Operators in quaternionic Hilbert space*

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Abstract. *In this paper a result on functional calculus for contractions, in the sense of B. Sz.-Nagy and C. Foiaş, in quaternionic Hilbert space is presented.*

Key words: *quaternionic Hilbert space, contraction, functional calculus, symplectic image*

Sažetak. *U ovom radu je prikazan jedan rezultat o funkcionalnom računu za kontrakcije u smislu B. Sz.-Nagya i C. Foiaşa u kvaternionskom Hilbertovom prostoru.*

Ključne riječi: *kvaternionski Hilbertov prostor, kontrakcija, funkcionalni račun, simplektička slika*

1. Introduction

A left vector space $\mathcal{H}$ over the noncommutative field of quaternions Q, complete in respect to the quaternion-valued inner product, is said to be **left quaternionic Hilbert space**. In the following we shall omit the adjective 'left'.

Given quaternionic Hilbert space $\mathcal{H}$, there exists one and only one ordered pair (H, κ), where H is a complex Hilbert space, equal as a set to $\mathcal{H}$, whose inner product is the complex part of inner product in $\mathcal{H}$, and $\kappa : H \to H$ is an antiunitary operator satisfying $\kappa^2 = -I$, defined as $\kappa\, x = kx$. Here k denotes an element in a basis of Q.

Conversely, an ordered pair (H, κ), of complex Hilbert space H, and an antiunitary operator $\kappa : H \to H$ with the property $\kappa^2 = -I$, determine uniquely quaternionic Hilbert space $\mathcal{H}$, equal as a set to H, with the inner product

$$(x|y) = \langle x|y\rangle + \langle x|\kappa\, y\rangle\, k,$$

where $\langle|\rangle$ denote the inner product in H. The Hilbert space H is said to be **symplectic image** of $\mathcal{H}$.

A subspace H_0 in H is a symplectic image of a subspace $\mathcal{H}_0$ in $\mathcal{H}$ if and only if H_0 reduces κ.

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The set $\mathcal{L}(\mathcal{H})$ of all linear bounded operators in $\mathcal{H}$ is a real Banach algebra.

An operator $\mathcal{A} \in \mathcal{L}(\mathcal{H})$ determines unique linear bounded operator $A : H \to H$ defined by $A\,x = \mathcal{A}\,x$ which commutes with κ. The operator A is said to be a **symplectic image** of operator $\mathcal{A}$.

Conversely, a linear bounded operator A in H, commuting with κ, is symplectic image of a linear bounded operator $\mathcal{A}$ in $\mathcal{H}$.

For each $\lambda \in Q$, there exists a $q \in Q \setminus \{0\}$ such that $q^{-1}\,\lambda\,q \in C^+$, where $C^+ = \{z \in C;\ \operatorname{Re} z \geq 0\}$. Here C denotes the set of complex numbers. Thus the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A} \in \mathcal{L}(\mathcal{H})$ is defined as the set of all $\lambda \in C^+$ such that $\lambda\mathcal{I} - \mathcal{A}$ is not boundedly invertible.

A contraction $\mathcal{T} \in \mathcal{L}(\mathcal{H})$ is said to be **completely nonunitary** (c.n.u.), if there exists no subspace $\mathcal{H}_0 \subset \mathcal{H}$, reducing $\mathcal{T}$, such that $\mathcal{T}|\mathcal{H}_0$ is unitary. Each contraction can be split into the direct sum of a unitary operator and a c.n.u. contraction. The spectrum of any contraction in $\mathcal{H}$ is a closed subset in the intersection of $\bar{D} = \{z \in C;\ |z| \leq 1\}$ and C^+.

2. Functional calculus

The spectral theorem for normal operators has been proved by O. Teichmüller in his thesis [3]. A functional calculus, based on it, has been established by K. Viswanath [4]. Riesz-Dunford calculus for nonnormal operators, using analytic functions on a neighbourhood of spectrum, also holds true in quaternionic case. In complex case B. Sz.-Nagy and C. Foiaş [2] have extended Riesz-Dunford functional calculus for c.n.u. contractions enlarging the set of admissible functions. In quaternionic case also Riesz-Dunford calculus for c.n.u. contractions can be extended [1] in the sense of B. Sz.-Nagy and C. Foiaş. In order to be more precise we need a definition.

Let K be the set of real numbers R or C. Let H^∞_K denote the Banach algebra of functions

$$u(z) = \sum_{n\geq 0} u_n\,z^n, \qquad u_n \in K,$$

analytic in the unit disc $D = \{z \in C;\ |z| < 1\}$, such that

$$\|u\|_\infty = \sup_{z\in D} |u(z)| < \infty.$$

Put

$$u_r(z) = u(r\,z) = \sum_{n=0}^{\infty} c_n\,r^n\,z^n, \qquad 0 < r < 1.$$

u_r is analytic in a neighbourhood of the closed unit disc $\bar{D}$. Thus, by means of Riesz-Dunford functional calculus, there exists

$$u_r(\mathcal{T}) = \sum_{n=0}^{\infty} c_n\,r^n\,\mathcal{T}^n$$

for each contraction $\mathcal{T}$, especially for c.n.u. contraction.

Theorem 1. *Let $\mathcal{T} \in \mathcal{L}(\mathcal{H})$ be a c.n.u. contraction. Then, for each $u \in H_R^\infty$, there exists*

$$\lim_{r \nearrow 1} u_r(\mathcal{T}) = u(\mathcal{T})$$

in the sense of the strong operator topology.

In the proof it is used, that the spectral measure of the minimal unitary dilation of $\mathcal{T}$ is absolutely continuous in respect to Lebesgue measure on $\{z \in C^+;\ |z| = 1\}$, the fact which is crucial in complex case, too.

This theorem really extends the Riesz-Dunford functional calculus, since there exist c.n.u. contractions having a nonvoid intersection between its spectrum and $\{z \in C^+;\ |z| = 1\}$. Moreover, the following holds true.

Theorem 2. *There exists a c.n.u. contraction $\mathcal{T}$ such that*

$$\sigma(\mathcal{T}) = \{z \in C^+;\ |z| = 1\}.$$

So defined functional calculus has the following properties.

Theorem 3. *Let $\mathcal{T} \in \mathcal{L}(\mathcal{H})$ be a c.n.u. contraction. The map*

$$u \mapsto u(\mathcal{T})$$

is a homomorphism of algebras with the following properties

1. *If T is symplectic image of $\mathcal{T}$, then $u(T)$ is a symplectic image of $u(\mathcal{T})$,*
2. *$u(\mathcal{T})^\star = u(\mathcal{T}^\star)$,*
3. *$\|u(\mathcal{T})\| \le \sup_{z \in D} |u(z)| = \|u\|_\infty$.*

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