Compact operators, the essential spectrum and the essential numerical range*

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Abstract. *Some properties of bounded operators on Hilbert space concerned with matrix representations in orthonormal bases are presented. In particular, the classes of operators with columns or diagonals converging to* 0 *are described.*

Key words: *Hilbert space, orthonormal basis, bounded operator, compact operator, essential spectrum, essential numerical range*

Sažetak. Kompaktni operatori, esencijalni spektar i esencijalni numerički rang. *U radu su izložena neka svojstva ograničenih operatora na Hilbertovom prostoru povezana s njihovim matričnim reprezentacijama u ortonormiranim bazama. Posebno, opisane su klase operatora u čijim matričnim prikazima stupci, odnosno dijagonale, konvergiraju u* 0*.*

Ključne riječi: *Hilbertov prostor, ortonormirana baza, ograničen operator, kompaktan operator, esencijalni spektar, esencijalni numerički rang*

1. Introduction

Let H be a complex infinite dimensional separable Hilbert space with the scalar product $(\cdot|\cdot)$. We denote by $B(H)$ and $K(H)$ the algebra of all bounded operators and the ideal of all compact operators on H, respectively. The quotient map from $B(H)$ onto the Calkin algebra $C(H) = B(H)/K(H)$ is denoted by π.

In this note several properties of bounded operators are discussed. Although most of the presented material is known, various results are collected and reinterpreted, which, we hope, may help to clarify the connections between some standard notions from the operator theory. Also, some proofs are new or simplified.

2. Compact operators

A bounded operator on H is compact if it maps a unit ball of H into the relatively compact set. Equivalently, $A \in B(H)$ is compact if and only if it satisfies

*The lecture presented at the MATHEMATICAL COLLOQUIUM in Osijek organized by Croatian Mathematical Society – Division Osijek, November 7, 1997.

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$\lim_n Ax_n = 0$ for each sequence (x_n) in H weakly converging to 0. Let us note that the later condition does not imply $w - s$ continuity of A. Indeed, the set of all operators $A : H \to H$ with the property that the inverse image under A of each (norm) open set is open in the weak operator topology, coincides with the ideal of finite rank operators ([11], Problem 130).

Since orthonormal sequences weakly converge to 0 we conclude: if $A \in B(H)$ is a compact operator and (e_n) is an orthonormal sequence in H, then $\lim_n Ae_n = 0$. Moreover, the converse is also true:

Theorem 1. *A bounded operator $A \in B(H)$ is compact if and only if it satisfies $\lim_n Ae_n = 0$ for each orthonormal sequence (e_n) in H.*

Theorem 1 is well known. Originally it was proved in [9] and, independently, in [15] (see also [13]). While the argument in [9] depends on the analysis of operator ranges, the proof in [15] uses Zorn's lemma. On the other hand, *Theorem 1* is reproved in [2] by elementary means. The argument is based on the simple geometrical property of the underlying Hilbert space ([12], p. 300): every sequence of unit vectors weakly converging to 0 contains a subsequence that is near to an orthonormal sequence. Also, it is observed in [2] that even a linear transformation $A : H \to H$ satisfying the condition in the above theorem must be a compact operator, i. e. the continuity assumption on A is superfluous.

Since we restrict ourselves to separable Hilbert spaces, one can restate *Theorem 1* in terms of orthonormal bases:

Corollary 1. *A bounded operator $A \in B(H)$ is compact if and only if satisfies $\lim_n Ae_n = 0$ for each orthonormal basis for H.*

3. The essential spectrum

A natural question is arising from Corollary 1: is it enough to require $\lim_n Ae_n = 0$ for **some** orthonormal basis in order to conclude $A \in K(H)$? The question is strongly suggested by the class of Hilbert-Schmidt operators. Namely, if there exists an orthonormal basis (e_n) for H such that $\sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty$ then $\sum_{n=1}^{\infty} \|Ae_n\|^2 = \sum_{n=1}^{\infty} \|Af_n\|^2 < \infty$ for each orthonormal basis (f_n).

Let us take a bounded operator $A \in B(H)$ such that there exists an orthonormal basis (e_n) for H with the property $\lim_n Ae_n = 0$.

First we note that the operator A^*A satisfies the same condition. Since A is compact precisely when A^*A is compact, we may suppose without loss of generality that A is a self-adjoint operator.

Now we look at $\pi(A) \in C(H)$. In order to prove that A is compact it is enough to conclude $\pi(A) = 0$. We note that $\pi(A)$ is a self-adjoint element of C^*-algebra $C(H)$, so its norm and spectral radius coincide. Therefore, one should prove that the essential spectrum $\sigma_e(A)$ contains 0 as its only point.

Let $\lambda \in \sigma_e(A)$, $\lambda \neq 0$. This means that the operator $A - \lambda I$ does not have the essential inverse (i. e. $\pi(A)$ is not invertible in $C(H)$). Now we observe that λ, belonging to the spectrum of the self-adjoint element, must be a real number; therefore, the operator $A - \lambda I$ is also self-adjoint. It follows that $A - \lambda I$ does not have either the left essential inverse. Precisely: there is no bounded operator $T \in B(H)$ such that $T(A-\lambda I)-I$ is compact. Then by Theorem 1.1 from [8] there exist an infinite dimensional orthogonal projection $P \in B(H)$ and a compact operator $K \in K(H)$ such that $P(A-\lambda I)=K$. Writing $P=\frac{1}{\lambda}(PA-K)$ we find $\lim_n Pe_n=0$ where (e_n) is the orthonormal basis from assumption on A.

After all, our initial question is reduced to the following: Let $P \in B(H)$ be an infinite dimensional (hence non-compact) orthogonal projection. Is it possible to find an orthonormal basis (a_n) for H such that $\lim_n Pa_n=0$?

According to [2], the answer is: yes, if P has an infinite dimensional kernel. To see this, we will briefly repeat the arguments from [2]. First, we exibit a useful example:

Let (b_n) be an orthonormal basis for H and let Q be a linear transformation on H whose matrix with respect to (b_n) is

$$Q=\begin{bmatrix} 1 & 0 & 0 & \dots \\ 0 & \begin{matrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{matrix} & 0 & \dots \\ 0 & 0 & \begin{matrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{matrix} & \\ \vdots & \vdots & & \ddots \end{bmatrix}.$$

It is easy to check that the above matrix defines a bounded operator $Q \in B(H)$ such that $Q^2=Q=Q^*$, so Q is an orthogonal projection. Further, Q is infinite dimensional since it consists of infinitely many one-dimensional projections. Obviously, $\lim_n Qb_n=0$.

Let us mention that the above projection was obtained from the construction similar to that used in [10] for a simple proof of Hummel's theorem. Hummel's theorem states that for each compact operator $K \in K(H)$ and $\varepsilon>0$ there exists an orthonormal basis (e_n) for H such that $\|Ke_n\|<\varepsilon$, $\forall n$. We also refer to [3] where the whole class of operators $A \in B(H)$ satisfying Hummel's condition is detected: for each $\varepsilon>0$ there exists an orthonormal basis (e_n) for H such that $\|Ae_n\|<\varepsilon$, $\forall n$ if and only if the left essential spectrum of A contains 0.

Now let $P \in B(H)$ be an arbitrary infinite dimensional orthogonal projection with an infinite dimensional kernel. Let Q and (b_n) be as above. Clearly, there exists a unitary operator $V \in B(H)$ such that $P=VQV^*$. It remains to define the orthonormal basis (a_n), $a_n=Vb_n$. Obviously, $\lim_n Pa_n=0$.

As a result of the above considerations we see that also some non-compact operators are able to convert orthonormal bases into convergent sequences. Actually, ([2], Theorem 4), the description of all bounded operators with this property goes back to Theorem 1.1 from [8]:

Theorem 2. *A bounded operator $A \in B(H)$ possesses an orthonormal basis (a_n) for H such that $\lim_n A_n=0$ if and only if 0 belongs to the left essential spectrum of A.*

4. The essential numerical range

Corollary 1 can be reformulated in the following way:

Corollary 2. *A bounded operator $A \in L(H)$ is compact if and only if it satisfies $\lim_n (Ae_n|e_n) = 0$, for each orthonormal basis (e_n) for H.*

In fact, *Corollary 1* follows immediately from *Corollary 2*. Conversely, *Corollary 2* can be deduced from *Corollary 1* as in [1], Lemma 3. First, it is enough to consider positive operators A satisfying the condition from *Corollary 2* and then it remains to apply *Corollary 1* to $\sqrt{A}$.

Analogously to discussion in *Section 3*, one may try to find those operators $A \in B(H)$ which satisfy the condition related to that in *Corollary 2*:

There exists an orthonormal basis (e_n) for H such that $\lim\limits_n (Ae_n|e_n) = 0$. $(*)$

Apparently, the class of operators satisfying $(*)$ is much larger than the ideal of all compact operators; an obvious example is the unilateral shift. Also, there are unitary operators with this property. To provide an example we can consider the operator whose matrix corresponding to an orthogonal decomposition of H is $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$.

It turns out that the essential numerical range of an operator serves as the most convenient tool in order to determine the class of all operators satisfying $(*)$. The essential numerical range is introduced in [16]. We denote it by W_e. According to [8, Section 5], the essential numerical range of an operator $A \in B(H)$ is connected with the usual numerical range W in the following way:

$$W_e(A) = \bigcap\nolimits_{K \in K(H)} W(A+K)^-$$

(here "$^-$" denotes the topological closure in the complex plane).

Besides this, Theorem 5.1 from [8] contains more precise characterization of the essential numerical range. Along this line our condition $(*)$ can also be described ([2], Theorem 5]:

Theorem 3. *Let $A \in B(H)$ be a bounded operator on a separable Hilbert space H. Then there exists an orthonormal basis (e_n) for H such that $\lim_n (Ae_n|e_n) = 0$ if and only if 0 belongs to the essential numerical range of A.*

It is worth noting that the essential numerical range plays an important role in solving several problems from the operator theory. Let us mention few of them.

(1) We say that a bounded operator $A \in B(H)$ has "the small entry property" if for each $\varepsilon > 0$ there exists an orthonormal basis (e_n) for H such that $|(Ae_n|e_m)| < \varepsilon$, $\forall n, m$. The small entry property is discussed in [17] using the Schur product of bounded operators.

(2) A bounded operator $A \in B(H)$ is called zero-diagonal if there exists an orthonormal basis (e_n) for H such that $(Ae_n|e_n) = 0$, $\forall n$. Zero-diagonal operators are investigated in [6]. As the starting point for the analysis of zero-diagonal operators we recommend a result from [7]: an operator on finite dimensional Hilbert space has the matrix with zero-diagonal if and only if its numerical range contains 0.

(3) An operator $A \in B(H)$ (necessarily self-adjoint) is called a self-commutator if there is a bounded operator $B \in B(H)$ such that $A = B^*B - BB^*$. The class of self-commutators is described in [14]. For the related results one may also consult [4] and [5].

In the next theorem we will summarize the results from the above mentioned articles. Although the following list of mutually equivalent conditions is far from complete, it indicates the importance of the essential numerical range. In the same time, all conditions listed below may be regarded as the reformulations of our property $(*)$.

Theorem 4. *For an operator $A \in B(H)$ the following conditions are mutually equivalent:*

(a) *There exists an orthonormal basis (e_n) for H such that $\lim_n (Ae_n|e_n) = 0$.*

(b) $0 \in W_e(A)$.

(c) *There exists an orthonormal sequence (a_n) in H such that $\lim_n (Aa_n|a_n) = 0$.*

(d) *There exists a sequence of unit vectors (x_n) in H weakly converging to 0 such that $\lim_n (Ax_n|x_n) = 0$.*

(e) *There exists an orthogonal projection $P \in B(H)$ with an infinite dimensional range such that PAP is a compact operator.*

(f) *For each $\varepsilon > 0$ there exists an orthonormal basis (e_n) for H such that $|(Ae_n|e_m)| < \varepsilon$, $\forall n, m$.*

(g) *For each $\varepsilon > 0$ and $p > 1$ there exists an orthonormal basis (e_n) for H such that $\sum_{n=1}^{\infty} |(Ae_n|e_n)|^p < \varepsilon$.*

(h) *There exists a sequence of zero-diagonal operators (A_n) in $B(H)$ such that $A = (\text{norm})\ \lim_n A_n$.*

(i) *There exist a zero-diagonal operator $T \in B(H)$ and a compact operator $K \in K(H)$ such that $A = T + K$.*

(j) *(A self-adjoint) There exists an operator $B \in B(H)$ such that $A = B^*B - BB^*$.*

(k) *(A self-adjoint) The spectrum of A has at least one nonnegative limit point and at least one nonpositive limit point.*

Proof. (a) $\Leftrightarrow$ (b) is the assertion of *Theorem 3* above.
(b), (c), (d) and (e) are equivalent by [8], Theorem 5.1.
(e) $\Leftrightarrow$ (f) $\Leftrightarrow$ (g) is proved in [17].
(h) $\Leftrightarrow$ (b) is Theorem 3 from [6].
(i) $\Rightarrow$ (a) by *Corollary 2* and *Theorem 3* above.
(a) $\Rightarrow$ (i): Let us take the orthonormal basis from (a) and define $K \in B(H)$ by $Ke_n = (Ae_n|e_n)e_n$, $\forall n$. Since $(Ae_n|e_n) \to 0$, K is compact. Obviously, $T = A - K$ is zero-diagonal.
(j) $\Leftrightarrow$ (k) $\Leftrightarrow$ (b) is proved in [14], see also [6].

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