Representations of vertex algebras*

Dražen Adamović†

Abstract. *In this paper we present some results on the representation theory of vertex operator (super) algebras associated to affine Lie algebras and Neveu-Schwarz algebra.*

Key words: *vertex operator algebra, affine Lie algebra, Neveu-Schwarz algebra, representation theory*

Sažetak. **Reprezentacije verteks algebri.** *U članku su prikazani neki rezultati o teoriji reprezentacija xerteks operator (super) algebri pridruženih afinim Liejevim algebrama i Neveu-Schwarz algebri.*

Ključne riječi: *verteks operator algebra, afina Liejeva algebra, Neveu-Schwarz algebra, teorija reprezentacija*

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1. Introduction

The theory of vertex algebras has developed rapidly in the last few years. These rich algebraic structures provide the proper formulation for the moonshine module construction for the Monster group ([5], [8]) and also give a lot of new insight into the representation theory of the infinite-dimensional Lie algebras and superalgebras (see [4], [12], [9], [14]). The modern notion of chiral algebra in conformal field theory [6] in physics essentially corresponds to the mathematical notion of vertex operator algebra. The axiomatic approach to vertex operator algebra theory was made in [8] and [7].

Much work on vertex operator algebras has been concentrated on the concrete examples of vertex operator algebras and the representation theory. It is of great importance to investigate vertex algebras which are constructed from infinite-dimensional Lie algebras. In this paper we shall present some results on vertex operator (super)algebras associated to the representations of affine Lie algebras and Neveu-Schwarz algebra.

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†Department of Mathematics, University of Zagreb, Bijenička cesta 30, HR-10 000 Zagreb, e-mail: adamovic@cromath.math.hr

2. General definitions

For a rational function $f(z,w)$, with possible poles only at $z=w, z=0$ and $w=0$, we denote by $\iota_{z,w}f(z,w)$ the power series expansion of $f(z,w)$ in the domain $\mid z\mid>\mid w\mid$. Set $\mathbb{Z}_+=\{0,1,2,\cdots\}$, $\mathbb{N}=\{1,2,3,\cdots\}$.

A superalgebra is an algebra V with a $\mathbb{Z}_2$-gradation $V=V_0\oplus V_1$. Elements in V_0 (resp. V_1) are called even (resp. odd). Let $\tilde{a}$ be 0 if $a\in V_0$, and 1 if $a\in V_1$.

Definition 1. *A vertex operator superalgebra (SVOA) is a $\frac{1}{2}\mathbb{Z}_+$-graded vector space $V=\bigoplus_{n\in\frac{1}{2}\mathbb{Z}_+}V(n)$ with a sequence of linear operators $\{a(n)|n\in\mathbb{Z}\}\subset End\ V$ associated to every $a\in V$, whose generating series $Y(a,z)=\sum_{n\in\mathbb{Z}}a(n)z^{-n-1}\in(End\ V)[[z,z^{-1}]]$, called the* vertex operators *associated to a, satisfy the following axioms:*

(A1) $Y(a,z)=0$ *iff* $a=0$.

(A2) *There is a* vacuum *vector, which we denote by* $\mathbf{1}$ *, such that*

$$Y(1,z)=I_V\ (I_V \text{ is the identity of } End\,V).$$

(A3) *There is a special element $\omega\in V$ (called the* Virasoro element*), whose vertex operator we write in the form*

$$Y(\omega,z)=\sum_{n\in\mathbb{Z}}\omega(n)z^{-n-1}=\sum_{n\in\mathbf{Z}}L_nz^{-n-2},$$

such that

$$L_0\mid_{V(n)}=nI\mid_{V(n)},$$

$$Y(L_{-1}a,z)=\frac{d}{dz}Y(a,z)\quad \text{for every } a\in V, \tag{1}$$

$$[L_m,L_n]=(m-n)L_{m+n}+\delta_{m+n,0}\frac{m^3-m}{12}c, \tag{2}$$

where c is some constant in $\mathbb{C}$, which is called the rank *of V.*

(A4) *The* Jacobi identity *holds for any $m,n\in\mathbb{Z}$, i.e.*

$$\begin{aligned}
& Res_{z-w}\big(Y(Y(a,z-w)b,w)\iota_{w,z-w}((z-w)^mz^n)\big)\\
= \; & Res_z\big(Y(a,z)Y(b,w)\iota_{z,w}(z-w)^mz^n\big)\\
& -(-1)^{\tilde{a}\tilde{b}}Res_z\big(Y(b,w)Y(a,z)\iota_{w,z}(z-w)^mz^n\big)
\end{aligned}$$

An element $a\in V$ is called *homogeneous* of degree n if a is in $V(n)$. In this case we write $\deg a=n$.

Define a natural $\mathbb{Z}_2$-gradation of V by letting

$$V_0=\bigoplus_{n\in\mathbb{Z}_+}V(n),\qquad V_1=\bigoplus_{n\in\frac{1}{2}+\mathbb{Z}_+}V(n).$$

$V=V_0+V_1$. V_0 (resp. V_1) is called the even (resp. odd) part of V. Elements in V_0 (resp. V_1) are called even (resp. odd).

Remark 1. *If in the definition of vertex operator superalgebra the odd subspace $V_1 = 0$ we get the usual definition of vertex operator algebra (VOA).*

Definition 2. *Given an SVOA V,* a representation of V (or V-module) *is a $\frac{1}{2}\mathbb{Z}_+$-graded vector space $M = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}_+} M(n)$ and a linear map*

$$V \longrightarrow (End\ M)[[z, z^{-1}]], \qquad a \longmapsto Y_M(a, z) = \sum_{n\in\mathbb{Z}} a(n)z^{-n-1},$$

satisfying

(R1) $a(n)M(m) \subset M(m + \deg a - n - 1)$ *for every homogeneous element a.*

(R2) $Y_M(\mathbf{1}, z) = I_M$, *and setting $Y_M(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, we have*
$[L_m, L_n] = (m - n)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}c,$
$Y_M(L_{-1}a, z) = \frac{d}{dz}Y_M(a, z)$ *for every $a \in V$.*

(R3) *The* Jacobi identity *holds for any $m, n \in \mathbb{Z}$, i.e.*

$$\begin{aligned} & Res_{z-w}\big(Y_M(Y(a, z - w)b, w)\iota_{w,z-w}((z - w)^m z^n)\big) \\ = \;& Res_z\big(Y_M(a, z)Y_M(b, w)\iota_{z,w}(z - w)^m z^n\big) \\ & -(-1)^{\tilde{a}\tilde{b}} Res_z\big(Y_M(b, w)Y_M(a, z)\iota_{w,z}(z - w)^m z^n\big) \end{aligned}$$

The notions of submodules, quotient modules, submodules generated by a subset, direct sums, irreducible modules, completely reducible modules, etc., can be introduced in the usual way. As a module over itself, V is called the *adjoint module*. A submodule of the adjoint module is called an *ideal* of V. Given an ideal I in V such that $\mathbf{1} \notin I, \omega \notin I$, the quotient V/I admits a natural SVOA structure.

Definition 3. *An SVOA is called* rational *if it has finitely many irreducible modules and every module is a direct sum of irreducibles.*

3. VOAs associated to affine Lie algebras

We shall start with recalling some basic facts about affine Lie algebras and their representations.

Let $\mathbf{g}$ be a finite-dimensional simple Lie algebra over $\mathbf{C}$, $\mathbf{h}$ its Cartan subalgebra, and Δ root system of $(\mathbf{g}, \mathbf{h})$. We fix a set of positive root Δ_+ in Δ. Let θ denote the maximal root in Δ and $(\cdot, \cdot)$ be nondegenerate symmetric bilinear form on $\mathbf{g}$ such that $(\theta, \theta) = 2$. Let $h^\vee$ denote the dual Coxeter number. The affine Lie algebra $\hat{\mathbf{g}}$ associated with $\mathbf{g}$ is defined as

$$\mathbf{g} \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}c \oplus \mathbb{C}d$$

where c is an element of the center of $\hat{\mathbf{g}}$ and the Lie algebra structure is given by

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + n(x, y)\delta_{n+m,0}c, \qquad [d, x \otimes t^n] = nx \otimes t^n$$

for $x, y \in \mathbf{g}$, $n, m \in \mathbb{Z}$. We will write $x(n)$ for $x \otimes t^n$. Let $\hat{\mathbf{g}}'$ be the subalgebra $\mathbf{g} \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}c$. Put

$$\hat{\mathbf{g}}_+ = \mathbf{g} \otimes \mathbf{C}[t]t, \qquad \hat{\mathbf{g}}_+ = \mathbf{g} \otimes \mathbf{C}[t^{-1}][t^{-1}].$$

It is clear that $\hat{\mathbf{g}}_+$ and $\hat{\mathbf{g}}_-$ are Lie subalgebras of $\hat{\mathbf{g}}$. If we identify $\mathbf{g}$ with $\mathbf{g} \otimes 1$, $\mathbf{g}$ becomes a subalgebra of $\hat{\mathbf{g}}$.

Let U be any $\mathbf{g}$–module and $\ell \in \mathbf{C}$. Set

$$c.v = \ell v, \qquad \hat{\mathbf{g}}_+.v = 0$$

for any $v \in U$. Then U becomes $P = \hat{\mathbf{g}}_+ + \mathbf{g} + \mathbf{C}c$-module. We define the induced $\hat{\mathbf{g}}'$-module (so called generalized Verma module) with $M(\ell, U) = U(\hat{\mathbf{g}}') \otimes_{U(P)} U$.

For $\lambda \in \mathbf{h}^*$ with $V(\lambda)$ we denote the irreducible highest weight $\mathbf{g}$–module with the highest weight λ. We will write $M(\ell, \lambda)$ for $M(\ell, V(\lambda))$. Let $L(\ell, \lambda)$ denote its irreducible quotient. In the case $\lambda = 0$, $V(0)$ is the trivial $\mathbf{g}$-module. Let $\mathbf{1}$ be the highest weight vector in $M(\ell, 0)$.

Let P_+ denote the set of all dominant integral weights for $\mathbf{g}$. Then $V(\lambda)$, $\lambda \in P_+$, are all finite-dimensional $\mathbf{g}$-modules. The $\hat{\mathbf{g}}'$-modules $L(\ell, \lambda)$ such that $\ell \in \mathbb{N}$ and $(\lambda, \theta) \leq \ell$ are are all integrable modules of level ℓ in the category $\mathcal{O}$ (cf. [K]).

Now we recall a construction of VOAs associated to the highest weight $\hat{\mathbf{g}}$-modules $M(\ell, 0)$ and $L(\ell, 0)$. Let

$$Y(x(-1)\mathbf{1}, z) = \sum_{n \in \mathbb{Z}} x(n) z^{-n-1}, \quad (x \in \mathbf{g}), \tag{3}$$

be the family of fields acting on $M(\ell, 0)$ defined with the action of $x(n)$.

If $\ell \neq -h^\vee$, $\hat{\mathbf{g}}$–module $M(\ell, 0)$ has the structure of vertex operator algebra, where

$$Y : M(\ell, 0) \to \mathrm{End}(M(\ell, 0))[[z, z^{-1}]$$

is a unique extension of the fields defined by (3). Moreover, on the irreducible $\hat{\mathbf{g}}$–module $L(\ell, 0)$ we have the structure of a simple VOA(cf. [9], [14], [13]).

Theorem 1. ([9], [12], [14]) *If $\ell \in \mathbb{N}$, then the vertex operator algebra $L(\ell, 0)$ is rational and the set*

$$\{L(\ell, \lambda) \mid \lambda \in P_+, \ (\lambda, \theta) \leq \ell\}$$

provides a complete list of irreducible $L(\ell, 0)$-modules.

Remark 2. *Vertex operator algebras $L(\ell, 0)$ of certain rational levels ℓ for affine Lie algebras $A_1^{(1)}$ and $C_n^{(1)}$ have been studied in [4] and [1]. The irreducible representations for these vertex operator algebras in the category $\mathcal{O}$ are given with the certain set of modular invariant representations for affine Lie algebra (see [4], [1], [2]).*

4. Neveu-Schwarz SVOAs

In this section we will be concentrated on the SVOAs associated to the Neveu-Schwarz algebra. We will recall the authors result on the rationality on the Neveu-Schwarz SVOAs.

Let us recall first that the Neveu-Schwarz algebra is the Lie superalgebra

$$\mathbf{NS} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \bigoplus \bigoplus_{m\in\frac{1}{2}+\mathbb{Z}} \mathbb{C}G_m \bigoplus \mathbf{C}C$$

with commutation relations $(m, n \in \mathbb{Z})$:

$$\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}C,\\
&[G_{m+\frac{1}{2}}, L_n] = (m+\frac{1}{2}-\frac{n}{2})G_{m+n+\frac{1}{2}},\\
&[G_{m+\frac{1}{2}}, G_{n-\frac{1}{2}}]_+ = 2L_{m+n} + \frac{1}{3}m(m+1)\delta_{m+n,0}C,\\
&[L_m, C] = 0, \quad [G_{m+\frac{1}{2}}, C] = 0.
\end{aligned}$$

Set

$$\mathbf{NS}_{\pm} = \bigoplus_{\mathbf{n}\in\mathbb{N}} \mathbb{C}\mathbf{L}_{\pm\mathbf{n}} \bigoplus \bigoplus_{\mathbf{m}\in\frac{1}{2}+\mathbb{Z}_+} \mathbb{C}\mathbf{G}_{\pm\mathbf{m}}.$$

Given complex numbers c and h, the Verma module $M_{c,h}$ over $\mathbf{NS}$ is the free $\mathbf{U}(\mathbf{NS}_-)$-module generated by $\mathbf{1}$, such that $\mathbf{NS}_+\mathbf{1} = 0, L_0\mathbf{1} = h\cdot\mathbf{1}$ and $C\cdot\mathbf{1} = c\cdot\mathbf{1}$. There exists a unique maximal proper submodule $J_{c,h}$ of $M_{c,h}$. Denote the quotient $M_{c,h}\,/\,J_{c,h}$ by $L_{c,h}$. Recall that $v \in M_{c,h}$ is called a singular vector if $\mathbf{NS}_+v = 0$ and v is an eigenvector of L_0. For example, $G_{-\frac{1}{2}}\mathbf{1}$ is a singular vector of $M_{c,0}$ for any c. Denote $M_{c,0}\,/\langle G_{-\frac{1}{2}}\mathbf{1}\rangle$ by M_c, where $\langle G_{-\frac{1}{2}}\mathbf{1}\rangle$ is the submodule of $M_{c,0}$ generated by the singular vector $G_{-\frac{1}{2}}\mathbf{1}$. For simplicity we denote $L_{c,0}$ by L_c.

Define the following fields

$$Y(L_{-2}\mathbf{1}, z) = L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, Y(G_{-\frac{3}{2}}\mathbf{1}, z) = G(z) = \sum_{n\in\mathbb{Z}} G_{n+\frac{1}{2}} z^{-n-2}. \tag{4}$$

Then there is a unique extension of the fields (4) such that M_c becomes SVOA and $\mathbf{NS}$–modules $M_{c,h}$, $L_{c,h}$ become modules for SVOA M_c . Moreover, on the irreducible $\mathbf{NS}$–module L_c we have the structure of simple SVOA. We are interested in the classification of irreducible L_c–modules.

Set

$$c_{p,q} = \frac{3}{2}(1 - \frac{2(p-q)^2}{pq}), \qquad h^{r,s}_{p,q} = \frac{(sp-rq)^2 - (p-q)^2}{8pq}.$$

Whenever we mention $c_{p,q}$ again, we always assume that $p, q \in \{2, 3, 4, \cdots\}, p-q \in 2\mathbb{Z}$, and that $(p-q)/2$ and q are relatively prime to each other. Set

$$S_{p,q} = \{h^{r,s}_{p,q} \mid 0 < r < p, 0 < s < q, r-s \in 2\mathbb{Z}\}.$$

For $c \neq c_{p,q}$, we have that $L_c = M_c$. So, in this case SVOA L_c is not rational (cf. [12]).

Theorem 2. ([3], Theorem 3.3) *The vertex operator superalgebra $L_{c_{p,q}}$ is rational. Moreover, the minimal series modules $L_{c,h_{r,s}}, 0 < r < p, 0 < s < q, r-s \in 2\mathbb{Z}$ are all the irreducible representations of L_c.*

Remark 3. *The proof of the* Theorem 2. *uses the representation theory of VOAs associated to affine Lie algebras* $A_1^{(1)}$ *from [4], and the Goddard-Kent-Olive construction [10].*

Remark 4. Theorem 2. *gives the affirmative answer on the Kac-Wang conjecture [[12],* Conjecture *3.1].*

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