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Some novel applications in relation with certain equations and inequalities in the complex plane*

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Abstract. The aim of this research is first to reveal some novel and nonlinear applications relating to some equations and inequalities in the complex plane and then to present a number of consequences thereof.

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1. Introduction and preliminaries

Let z and w be the independent and dependent complex variables, respectively. Consider the nth order ordinary differential equation:

$$\sum_{j=0}^{m} \sum_{l=1}^{k} \sum_{i=0}^{n} \psi_i(z) \left[w(z)\right]^j \left[w^{(i)}(z)\right]^l = \phi(z), \tag{1}$$

where $\psi_i(z)$ and $\phi(z)$ are continuous functions in some domains of the complex plane and n, m and k are positive integers. Moreover, $w(z)$ is also a differentiable function in any domain of the complex plane. As known from the literature, ordinary differential equations arise and are used in many different fields of science.

In this study, a number of relations between the solutions of some ordinary differential equations of the form (1) and some complex inequalities are established without actually finding any solutions of the differential equations under consideration.

Now we shall introduce some notations, definitions and information which will be used in getting the main results.

Let us denote by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{U}$, $\mathbb{D}$, $\mathbb{Z}^+$ and $\mathbb{Z}^-$, the set of real numbers, the set of complex numbers, the unit open disk, i.e., $\mathbb{U} = \left\{z \in \mathbb{C} : |z| < 1\right\}$ and the punctured unit disk, i.e., $\mathbb{D} = \mathbb{U} - \{0\}$, the set of positive integers and the set of negative

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integers, respectively. In addition, let $\mathcal{H}^{\kappa}_{p,q}$ denote the family of functions $f(z)$ which are non-normalized by the Taylor-Maclaurin series:

$$\begin{gathered} f(z) = \kappa z^q + a_{p+1} z^{p+1} + a_{p+2} z^{p+2} + \cdots + a_{p+n} z^{p+n} + \cdots , \\ \kappa \neq 0,\ q \in \mathbb{Z} := Z^+ \cup \mathbb{Z}^-,\ a_n \in \mathbb{C}, n, p \in \mathbb{Z}^+, \end{gathered} \tag{2}$$

which are analytic and multivalent (or meromorphically multivalent) in the domain:

$$\mathbb{E} := \begin{cases} \mathbb{U}, & \text{when} \quad q \in \mathbb{Z}^+ \\ \mathbb{D}, & \text{when} \quad q \in \mathbb{Z}^- \end{cases} .$$

For the main result, we also need to recall the following assertion given in [6] by M. Nunokawa.

Lemma 1. *Let $p(z)$ be an analytic function in the disk $\mathbb{U}$ with $p(0) = 1$. If there exists a point z_0 in $\mathbb{U}$ such that*

$$\Re e\Big(p(z)\Big) > 0 \quad \Big(|z| < |z_0|\Big) \quad , \quad \Re e\Big(p(z_0)\Big) = 0 \quad \textit{and} \quad p(z_0) \neq 0, \tag{3}$$

then

$$p(z_0) = ia \quad \textit{and} \quad \left.\frac{zp'(z)}{p(z)}\right|_{z=z_0} = ic\Big(a + \frac{1}{a}\Big), \tag{4}$$

where $c \geq \frac{1}{2}$ and $a \in \mathbb{R}^* := \mathbb{R} - \{0\}$.

2. The main result and conclusions

We begin by stating and then by proving the following main result concerning several useful consequences dealing with the theories of certain functions and differential equations with complex variables.

Theorem 1. *Let $\psi(z)$ be an analytic function that satisfies any of the cases in inequalities*

$$\Big|\psi(z)\Big| < \begin{cases} \dfrac{\lambda}{4(1-\lambda)}, & \textit{when} \quad 0 \leq \lambda \leq \dfrac{1}{2} \\ \dfrac{1-\lambda}{4\lambda}, & \textit{when} \quad \dfrac{1}{2} \leq \lambda < 1 \end{cases} \tag{5}$$

and

$$\Re e\Big(\psi(z)\Big) > \begin{cases} \dfrac{\lambda}{4(\lambda-1)}, & \textit{when} \quad 0 \leq \lambda \leq \dfrac{1}{2} \\ \dfrac{\lambda-1}{4\lambda}, & \textit{when} \quad \dfrac{1}{2} \leq \lambda < 1 \end{cases} . \tag{6}$$

If a function $w := w(z)$ is a unique solution of the complex differential equation

$$zw\frac{d^2w}{dz^2} - z\left(\frac{dw}{dz}\right)^2 + \Big(1 - \psi(z)\Big)w\frac{dw}{dz} = 0 \ , \tag{7}$$

then

$$\Re e\left(\frac{zw'(z)}{w(z)}\right) \begin{cases} > q\lambda, & \textit{when} \quad q \in \mathbb{Z}^+ \\ < q\lambda, & \textit{when} \quad q \in \mathbb{Z}^- \end{cases} , \tag{8}$$

where

$$z \in \mathbb{E}\ ,\ w(z) = \big(1-\alpha\big) f(z) + \alpha z f'(z)\ ,\ \alpha \in \mathbb{R}\ \ \mathit{and}\ \ f(z) \in \mathcal{H}^{\kappa}_{p,q}. \tag{9}$$

Proof. In view of (2) and (9), let us define a function $p(z)$ by

$$p(z) = \frac{\dfrac{1}{q} \cdot \dfrac{zw'(z)}{w(z)} - \lambda}{1-\lambda}, \tag{10}$$

where $z \in \mathbb{E}$, $0 \leq \lambda < 1$, $\alpha \in \mathbb{R}$, $w(z) = \big(1-\alpha\big) f(z) + \alpha z f'(z)$ and $f(z) \in \mathcal{H}^{\kappa}_{p,q}$. It is obvious that the function $p(z)$ has the following Taylor-Maclaurin series:

$$p(z) = 1 + h_1 z + h_2 z^2 + h_3 z^3\ + \cdots, \quad z \in \mathbb{U},$$

and hence $p(z)$ is analytic in $\mathbb{U}$ and $p(0) = 1$. By differentiating both sides of (10) with respect to the complex variable z and simplifying, one obtains

$$\frac{\left(\dfrac{zw'(z)}{w(z)}\right)'}{\dfrac{zw'(z)}{w(z)}} = \frac{(1-\lambda)p'(z)}{\lambda + (1-\lambda)p(z)}$$

or, equivalently,

$$1 + z\left(\frac{w''(z)}{w'(z)} - \frac{w'(z)}{w(z)}\right) = \psi(z)\left(= \frac{(1-\lambda)zp'(z)}{\lambda + (1-\lambda)p(z)}\right), \tag{11}$$

where $z \in \mathbb{U}$. From equation (11) it is clear that $\psi(z)$ satisfies the complex differential equation given by (7).

Suppose now that there exists a point z_0 in $\mathbb{U}$ such that

$$\Re e\Big(p(z)\Big) > 0\ \ \Big(|z| < |z_0|\Big)\ \ ,\ \ \Re e\Big(p(z_0)\Big) = 0\ \ \text{and}\ \ p(z_0) \neq 0.$$

By applying the assertions of Lemma 1, given by (4), namely, by means of

$$p(z_0) = ia \quad \text{and} \quad \left.\frac{zp'(z)}{p(z)}\right|_{z=z_0} = \ ic\Big(a + \frac{1}{a}\Big), \quad c \geq \frac{1}{2}, a \in \mathbb{R}^*$$

and by using the inequalities:

$$0 \leq \lambda \leq \frac{1}{2} \quad \Longrightarrow \quad \frac{1+a^2}{a^2 + \left(\dfrac{\lambda}{1-\lambda}\right)^2} \geq 1$$

and

$$\frac{1}{2} \leq \lambda < 1 \quad \Longrightarrow \quad \frac{1+a^2}{1 + a^2\left(\dfrac{1-\lambda}{\lambda}\right)^2} \geq 1,$$

we obtain that

$$\begin{aligned}\Big|\psi(z_0)\Big| &= \left|\frac{(1-\lambda)zp'(z)}{\lambda+(1-\lambda)p(z)}\Big|_{z=z_0}\right| \\ &= \left|\frac{zp'(z)}{p(z)}\cdot\frac{(1-\lambda)p(z)}{\lambda+(1-\lambda)p(z)}\Big|_{z=z_0}\right| \\ &= \left|ic\Big(a+\frac{1}{a}\Big)\cdot\frac{ia(1-\lambda)}{\lambda+ia(1-\lambda)}\right| \\ &\geq \begin{cases}\dfrac{\lambda}{4(1-\lambda)}, & \text{when } \ 0\leq\lambda\leq\dfrac{1}{2} \\ \dfrac{1-\lambda}{4\lambda}, & \text{when } \ \dfrac{1}{2}\leq\lambda<1\end{cases}\end{aligned} \tag{12}$$

and

$$\begin{aligned}\Re e\Big(\psi(z_0)\Big) &= \Re e\left[\frac{zp'(z)}{p(z)}\cdot\frac{(1-\lambda)p(z)}{\lambda+(1-\lambda)p(z)}\Big|_{z=z_0}\right] \\ &= \Re e\left[ic\Big(a+\frac{1}{a}\Big)\cdot\frac{ia(1-\lambda)}{\lambda+ia(1-\lambda)}\right] \\ &\leq \begin{cases}\dfrac{\lambda}{4(1-\lambda)}, & \text{when } \ 0\leq\lambda\leq\dfrac{1}{2} \\ \dfrac{1-\lambda}{4\lambda}, & \text{when } \ \dfrac{1}{2}\leq\lambda<1\end{cases},\end{aligned} \tag{13}$$

where $c \geq 1/2$ and $a \in \mathbb{R}^*$. Clearly, each of the cases in the inequalities given by (12) and (13) contradicts the corresponding cases in assumptions (5) and (6) of Theorem 1, respectively. Hence, equality (10) requires the inequality

$$\Re e\Big(p(z)\Big) = \Re e\left(\frac{\frac{zw'(z)}{qw(z)}-\lambda}{1-\lambda}\right) > 0, \quad z \in \mathbb{U},$$

to hold, which evidently yields inequality (8). This completes the proof of Theorem 1. □

When one focuses on Theorem 1, it is easily seen that it includes a number of consequences containing both some certain complex functions and certain complex differential equations. By choosing suitable values of parameters in Theorem 1, several useful and comprehensive results can be easily derived. In particular, for some of them, which can be related with the condition in (6), one may compare these results with the earlier ones given in [5]. But, although it is not possible to determine all of them, we want to point out only four of those establishing the relation between certain types of ordinary differential equations and multivalent (or meromorphicaly multivalent) functions; namely, Corollaries 1-4 given below. Other corollaries of the main result and construction of some elementary examples are left to the attention of researchers who have been working on the theories of differential equations or complex functions. Specifically, for a number of its basic consequences in relation with the analytic and geometric function theory, which is a special field (of complex analysis) appertaining to the relations between the analytic properties of a function $f(z)$ in $\mathcal{H}^{\kappa}_{p,q}$ and the geometric properties of the image domain $f\big(\mathbb{E}\big)$, one may refer to the works in [1-4, 7].

Firstly, by letting $\alpha := 0$, $\kappa := 1$ and $q := p$ in Theorem 1, the following result related to multivalently starlikeness can be obtained.

Corollary 1. *Let $\psi(z)$ be an analytic function that satisfies any of the cases in the inequalities given by (5) and (6), and let the function $w \equiv w(z) := f(z) \in \mathcal{T}(p) := \mathcal{H}^{1}_{p,p}$ satisfy the complex differential equation given by (7). Then*

$$\Re e \left(\frac{zw'(z)}{w(z)} \right) > p\lambda, \quad 0 \le \lambda < 1, p \in \mathbb{Z}^+, z \in \mathbb{U},$$

i.e., $w \in \mathcal{T}(p)$ is a multivalently starlike function of order $p\lambda$ in the unit open disk $\mathbb{U}$.

Secondly, by setting $\alpha := 0$, $\kappa := 1$ and $q := -p$ in Theorem 1, the following result concerning with multivalently meromorphic starlikeness can be derived.

Corollary 2. *Let $\psi(z)$ be an analytic function that satisfies any of the cases in the inequalities given by (5) and (6), and let the function $w \equiv w(z) := f(z) \in \mathcal{M}(p) := \mathcal{H}^{1}_{p,-p}$ satisfy the complex differential equation given by (7). Then*

$$\Re e \left(-\frac{zw'(z)}{w(z)} \right) > p\lambda, \quad 0 \le \lambda < 1, p \in \mathbb{Z}^+, z \in \mathbb{D},$$

i.e., $w \in \mathcal{M}(p)$ is a multivalently meromorphic starlike function of order $p\lambda$ in the punctured unit disk $\mathbb{D}$.

Thirdly, by taking $\alpha := 1$, $\kappa := 1$ and $q := p$ in Theorem 1, the following result appertaining to multivalently convexity can be given.

Corollary 3. *Let $\psi(z)$ be an analytic function that satisfies any one of the cases in the inequalities given by (5) and (6), and let the function $w \equiv w(z) := f(z) \in \mathcal{T}(p)$ satisfy the complex differential equation*

$$zw\frac{d^2\big(zw'\big)}{dz^2} - z\left(\frac{d\big(zw'\big)}{dz}\right)^2 + \Big(1-\psi(z)\Big)w\frac{d\big(zw'\big)}{dz} = 0. \tag{14}$$

Then

$$\Re e \left(\frac{zw''(z)}{w'(z)} \right) > p\lambda - 1, \quad 0 \le \lambda < 1, p \in \mathbb{Z}^+, z \in \mathbb{U},$$

i.e., $w \in \mathcal{T}(p)$ is a multivalently convex function of order $p\lambda$ in the unit open disk $\mathbb{U}$.

Finally, by putting $\alpha := 1$, $\kappa := 1$ and $q := -p$ in Theorem 1, the following result regarding multivalently meromorphic convexity can be revealed.

Corollary 4. *Let $\psi(z)$ be an analytic function that satisfies any of the cases in the inequalities given by (5) and (6), and let the function $w \equiv w(z) := f(z) \in \mathcal{M}(p)$ satisfy the complex differential equation given by (14). Then*

$$\Re e\left(-\frac{zw''(z)}{w'(z)}\right) > 1 + p\lambda, \quad 0 \leq \lambda < 1, p \in \mathbb{Z}^{+}, z \in \mathbb{D},$$

i.e., $w \in \mathcal{M}(p)$ is a multivalently meromorphic convex function of order $p\lambda$ in the punctured unit disk $\mathbb{D}$.

Concluding remarks: Note that the differential equation given by (7) is one of the special forms of the general form given by (1) and is, at the same time, associated with certain differential equations in the literature, which are well-known Painlave or Hiller equations. So, in view of the main theorem, not only some interesting results in relation with Painlave or Hiller equations, but also several properties, such as modules, real and imaginary parts and arguments, of functions which are solutions of certain types of Painlave or Hiller equations can be revealed and all these are left to the researchers in the related field.

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