastic processes with $E\xi_k(t) = 0$ and $E\xi_k(t+\tau)\xi_l(t) = r_{kl}(\tau)$.*

Let

$$\hat{r}_{kl}(\tau) = \frac{1}{T} \int_0^T \xi_k(t+\tau)\xi_l(t)\, dt, \qquad 0 \le t \le T$$

be the estimators of covariance functions $r_{kl}(\tau)$. Set $X_{kl}(\tau) = \hat{r}_{kl}(\tau) - r_{kl}(\tau)$. It is easy to see, that $X_{kl}(\tau)$, $0 \le \tau \le T$, $k, l = 1, 2, \dots, n$, belong to $\mathrm{SG}_{\Xi}(\Omega)$, where $\Xi = \{\xi_k(t), k = 1, \dots, n, 0 \le t \le 2T\}$.

Let A be a symmetric positive semi-definite matrix, $\eta(\tau) = \vec{X}^T(\tau) A \vec{X}(\tau)$, where $\vec{X}(\tau)$ is the vector with components $X_{kl}(\tau)$. It follows from Lemma 4.*, that for $x > 2$ the following inequality holds:*

$$\Pr\left\{\frac{\eta(\tau)}{E\eta(\tau)} > x\right\} \le W(x), \tag{18}$$

where $W(x)$ is defined in (17).

This inequality enables us to construct confidence ellipsoids for $r_{kl}(\tau)$.

For example, let $n = 2$ and $\vec{X}^T(\tau) = (X_{11}(\tau), X_{22}(\tau))$, $B(\tau) > 0$ be the covariance matrix of $\vec{X}(\tau)$. It is easy to prove that the confidence ellipsoid with the minimal area is the ellipsoid

$$\frac{\vec{X}^T(\tau) A \vec{X}(\tau)}{E\vec{X}^T(\tau) A \vec{X}(\tau)} = \delta,$$

where $A = B^{-1}(\tau)$. In this case

$$\vec{X}^T(\tau) A \vec{X}(\tau) = \vec{X}^T(\tau) B^{-1}(\tau) \vec{X}(\tau),$$

$$E\vec{X}^T(\tau) A \vec{X}(\tau) = Sp(BA) = Sp(BB^{-1}) = 2,$$

$$b_{ij}(\tau) = \frac{2}{T^2} \int_0^T (T-u)(r_{ij}^2(u) + r_{ij}(u-\tau) r_{ij}(u+\tau))\, du, \quad i, j = 1, 2.$$

Example 2. *Let $\xi = \{\xi(t), t \in [0, 2T]\}$ be a Gaussian stationary stochastic process with $E\xi(t) = m$ and $E(\xi(t+\tau) - m)(\xi(t) - m) = r(\tau)$. Let*

$$\hat{m}_0 = \frac{1}{T}\int_0^T \xi(t)\,dt, \qquad \hat{m}_\tau = \frac{1}{T}\int_0^T \xi(t+\tau)\,dt,$$

$$\hat{r}_\tau = \frac{1}{T}\int_0^T (\xi(t+\tau) - \hat{m}_\tau)(\xi(t) - \hat{m}_0)\,dt, \quad 0 \le \tau \le T,$$

be the estimators of covariance function $r(\tau)$ and the expectation m.

Set

$$\eta_1 = \hat{r}(\tau) + \frac{2}{T^2}\int_0^T (T-u) r(u+\tau)\,du,$$

$$\eta_2 = (\hat{m}_0 - m)^2 - \frac{2}{T^2}\int_0^T (T-u) r(u)\,du.$$

It is sufficiently easy to prove, that $\eta_i \in \mathrm{SG}_\Xi(\Omega)$, $i = 1, 2$, where $\Xi = \{\xi(t), t \in [0, 2T]\}$.

Let $\eta = a_{11}\eta_1^2 + a_{22}\eta_2^2 + 2a_{12}\eta_1\eta_2$, where $A = |a_{ij}|_{i,j=1}^2$ is a symmetric positive semi-definite matrix. It follows from Lemma 4.*, that for $x > 2$ the inequality holds*

$$\Pr\left\{\frac{\eta}{E\eta} > x\right\} \le W(x), \tag{19}$$

where $W(x)$ is defined in (17). This inequality enables us to construct confidence ellipsoids for $(r(\tau), m)$.

5. Random processes from $L_U(\Omega)$ and $\mathrm{SG}_\Xi(\Omega)$ spaces

Let (T, ρ) be a metric space, $\mu(\cdot)$ be a Borel measure in (T, ρ), $\mu(T) < \infty$.

Definition 5. *A random process $X = (X(t), t \in T)$ belongs to the space $L_U(\Omega)$, if the random variable $X(t) \in L_U(\Omega)$ for all $t \in T$ and $\sup_{t \in T}\|X(t)\| < \infty$ ($\|\cdot\|$ is the Luxemburg norm).*

Definition 6. *A random process $X = (X(t), t \in T)$ is called square–Gaussian, if for all $t \in T$ $X(t)$ it belongs to a space $\mathrm{SG}_\Xi(\Omega)$ and $\sup_{t\in T}\big(E(X(t))^2\big)^{\frac{1}{2}} < \infty$.*

Lemma 5. *A square–Gaussian random process $X = (X(t), t \in T)$ belongs to the space $L_U(\Omega)$, and for all $t_i \in T$, $i = 1, 2, \ldots, n$ the following inequality*

$$\left\|\sum_{i=1}^n \lambda_i X(t_i)\right\| \le c_1\left(\mathrm{Var}\sum_{i=1}^n \lambda_i X(t_i)\right)^{\frac{1}{2}} \tag{20}$$

holds, where c_1 is defined in (6).

Lemma 5. follows from *Lemma 2.*

Theorem 1. ([1, 4]) *Let $X = (X(t), t \in T)$ be a separable random process, $X \in L_U(\Omega)$, ε_k, $k = 1, 2, \dots$ be a monotonically decreasing sequence such that $\varepsilon_1 = \sup_{t,s\in T} \rho(t,s)$, $\varepsilon_k \to 0$, as $k \to \infty$, $B(t, \varepsilon_k)$ be an open ball with the centre t and the radius ε_k, $\mu_k(t) = \mu(B(t, \varepsilon_k))$,*

$$\sigma_k(t) = \sup_{u\in B(t,\varepsilon_k)} \|X(u) - X(t)\|, \quad \sup_{t\in T} \sum_{l=1}^{\infty} \sigma_l(t) \ln\left(\frac{1}{\mu_{l+1}(t)} + 1\right) < \infty.$$

Then the following inequality

$$\sup_{t\in T} |X(t) - \int_S X(u) \frac{d\mu(u)}{\mu(T)}| \le \eta R \tag{21}$$

holds, where

$$R = 4 \sup_{t\in T} \sum_{l=1}^{\infty} \sigma_l(t) \ln\left(\frac{1}{\mu_{l+1}(t)} + 1\right),$$

and $\eta > 0$ is a random variable such that for $x > 1$

$$\Pr\{\eta > x\} \le \frac{2e}{e-1} \frac{(\mu(T))^{2x}}{(1 + \mu^2(T))^x}. \tag{22}$$

From the *Theorem 1.* there follows the following theorem.

Theorem 2. *Let $T = [0, T]$, $\rho(t, s) = |t - s|$, $\mu(\cdot)$ be the Lebesque measure, $X = (X(t), t \in T)$ be a separable random process from $\mathrm{SG}_\Xi(\Omega)$,*

$$\sigma(h) = c_1 \sup_{|t-s|\le h} \big(E|X(t) - X(s)|^2\big)^{\frac{1}{2}},$$

where c_1 is defined in (6).

Then for all $0 < p < 1$

$$\sup_{t\in[0,T]} |X(t) - \frac{1}{T} \int_0^T X(u)\, du| \le \eta_1 R_p, \tag{23}$$

where

$$R_p = \frac{4}{p(1-p)} \int_0^{p\sigma(T)} \ln\left(\frac{1}{2\sigma^{(-1)}(u)} + 1\right) du,$$

and $\eta_1 > 0$ is a random variable such that for $x > 1$

$$\Pr\{\eta_1 > x\} \le \frac{2e}{(e-1)} \left(\frac{T^2}{1 + T^2}\right)^x. \tag{24}$$

Proof. In this case $\varepsilon_1 = T$, $\mu_k(t) = \min(2\varepsilon_k, T)$,

$$\int_T X(u) \frac{d\mu(u)}{\mu(T)} = \frac{1}{T} \int_0^T X(u)\, du,$$

$$\|X(t) - X(s)\| \le c_1 \big(E|X(t) - X(s)|^2\big)^{\frac{1}{2}}, \quad \text{(Lemma 2.)},$$

$$\sigma_k(t) \le c_1 \sup_{|t-s|<\varepsilon_k} \left(E|X(t)-X(s)|^2\right)^{\frac{1}{2}} \le \sigma(\varepsilon_k).$$

Therefore

$$\sum_{l=1}^{\infty} \sigma_l(t) \ln\left(\frac{1}{\mu_{l+1}(t)}+1\right) \le \sum_{l=1}^{\infty} \sigma_(\varepsilon_l) \ln\left(\frac{1}{\min(2\varepsilon_{l+1},T)}+1\right).$$

Now we choose the sequence ε_k such that

$$\sigma_(\varepsilon_k) = p^{k-1}\delta_1,\ 0<p<1,\ (\varepsilon_k = \sigma^{(-1)}(p^{k-1}\delta_1)),\ \delta_1 = \sigma_(\varepsilon_1) = \sigma(T).$$

Therefore,

$$\begin{aligned} \textstyle\sum_{l=1}^{\infty} \sigma_l(t) \ln\left(\frac{1}{\mu_{l+1}(t)}+1\right) \le & \textstyle\sum_{l=1}^{\infty} \delta_1 p^{l-1} \ln\left(\frac{1}{2\sigma^{(-1)}(p^l\delta_1)}+1\right) \\ \le & \textstyle\sum_{l=1}^{\infty} \frac{1}{p(1-p)} \int_{p^{l+1}\delta_1}^{p^l\delta_1} \ln\left(\frac{1}{2\sigma^{(-1)}(u)}+1\right) du \\ = & \textstyle\frac{1}{p(1-p)} \int_0^{p\sigma(T)} \ln\left(\frac{1}{2\sigma^{(-1)}(u)}+1\right) du. \end{aligned}$$

From this inequality and (21) there follows the inequality (23). □

6. Distribution of suprema of quadratic forms from random processes from $\mathrm{SG}_\Xi(\Omega)$

Let $X_i = \{X_i(t), t \in [0,T]\}$, $i = 1,\dots,m$ be a separable random process from $\mathrm{SG}_\Xi(\Omega)$, and let V be a symmetric positive semi-definite matrix,

$$Y(t) = \vec{X}^T(t)V\vec{X}(t) = (V\vec{X}(t),\vec{X}(t)),$$

where $\vec{X}^T(t) = (X_1(t), X_2(t),\dots,X_m(t))$.

Let S be the ortogonal matrix which reduces the matrix V to the diagonal form.

$$PVP^T = D = \|d_i^2\delta_{ij}\|_{i,j=1}^m,$$

(δ_{ij} is the Kronecker delta), $\vec{Z}(t) = P\vec{X}(t)$, $\vec{Z}^T(t) = (Z_1(t), Z_2(t),\dots,Z_m(t))$.

$$\sigma_i(t) = c_1 \sup_{|t-s|<h} \left(E|Z_i(t)-Z_i(s)|^2\right)^{\frac{1}{2}}, \tag{25}$$

where c_1 is defined in (6).

Theorem 3. *If for any $0<p<1$, $i=1,2,\dots,m$,*

$$R_{ip} = \frac{4}{p(1-p)} \int_0^{p\sigma_i(T)} \ln\left(\frac{1}{2\sigma_i^{(-1)}(u)}+1\right) du < \infty, \tag{26}$$

then for any $o<\alpha<1$,

$$x > \max\left(\frac{\delta_2}{(1-\alpha)^2}, \frac{2\delta_1}{\alpha^2}\right)$$

the following inequality

$$\Pr\Big\{\sup_{0\le t\le T} Y(t) > x\Big\} \le W_1\left(\frac{x\alpha^2}{\delta_1}\right) + mW_2\left(\frac{x^{\frac12}(1-\alpha)}{\delta_2^{\frac12}}\right) \tag{27}$$

holds, where

$$\begin{aligned} W_1(x) &= \left(\tfrac{e}{4}\right)^{\frac18} \frac{x^{\frac14}\left(\sqrt{\frac{x}{2}}-1\right)}{\sinh\left(\sqrt{\frac{x}{2}}-1\right)}, \\ W_2(x) &= \tfrac{2e}{e-1}\left(\tfrac{T^2}{1+T^2}\right)^x, \\ \delta_1 &= \tfrac{1}{T^2}\int_0^T\int_0^T E(V\vec{X}(t), X(u))\,dt\,du, \\ \delta_2 &= \textstyle\sum_{j=1}^m d_j^2 R_{jp}^2. \end{aligned}$$

Proof. It is easy to see that

$$\begin{aligned} Y(t) &= (V\vec{X}(t), \vec{X}(t)) = (S^T DS\vec{X}(t), \vec{X}(t)) = (DS\vec{X}(t), S\vec{X}(t)) = (D\vec{Z}(t), \vec{Z}(t)) \\ &= \sum_{i=1}^m d_i^2 Z_i^2(t) = \sum_{i=1}^m d_i^2(\theta_i + \eta_i(t))^2, \end{aligned} \tag{28}$$

where

$$\theta_i = \frac{1}{T}\int_0^T Z_i(t)\,dt, \qquad \eta_i(t) = Z_i(t) - \theta_i.$$

From (28) the following inequality follows

$$Y^{\frac12}(t) \le \left(\sum_{i=1}^m d_i^2\theta_i^2\right)^{\frac12} + \left(\sum_{i=1}^m d_i^2\eta_i^2(t)\right)^{\frac12}. \tag{29}$$

It follows from *Theorem 2.* ($\eta_i(t) \in \mathrm{SG}_\Xi(\Omega)$), that

$$\sup_{0\le t\le T} |\eta_i(t)| \le \eta_i R_{ip}, \tag{30}$$

where $\eta_i \ge 0$, $i = 1, 2, \ldots, m$ are random variables such, that for $x > 1$

$$\Pr\{\eta_i > x\} \le W_2(x). \tag{31}$$

The following equalities are evident

$$\begin{aligned} \sum_{i=1}^m d_i^2\theta_i^2 &= \textstyle\sum_{i=1}^m d_i^2 \frac{1}{T}\int_0^T Z_i(t)\,dt\,\frac{1}{T}\int_0^T Z_i(u)\,du \\ &= \tfrac{1}{T^2}\int_0^T\int_0^T (D\vec{Z}(t), \vec{Z}(u))\,dt\,du \\ &= \tfrac{1}{T^2}\int_0^T\int_0^T (V\vec{X}(t), \vec{X}(u))\,dt\,du. \end{aligned} \tag{32}$$

It follows from (32), (30) and (29), that

$$\Big(\sup_{0\le t\le T} Y(t)\Big)^{\frac12} \le \left(\frac{1}{T^2}\int_0^T\int_0^T (V\vec{X}(t), \vec{X}(u))\,dt\,du\right)^{\frac12} + \left(\sum_{i=1}^m d_i^2 R_{ip}^2\eta_i^2\right)^{\frac12}.$$

Therefore, for any $0 \le \alpha \le 1$, $x > 0$, the following inequalities hold:

$$\begin{aligned} &\Pr\{\sup_{0\le t\le T} Y(t) > x\} = \Pr\{(\sup_{0\le t\le T} Y(t))^{\frac12} > x^{\frac12}\} \\ &\le \Pr\left\{\left(\tfrac{1}{T^2}\int_0^T\int_0^T \big(V\vec{X}(t), \vec{X}(u)\big)\,dt\,du\right)^{\frac12} > \alpha x^{\frac12}\right\} \\ &\quad + \Pr\left\{\left(\textstyle\sum_{i=1}^m d_i^2 R_{ip}^2 \eta_i^2\right)^{\frac12} > (1-\alpha)x^{\frac12}\right\} = A_1 + A_2. \end{aligned} \tag{33}$$

For any $\alpha_i > 0$, $i = 1, 2, \ldots, m$, $\sum_{i=1}^m \alpha_i = 1$ it holds

$$\begin{aligned} A_2 &= \Pr\left\{\textstyle\sum_{i=1}^m d_i^2 R_{ip}^2 \eta_i^2 > (1-\alpha)^2 x\right\} \le \textstyle\sum_{i=1}^m \Pr\{d_i^2 R_{ip}^2 \eta_i^2 > \alpha_i (1-\alpha)^2 x\} \\ &= \textstyle\sum_{i=1}^m \Pr\left\{\eta_i > \frac{(1-\alpha)\alpha_i^{\frac12} x^{\frac12}}{d_i R_{ip}}\right\}. \end{aligned} \tag{34}$$

Set

$$\alpha_i = \frac{d_i^2 R_{ip}^2}{\sum_{j=1}^m d_j^2 R_{jp}^2}.$$

It follows from (34), that for $x > \frac{\delta_2}{(1-\alpha)^2}$

$$A_2 \le \sum_{i=1}^m \Pr\left\{\eta_i > \frac{(1-\alpha)^{\frac12} x^{\frac12}}{\delta_2^{\frac12}}\right\} \le m W_2\left(\frac{x^{\frac12}(1-\alpha)}{\delta_2^{\frac12}}\right). \tag{35}$$

It follows from (32) and the *Lemma 4.* ($\theta_i \in \mathrm{SG}_\Xi(\Omega)$) that, for $x > \frac{2\delta_1}{\alpha^2}$,

$$A_1 \le W_1\left(\frac{\alpha^2 x}{\delta_1}\right). \tag{36}$$

The inequality (27) follows from (35) and (36). □

Corollary 1. *Let a process $Y = \{Y(t), t \in [0,T]\}$ satisfy the assumptions of* Theorem 3. *Then for $x > \Delta_T^2 \max\big(1, \ln^2\big(1 + \frac{1}{T^2}\big)\big)$ it holds*

$$\Pr\Big\{\sup_{0\le t\le T} Y(t) > x\Big\} \le \big(\Delta_1 x^{\frac34} - \Delta_2 x^{\frac14} + \Delta_3\big)\exp\left\{-\frac{x^{\frac12}}{\Delta_T}\right\}, \tag{37}$$

where

$$\Delta_T = \delta_1^{\frac12}\sqrt{2} + \frac{\delta_2^{\frac12}}{\ln\big(\frac{1+T^2}{T^2}\big)}, \quad \Delta_1 = e^{\frac98} 2\Delta_T^{-\frac32}, \quad \Delta_2 = e^{\frac98}\Delta_T^{-\frac12}, \quad \Delta_3 = m\frac{2e}{e-1},$$

and δ_1, δ_2 are defined in (27).

Proof. It is easy to prove that for $x > 0$, $\frac{x}{\sinh x} \le \frac{2x+1}{e^x}$; therefore, it follows from (27) that for $x > \max\big(\frac{\delta_2}{(1-\alpha)^2}, \frac{2\delta_1}{\alpha^2}\big)$

$$\begin{aligned} \Pr\{\sup_{0\le t\le T} Y(t) > x\} \;\le\; & e\big(\tfrac{e}{4}\big)^{\frac18}\left(\frac{x^{\frac12}\alpha}{\delta_1^{\frac12}}\right)^{\frac12}\left(\sqrt{2}\frac{x^{\frac12}\alpha}{\delta_1^{\frac12}} - 1\right)\exp\left\{-\big(\tfrac{x}{2\delta_1}\big)^{\frac12}\alpha\right\} \\ & + m\tfrac{2e}{e-1}\exp\left\{-\ln\big(\tfrac{1+T^2}{T^2}\big)\big(\tfrac{x}{\delta_2}\big)^{\frac12}(1-\alpha)\right\}. \end{aligned} \tag{38}$$

Now (37) follows from (38) if we set $\alpha = \frac{(2\delta_1)^{1/2}}{\Delta_T}$. □

Example 3. *Let $\eta(t)$ be a random process from* Example 1. *The inequality (37) holds for $\eta(\tau)$. These inequalities enable us to construct uniform confidence ellipsoids for $r_{kl}(\tau)$, $0 \le \tau \le T$.*

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