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# A combinatorial method for determining the spectrum of linear combinations of finitely many diagonalizable matrices that mutually commute

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**Abstract.** Let $X_i$, $i = 1, 2, ..., m$, be diagonalizable matrices that mutually commute. This paper provides a combinatorial method to handle the problem when a linear combination matrix $X = \sum_{i=1}^{m} c_i X_i$ is a matrix such that $\sigma(X) \subseteq \{\lambda_1, \lambda_2, ..., \lambda_n\}$, where $c_i$, $i = 1, 2, ..., m$, are nonzero complex scalars and $\sigma(X)$ denotes the spectrum of the matrix $X$. If the spectra of matrices $X$ and $X_i$, $i = 1, 2, ..., m$, are chosen as subsets of some particular sets, then this problem is equivalent to the problem of characterizing all situations in which a linear combination of some commuting special types of matrices, e.g. the matrices such that $A^k = A$, $k = 2, 3, ...$, is also a special type of matrix. The method developed in this note makes it possible to solve such characterization problems for linear combinations of finitely many special types of matrices. Moreover, the method is illustrated by considering the problem, which is one of the open problems left in [Linear Algebra Appl. 437 (2012) 2091-2109], of characterizing all situations in which a linear combination $X = c_1X_1 + c_2X_2 + c_3X_3$ is a tripotent matrix when $X_1$ is an involutory matrix and both $X_2$ and $X_3$ are tripotent matrices that mutually commute. The results obtained cover those established in the reference above.

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**Key words**: diagonalizable matrices, commutativity, spectrum, linear combination, systems of linear equations

## 1. Introduction

Let $\mathbb{N}$, $\mathbb{C}$, and $\mathbb{C}_n$ denote the sets of natural numbers, complex numbers, and $n \times n$ complex matrices, respectively. $\mathbf{0}$ and $I_n$ will be a zero matrix of appropriate size and the identity matrix of order $n$, respectively. Let $\Omega_k$ denote the set of roots of unity of order $k$. The symbols $C(A)$, $A^*$, $A^\dagger$, and $A^\sharp$, will stand for column space, conjugate transpose, Moore-Penrose inverse, and group inverse of $A \in \mathbb{C}_n$, respectively. For details on generalized inverses of matrices, see [11]. A matrix $A \in \mathbb{C}_n$ is said to be idempotent (projector), orthogonal projector, involutory, tripotent, quadripotent, $k$-potent, normal, range Hermitian (EP), partial isometry, and $\{\alpha, \beta\}$-quadratic if $A^2 = A$, $A^2 = A = A^*$, $A^{-1} = A$, $A^3 = A$, $A^4 = A$, $A^k = A$ $(k \in \mathbb{N}, k \geq 2)$, $AA^* = A^*A$, $C(A) = C(A^*)$ (or $AA^\dagger = A^\dagger A$), $AA^*A = A$ (or $A^\dagger = A^*$), and $(A - \alpha I_n)(A - \beta I_n) = \mathbf{0}$ such that $\alpha, \beta \in \mathbb{C}$ and $\alpha \neq \beta$, respectively; and for the

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sake of simplicity, the sets of the aforesaid matrices will be denoted by $\mathbb{C}_n^{P}$, $\mathbb{C}_n^{OP}$, $\mathbb{C}_n^{I}$, $\mathbb{C}_n^{T}$, $\mathbb{C}_n^{Q}$, $\mathbb{C}_n^{k-P}$,$\mathbb{C}_n^{N}$, $\mathbb{C}_n^{EP}$, $\mathbb{C}_n^{PI}$, and $\mathbb{C}_n^{(\alpha,\beta)}$, respectively.

Let us recall the matrices introduced and characterized by some authors in recent years. A matrix $A \in \mathbb{C}_n$ is called generalized projector, hypergeneralized projector, $k$-generalized projector, $k$-hypergeneralized projector, group involutory, $\{k\}$-group periodic, and $\{K, s+1\}$-potent if $A^2 = A^*$, $A^2 = A^\dagger$, $A^k = A^*$, $A^k = A^\dagger$, $A^\sharp = A$, $A^\sharp = A^{k-1}$, and $KA^{s+1}K = A$, respectively, where $k, s \in \mathbb{N}$ and $K \in \mathbb{C}_n^{I}$. The sets of aforesaid matrices will be denoted by $\mathbb{C}_n^{GP}$, $\mathbb{C}_n^{HGP}$, $\mathbb{C}_n^{k-GP}$, $\mathbb{C}_n^{k-HGP}$, $\mathbb{C}_n^{GrI}$, $\mathbb{C}_n^{k-GrP}$, and $\mathbb{C}_n^{\{K,s+1\}}$, respectively. Groß and Trenkler established the results $\mathbb{C}_n^{GP} = \mathbb{C}_n^{Q} \cap \mathbb{C}_n^{N} \cap \mathbb{C}_n^{PI} = \mathbb{C}_n^{Q} \cap \mathbb{C}_n^{N}$ [22, Theorem 1] and $\mathbb{C}_n^{HGP} = \mathbb{C}_n^{Q} \cap \mathbb{C}_n^{EP}$ [22, Theorem 2], and also considered the problem of when the sum and the difference of two generalized projectors is also a generalized projector [22, Theorems 5 and 6]. A strengthened version of Groß and Trenkler's initial result $\mathbb{C}_n^{GP} = \mathbb{C}_n^{Q} \cap \mathbb{C}_n^{N} \cap \mathbb{C}_n^{PI} = \mathbb{C}_n^{Q} \cap \mathbb{C}_n^{N} = \mathbb{C}_n^{Q} \cap \mathbb{C}_n^{PI}$ was given by Baksalary and Lui [7, Theorem]. Bru and Thome proved that $A \in \mathbb{C}_n^{GrI} \Leftrightarrow A \in \mathbb{C}_n^{T}$ [17, Theorem 8]. Benítez and Thome established the result $A \in \mathbb{C}_n^{k-GP} \Leftrightarrow A \in \mathbb{C}_n^{N}$ and $\sigma(A) \in \{0\} \cup \Omega_{k+1} \Leftrightarrow A \in \mathbb{C}_n^{N}$ and $A \in \mathbb{C}_n^{(k+2)-P}$ [13, Theorem 1], and they also indicated that if $A \in \mathbb{C}_n^{k-GP}$, then $A^\dagger = A^\sharp = A^* = A^{m(k+1)+k}$, where $m \in \mathbb{N}$. The same authors characterized the set $\mathbb{C}_n^{k-GrP}$ in many aspects, including some of its topological properties [14, Theorem 2.4]. Deng et al. proved that $A \in \mathbb{C}_n^{k-HGP} \Leftrightarrow A \in \mathbb{C}_n^{EP}$ and $A \in \mathbb{C}_n^{(k+2)-P}$ [20, Theorem 2.4], and showed that $\mathbb{C}_n^{k-GP} \subset \mathbb{C}_n^{k-HGP} \subset \mathbb{C}_n^{(k+2)-P}$ [20, Corollary 2.7].

Some of these special types of matrices mentioned above play an important role in applied sciences. For instance, idempotent and tripotent matrices are used in statistical theory [21, Section 12.4] and quadratic forms [10], involutory matrices are used in quantum mechanics [1, p. 495] and cryptography [26, Section 6.11], and $k$-potent matrices are used in digital image encryption [34]. Further examples can be found in [14, 19, 23, 24, 25, 29, 31].

The problems of characterizing all situations in which a linear combination of some special types of matrices is also a special type of matrix have been studied by many authors in recent years. Xu and Xu summarized some of these results in a table [36, Table 1.1]. We want to give the summary of the known results by updating this table. Linear combinations considered in the literature are as follows:

$$X = c_1 X_1 + c_2 X_2 \tag{1}$$

and

$$X = c_1 X_1 + c_2 X_2 + c_3 X_3, \tag{2}$$

where $X_j \in \mathbb{C}_n$ and $c_j \in \mathbb{C} \setminus \{0\}$, $j = 1, 2, 3$. The results concerning linear combinations of the form (1) and linear combinations of the form (2) are listed in Table 1 and Table 2, respectively. Please note that $i \in \{1, 2\}$ in Table 1 and $j \in \{1, 2, 3\}$ in Table 2.

Two common properties of special types of matrices mentioned above are that they are diagonalizable and their spectra are subsets of some particular sets. In addition to that, under the assumption of mutual commutativity the problem of characterizing linear combinations of these special types of matrices is equivalent to the problem of when a linear combination of diagonalizable matrices whose spectra

| | $X_1X_2 = X_2X_1$ | $X_1X_2 \neq X_2X_1$ |
|---|---|---|
| $X \in \mathbb{C}_n^P$ | $X_i \in \mathbb{C}_n^P$ [2] | $X_i \in \mathbb{C}_n^P$ [2] |
| | $X_1 \in \mathbb{C}_n^P$ and $X_2 \in \mathbb{C}_n^T$ [6] | $X_1 \in \mathbb{C}_n^P$ and $X_2 \in \mathbb{C}_n^T$ [6] |
| | $X_i \in \mathbb{C}_n^T$ [30] | |
| | $X_1 \in \mathbb{C}_n^P$ and $X_2 \in \mathbb{C}_n^{k-P}$ [12] | $X_1 \in \mathbb{C}_n^P$ and $X_2 \in \mathbb{C}_n^{k-P}$ [15] |
| | $X_i \in \mathbb{C}_n^I$ [31] | $X_i \in \mathbb{C}_n^I$ [31] |
| | $X_1 \in \mathbb{C}_n^T$ and $X_2 \in \mathbb{C}_n^I$ [35] | $X_1 \in \mathbb{C}_n^T$ and $X_2 \in \mathbb{C}_n^I$ [35] |
| $X \in \mathbb{C}_n^I$ | $X_i \in \mathbb{C}_n^P$ [31] | $X_i \in \mathbb{C}_n^P$ [31] |
| | $X_i \in \mathbb{C}_n^I$ [31] | $X_i \in \mathbb{C}_n^I$ [31] |
| | $X_i \in \mathbb{C}_n^T$ [31] | |
| | $X_1 \in \mathbb{C}_n^T$ and $X_2 \in \mathbb{C}_n^I$ [35] | $X_1 \in \mathbb{C}_n^T$ and $X_2 \in \mathbb{C}_n^I$ [35] |
| $X \in \mathbb{C}_n^T$ ($X \in \mathbb{C}_n^{GrI}$) | $X_i \in \mathbb{C}_n^P$ [18, 4] | $X_i \in \mathbb{C}_n^P$ [18, 4] |
| | $X_i \in \mathbb{C}_n^I$ [31] | |
| | $X_i \in \mathbb{C}_n^T$ [5, 30] | |
| $X \in \mathbb{C}_n^{k-P}$ | $X_i \in \mathbb{C}_n^I$ [32] | |
| $X \in \mathbb{C}_n^{GP}$ | $X_i \in \mathbb{C}_n^{GP}$ [3, 13] | $X_i \in \mathbb{C}_n^{GP}$ [3, 13] |
| $X \in \mathbb{C}_n^{HGP}$ | $X_i \in \mathbb{C}_n^{HGP}$ [9] | |
| $X \in \mathbb{C}_n^{k-GP}$ ($X \in \mathbb{C}_n^{(k+2)-P}$) | $X_i \in \mathbb{C}_n^{k-GP}$ [13] | |
| | $X_i \in \mathbb{C}_n^{OP}$ [13] | |
| $X \in \mathbb{C}_n^{k-GrP}$ ($X \in \mathbb{C}_n^{(k+1)-P}$) | $X_i \in \mathbb{C}_n^P$ [14] | $X_i \in \mathbb{C}_n^P$ [14] |
| $X \in \mathbb{C}_n^{\{K,s+1\}}$ | $X_i \in \mathbb{C}_n^{\{K,s+1\}}$ [29] | |
| $X \in \mathbb{C}_n^{(\alpha_3,\beta_3)}$ | $X_i \in \mathbb{C}_n^{(\alpha_i,\beta_i)}$ [33] | $X_i \in \mathbb{C}_n^{(\alpha_i,\beta_i)}$ [33] |

**Table 1:** *Summary of results related to linear combinations of the form (1)*

| | $X_1X_2 = X_2X_1$ $X_1X_3 = X_3X_1$ $X_2X_3 = X_3X_2$ | $X_1X_2 = X_2X_1$ $X_1X_3 = X_3X_1$ $X_2X_3 \neq X_3X_2$ | $X_1X_2 = X_2X_1$ $X_1X_3 \neq X_3X_1$ $X_2X_3 = X_3X_2$ | $X_1X_2 = X_2X_1$ $X_1X_3 \neq X_3X_1$ $X_2X_3 \neq X_3X_2$ |
|---|---|---|---|---|
| $X \in \mathbb{C}_n^P$ | $X_j \in \mathbb{C}_n^P$ [8, 19] | $X_j \in \mathbb{C}_n^P$ [8, 19] | $X_j \in \mathbb{C}_n^P$ [8] | $X_j \in \mathbb{C}_n^P$ [8] |
| $X \in \mathbb{C}_n^T$ | $X_i \in \mathbb{C}_n^I$ and $X_3 \in \mathbb{C}_n^T$ [36] | | | |

**Table 2:** *Summary of results related to linear combinations of the form (2)*

are subsets of some particular sets is a matrix such that its spectrum is subset of a particular set. For instance, the problem of idempotency of linear combinations of two commuting idempotent matrices solved in [2] is equivalent to the problem when a linear combination matrix $P = c_1P_1 + c_2P_2$ is a matrix such that $\sigma(P) \subseteq \{1, 0\}$, where matrices $P_1$ and $P_2$ are commuting and diagonalizable, $\sigma(P_1) \subseteq \{1, 0\}$ and $\sigma(P_2) \subseteq \{1, 0\}$.

In this paper, a combinatorial method is developed to handle the problem when a linear combination of the form $X = \sum_{i=1}^{m} c_iX_i$ is a matrix such that $\sigma(X) \subseteq \{\lambda_1, \lambda_2, ..., \lambda_{n_x}\}$, where $c_i \in \mathbb{C} \setminus \{0\}$ and matrices $X_i \in \mathbb{C}_n$, $i = 1, 2, ..., m$, are diagonalizable and mutually commute. This method is based on solving the systems of linear equations whose coefficients are taken from the spectra of matrices $X_i$, $i = 1, 2, ..., m$. By means of the method, the problems of characterizing linear combinations of special types of matrices can be solved easily for linear combinations of finitely many special types of matrices. Furthermore, the method is illustrated by considering the problem, which is one of the open problems left in [36], of characterizing all situations wherein a linear combination of the form (2) is a tripotent matrix when $X_1$ is an involutory matrix and both $X_2$ and $X_3$ are tripotent matrices that mutually commute.

## 2. The method

Let $X_i \in \mathbb{C}_n$, $i = 1, 2, ..., m$, be mutually commuting diagonalizable matrices such that $\sigma(X_i) \subseteq \{\lambda_1^i, \lambda_2^i, ..., \lambda_{n_i}^i\}$, $n_i \leq n$ and $i = 1, 2, ..., m$. The problem considered is to characterize all situations in which linear combination of the form

$$c_1 X_1 + c_2 X_2 + \cdots + c_m X_m = X \tag{3}$$

is a matrix such that $\sigma(X) \subseteq \{\lambda_1, \lambda_2, ..., \lambda_{n_x}\}$, where $n_x \leq n$ and $c_i \in \mathbb{C} \setminus \{0\}$, $i = 1, 2, ..., m$. Since the matrices $X_i$, $i = 1, 2, ..., m$, are diagonalizable and mutually commute, they are simultaneously diagonalizable [27, Theorem 1.3.21], and clearly the matrix $X$ is also simultaneously diagonalizable with the matrices $X_i$, $i = 1, 2, ..., m$. Thus, there is a nonsingular matrix $S \in \mathbb{C}_n$ such that

$$c_1 S^{-1} X_1 S + c_2 S^{-1} X_2 S + \cdots + c_m S^{-1} X_m S = S^{-1} X S,$$

or equivalently

$$c_1 \Lambda_1 + c_2 \Lambda_2 + \cdots + c_m \Lambda_m = \Lambda, \tag{4}$$

where $\Lambda_i$ and $\Lambda$ are the corresponding diagonal matrices of the matrices $X_i$ and $X$, respectively. Without loss of generality, including multiplicities of eigenvalues, the explicit form of the linear combination (4) can be written as follows:

$$c_1 \begin{pmatrix} \lambda_1^1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_{n_1}^1 \end{pmatrix} + \cdots + c_m \begin{pmatrix} \lambda_1^m & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_{n_m}^m \end{pmatrix} = \begin{pmatrix} \lambda_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_{n_x} \end{pmatrix}. \tag{5}$$

It is seen from equation (5) that the characterization of the linear combination (3) is based on solving the systems of linear equations, whose unknowns are the scalars $c_i$; hereafter systems of linear equations will be shortly called linear systems. Therefore, firstly, all possible left-hand sides of linear equations should be determined, and then all linear systems formed by these left-hand sides need to be solved. Note that the coefficients of the unknowns $c_i$ are chosen from the corresponding sets $\sigma(X_i)$, $i = 1, 2, ..., m$, and the right-hand sides of linear equations are chosen from the set $\sigma(X)$. Accordingly, there are $n_i$ different coefficients for every unknown $c_i$, and thus there are $n_1 n_2 \cdots n_m$ possible left-hand sides of linear equations. Linear systems are formed by considering these $n_1 n_2 \cdots n_m$ different left-hand sides first one-by-one, then two-by-two, and by continuing the process finally $m$-by-$m$, and multiple right-hand sides of linear systems which have the same number of linear equations are the same. Since the solutions of linear systems which have more than $m$ linear equations are intersections of the solutions of linear systems which have $m$ linear equations, it is unnecessary to consider such linear systems having more than $m$ linear equations. Hence, to obtain all results of the characterization problem of a linear combination of the form (3) it is sufficient to solve $\binom{n_1n_2\cdots n_m}{1}+\binom{n_1n_2\cdots n_m}{2}+\cdots+\binom{n_1n_2\cdots n_m}{m}$ linear systems. One of the advantages of this method is to give an upper bound for the number of the results that will be obtained before starting the characterization.

It is worth mentioning the effects of some special cases of the spectra $\sigma(X_i)$, $i=1,2,...,m$, on the solutions. When the intersections of the spectra contain more than two elements, linear systems that have a permutation between their solutions arise, and the spectra containing some eigenvalues with their opposite signs lead to the existence of linear systems whose solutions are minus times of each other. Moreover, if all spectra including $\sigma(X)$ contain all eigenvalues with their opposite signs, then there exists a minus times of each linear equation. In that case, in the process of forming of the linear systems it is sufficient to get one of these two left-hand sides, which are minus times of each other. Hence, the numbers of the left-hand sides are halved, and thus the number of linear systems that need to be solved decreases to $\binom{n_1n_2\cdots n_m/2}{1}+\binom{n_1n_2\cdots n_m/2}{2}+\cdots+\binom{n_1n_2\cdots n_m/2}{m}$.

We want to close this section by mentioning the relationship between linear systems and diagonal matrices $\Lambda_i$, $i=1,2,...,m$. Obviously, there are diagonal matrices $\Lambda_i$, $i=1,2,...,m$, corresponding to each linear system. For instance, corresponding diagonal matrices of the linear system

$$\begin{pmatrix} \lambda_1^1 & \lambda_3^2 & \cdots & \lambda_8^m \\ \lambda_2^1 & \lambda_5^2 & \cdots & \lambda_{13}^m \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_1 & \cdots & \lambda_{n_x} \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{n_x} \end{pmatrix}$$

are $\Lambda_1=\lambda_1^1 I_{r_1}\oplus\lambda_2^1 I_{r_2}$, $\Lambda_2=\lambda_3^2 I_{r_1}\oplus\lambda_5^2 I_{r_2}$, $\cdots$, and $\Lambda_m=\lambda_8^m I_{r_1}\oplus\lambda_{13}^m I_{r_2}$, where the symbol $\oplus$ denotes the direct sum of matrices. In addition, as pointed out in the previous paragraph, it is sufficient to get one of the two left-hand sides, which are minus times of each other, while forming linear systems in the case when all the spectra contain all the eigenvalues with their opposite signs. However, even though one of two left-hand sides is considered while forming linear systems, assuming that there are also the corresponding blocks of the left-hand sides which are not considered in the corresponding diagonal matrices of linear systems makes us obtain the results for more general matrices.

## 3. Tripotency of linear combinations of an involutory matrix and two tripotent matrices that mutually commute

In this section, the problem of characterizing the tripotency of linear combinations of an involutory matrix and two tripotent matrices that mutually commute is considered via the method explained in the previous section.

### 3.1. Preliminaries

Let $T_1\in\mathbb{C}_n^I$ and $T_2$, $T_3\in\mathbb{C}_n^T$ be nonzero matrices that mutually commute. It is known that $\sigma(T_1)\subseteq\{1,-1\}$, $\sigma(T_2)\subseteq\{1,-1,0\}$, and $\sigma(T_3)\subseteq\{1,-1,0\}$ [16, Proposition 5.5.21]. Let us consider the problem of characterizing the tripotency of linear combination of the form

$$c_1T_1 + c_2T_2 + c_3T_3 = T, \tag{6}$$

where $c_1$, $c_2$, and $c_3$ are nonzero complex scalars. In other words, the problem tackled here is to characterize all situations in which a linear combination of the form (6) is a matrix such that $\sigma(T) \subseteq \{1, -1, 0\}$. Accordingly, there are $2 \cdot 3 \cdot 3 = 18$ possible left-hand sides. However, since all spectra contain all eigenvalues with their opposite signs, it is sufficient to consider $\frac{18}{2} = 9$ left-hand sides. Thus, there are $\binom{9}{1} + \binom{9}{2} + \binom{9}{3} = 129$ linear systems that need to be solved. These linear systems are in the forms $Ac = b_i$, $i = 1, 2, 3$, where $A$ is the coefficient matrix of appropriate size, $c = \begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix}^T$, and $b_i$, $i = 1, 2, 3$, are the multiple right-hand sides such that

$$\begin{aligned} b_1 &= \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}, \\ b_2 &= \begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 \end{pmatrix}, \\ b_3 &= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 \end{pmatrix}. \end{aligned}$$

All possible coefficient matrices of linear systems are listed in Table 3.

| | | | |
|---|---|---|---|
| 1) $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ | 2) $\begin{pmatrix} -1 & 1 & 1 \end{pmatrix}$ | 3) $\begin{pmatrix} 1 & -1 & 1 \end{pmatrix}$ | 4) $\begin{pmatrix} -1 & -1 & 1 \end{pmatrix}$ |
| 5) $\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}$ | 6) $\begin{pmatrix} -1 & 0 & 1 \end{pmatrix}$ | 7) $\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$ | 8) $\begin{pmatrix} -1 & 1 & 0 \end{pmatrix}$ |
| 9) $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$ | 10) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$ | 11) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$ | 12) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}$ |
| 13) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ | 14) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ | 15) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ | 16) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ |
| 17) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ | 18) $\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$ | 19) $\begin{pmatrix} -1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}$ | 20) $\begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ |
| 21) $\begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ | 22) $\begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ | 23) $\begin{pmatrix} -1 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ | 24) $\begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ |
| 25) $\begin{pmatrix} 1 & -1 & 1 \\ -1 & -1 & 1 \end{pmatrix}$ | 26) $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ | 27) $\begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ | 28) $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ |
| 29) $\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ | 30) $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ | 31) $\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ | 32) $\begin{pmatrix} -1 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ |
| 33) $\begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ | 34) $\begin{pmatrix} -1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ | 35) $\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ | 36) $\begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ |
| 37) $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ | 38) $\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ | 39) $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ | 40) $\begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ |
| 41) $\begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ | 42) $\begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ | 43) $\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$ | 44) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ |
| 45) $\begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ | 46) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$ | 47) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}$ | 48) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ |
| 49) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ | 50) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ | 51) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ | 52) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ |
| 53) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & -1 & 1 \end{pmatrix}$ | 54) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ | 55) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ | 56) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ |
| 57) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ | 58) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ | 59) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ | 60) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ |
| 61) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ | 62) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ | 63) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ | 64) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ |
| 65) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ | 66) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ | 67) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ | 68) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ |
| 69) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ | 70) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ | 71) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$ | 72) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ |
| 73) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ | 74) $\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & -1 & 1 \end{pmatrix}$ | 75) $\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ | 76) $\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ |
| 77) $\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ | 78) $\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ | 79) $\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ | 80) $\begin{pmatrix} -1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ |
| 81) $\begin{pmatrix} -1 & 1 & 1 \\ -1 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ | 82) $\begin{pmatrix} -1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ | 83) $\begin{pmatrix} -1 & 1 & 1 \\ -1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ | 84) $\begin{pmatrix} -1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ |

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|---|---|---|---|
| 85) $\begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ | 86) $\begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ | 87) $\begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ | 88) $\begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ |
| 89) $\begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ | 90) $\begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ | 91) $\begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ | 92) $\begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$ |
| 93) $\begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ | 94) $\begin{pmatrix} -1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ | 95) $\begin{pmatrix} 1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ | 96) $\begin{pmatrix} 1 & -1 & 1 \\ -1 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ |
| 97) $\begin{pmatrix} 1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ | 98) $\begin{pmatrix} 1 & -1 & 1 \\ -1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ | 99) $\begin{pmatrix} 1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ | 100) $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ |
| 101) $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ | 102) $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ | 103) $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ | 104) $\begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ |
| 105) $\begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ | 106) $\begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ | 107) $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$ | 108) $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ |
| 109) $\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ | 110) $\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ | 111) $\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ | 112) $\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ |
| 113) $\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ | 114) $\begin{pmatrix} -1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ | 115) $\begin{pmatrix} -1 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ | 116) $\begin{pmatrix} -1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ |
| 117) $\begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$ | 118) $\begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ | 119) $\begin{pmatrix} -1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ | 120) $\begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ |
| 121) $\begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ | 122) $\begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ | 123) $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$ | 124) $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ |
| 125) $\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ | 126) $\begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & -1 & 0 \end{pmatrix}$ | 127) $\begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ | 128) $\begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ |
| 129) $\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ | | | |

**Table 3:** *List of coefficient matrices of linear systems*

Furthermore, as pointed out in the last paragraph of Section 2, it is assumed that the corresponding blocks of the left-hand sides which are not considered are also included in the corresponding diagonal matrices of each linear system listed in Table 3. For instance, the corresponding diagonal matrices of the linear system 48) are

$$\begin{aligned}
\Lambda_1 &= I_{r_1} \oplus -I_{r_2} \oplus -I_{r_3} \oplus I_{r_4} \oplus I_{r_5} \oplus -I_{r_6}, \\
\Lambda_2 &= I_{r_1} \oplus -I_{r_2} \oplus I_{r_3} \oplus -I_{r_4} \oplus \mathbf{0}_{r_5} \oplus \mathbf{0}_{r_6}, \\
\Lambda_3 &= I_{r_1} \oplus -I_{r_2} \oplus I_{r_3} \oplus -I_{r_4} \oplus I_{r_5} \oplus -I_{r_6},
\end{aligned}$$

where $\sum_{i=1}^{6} r_i = n$.

Direct calculations show that the matrix $T$ of the form (6) is tripotent if and only if

$$\begin{aligned}
&(c_1^3T_1^3 - c_1T_1) + (c_2^3T_2^3 - c_2T_2) + (c_3^3T_3^3 - c_3T_3) \\
&\quad + c_1^2c_2(T_1^2T_2 + T_2T_1^2 + T_1T_2T_1) + c_1^2c_3(T_1^2T_3 + T_3T_1^2 + T_1T_3T_1) \\
&\quad + c_1c_2^2(T_1T_2^2 + T_2^2T_1 + T_2T_1T_2) + c_2^2c_3(T_3T_2^2 + T_2^2T_3 + T_2T_3T_2) \\
&\quad + c_1c_3^2(T_1T_3^2 + T_3^2T_1 + T_3T_1T_3) + c_2c_3^2(T_2T_3^2 + T_3^2T_2 + T_3T_2T_3) \\
&\quad + c_1c_2c_3(T_1T_2T_3 + T_1T_3T_2 + T_2T_3T_1 + T_2T_1T_3 + T_3T_1T_2 + T_3T_2T_1) = \mathbf{0}.
\end{aligned} \tag{7}$$

Under the assumption of mutual commutativity, if $T_i \in \mathbb{C}_n^I$, $i = 1, 2, 3$, then equation

(7) reduces to

$$(c_1^3 - c_1 + 3c_1c_2^2 + 3c_1c_3^2)T_1 + (c_2^3 - c_2 + 3c_2c_1^2 + 3c_2c_3^2)T_2 + (c_3^3 - c_3 + 3c_3c_1^2 + 3c_3c_2^2)T_3 + 6c_1c_2c_3T_1T_2T_3 = \mathbf{0}, \quad (8)$$

if $T_1$ and $T_2 \in \mathbb{C}_n^I$ and $T_3$ is a singular tripotent matrix, then equation (7) reduces to

$$(c_1^3 - c_1 + 3c_1c_2^2)T_1 + (c_2^3 - c_2 + 3c_2c_1^2)T_2 + (c_3^3 - c_3 + 3c_3c_1^2 + 3c_3c_2^2)T_3 + 3c_1c_3^2T_1{T_3}^2 + 3c_2c_3^2T_2{T_3}^2 + 6c_1c_2c_3T_1T_2T_3 = \mathbf{0}, \quad (9)$$

and if $T_1$ and $T_3 \in \mathbb{C}_n^I$ and $T_2$ is a singular tripotent matrix, then equation (7) reduces to

$$(c_1^3 - c_1 + 3c_1c_3^2)T_1 + (c_2^3 - c_2 + 3c_2c_1^2 + 3c_2c_3^2)T_2 + (c_3^3 - c_3 + 3c_3c_1^2)T_3 + 3c_1c_2^2T_1{T_2}^2 + 3c_3c_2^2T_3{T_2}^2 + 6c_1c_2c_3T_1T_2T_3 = \mathbf{0}, \quad (10)$$

and if $T_1 \in \mathbb{C}_n^I$ and $T_2$ and $T_3$ are singular tripotent matrices, then equation (7) reduces to

$$(c_1^3 - c_1)T_1 + (c_2^3 - c_2 + 3c_2c_1^2)T_2 + (c_3^3 - c_3 + 3c_3c_1^2)T_3 + 3c_1c_2^2T_1{T_2}^2 + 3c_1c_3^2T_1{T_3}^2 + 3c_2c_3^2T_2{T_3}^2 + 3c_3c_2^2T_3{T_2}^2 + 6c_1c_2c_3T_1T_2T_3 = \mathbf{0}. \quad (11)$$

Finally, let us introduce the symbol $|\cdot|$ that will be used to express the results, which are minus times of each other, all together in a single item of a theorem. This symbol is used in both the scalars $c_i$ and the matrices $T_i$, $i = 1, 2, 3$, and it denotes neither the absolute value of the scalars nor the determinants of the matrices. However, this symbol acts as a real absolute value function. Namely, $|c_i|_i := \pm c_i$ and $|T_i|_i := \pm T_i$, $i = 1, 2, 3$. Nevertheless, the scalars $c_i$ and the matrices $T_i$, which are indexed with the same number, are positive (or negative) at the same time. To illustrate, let us consider the following expression

$$(|c_1|, |c_2|, |c_3|) \in \{(3, -1, 4), (-3, 1, -4)\} \text{ and } |T_1| - |T_2| + 2|T_3| - 3|T_1||T_2||T_3| = \mathbf{0}. \quad (12)$$

Expression (12) consists of the combination of the results listed in Table 4.

| | |
|---|---|
| 1) $(c_1, c_2, c_3) \in \{(3, -1, 4), (-3, 1, -4)\}$ and $T_1 - T_2 + 2T_3 - 3T_1T_2T_3 = \mathbf{0}$ | 2) $(-c_1, -c_2, -c_3) \in \{(3, -1, 4), (-3, 1, -4)\}$ and $-T_1 + T_2 - 2T_3 + 3T_1T_2T_3 = \mathbf{0}$ |
| 3) $(-c_1, c_2, c_3) \in \{(3, -1, 4), (-3, 1, -4)\}$ and $-T_1 - T_2 + 2T_3 + 3T_1T_2T_3 = \mathbf{0}$ | 4) $(c_1, -c_2, -c_3) \in \{(3, -1, 4), (-3, 1, -4)\}$ and $T_1 + T_2 - 2T_3 - 3T_1T_2T_3 = \mathbf{0}$ |
| 5) $(c_1, -c_2, c_3) \in \{(3, -1, 4), (-3, 1, -4)\}$ and $T_1 + T_2 + 2T_3 + 3T_1T_2T_3 = \mathbf{0}$ | 6) $(-c_1, c_2, -c_3) \in \{(3, -1, 4), (-3, 1, -4)\}$ and $-T_1 - T_2 - 2T_3 - 3T_1T_2T_3 = \mathbf{0}$ |
| 7) $(c_1, c_2, -c_3) \in \{(3, -1, 4), (-3, 1, -4)\}$ and $T_1 - T_2 - 2T_3 + 3T_1T_2T_3 = \mathbf{0}$ | 8) $(-c_1, -c_2, c_3) \in \{(3, -1, 4), (-3, 1, -4)\}$ and $-T_1 + T_2 + 2T_3 - 3T_1T_2T_3 = \mathbf{0}$ |

**Table 4:** *The results contained in expression* (12)

Note that the results 1), 3), 5), and 7) are the same as the results 2), 4), 6), and 8), respectively.

So far it is enough to give the main results.

### 3.2. Results

As we have already pointed out, the main results deal with the tripotency of linear combinations of the form (6), where $T_1$ is an involutory matrix and both $T_2$ and $T_3$ are tripotent matrices that mutually commute. The results have been presented for three separate situations:

(i) all of the matrices $T_1, T_2$, and $T_3$ are involutory,

(ii) either the matrix $T_2$ or $T_3$ is a singular tripotent matrix, while the rest are involutory,

(iii) both the matrices $T_2$ and $T_3$ are singular tripotent matrices, while the matrix $T_1$ is involutory.

In fact, the results for situations (i) and (ii) have also been given by Xu and Xu in [36]. However, Kişi stated in [28] that there are some missing results in [36], and these missing results are also included in this note.

Although the results have been presented in three separate theorems, their proofs will be given together.

**Theorem 1.** *Let $T_1, T_2$, and $T_3 \in \mathbb{C}_n$ be involutory matrices that mutually commute, and let $T$ be a linear combination of the form* (6) *with $c_1, c_2, c_3 \in \mathbb{C} \setminus \{0\}$. Then, $T$ is tripotent if and only if*

*(a)* $|c_1| + |c_2| + |c_3| \in \{1, -1, 0\}$ *and* $|T_1| = |T_2| = |T_3|$,

*(b)* $(c_i + |c_j|, c_k) \in \left\{(\frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2}), (0, 1), (0, -1)\right\}$ *and*

$$T_i = |T_j| \neq \pm T_k,$$

*(c)* $(|c_1|, |c_2|, |c_3|) \in \left\{ \begin{array}{l} (-\frac{1}{2}, -\frac{1}{2}, 1), (\frac{1}{2}, \frac{1}{2}, -1), (\frac{1}{2}, 1, -\frac{1}{2}), (-\frac{1}{2}, -1, \frac{1}{2}) \\ (1, \frac{1}{2}, -\frac{1}{2}), (-1, -\frac{1}{2}, \frac{1}{2}), (1, 1, -1), (-1, -1, 1) \end{array} \right\}$ *and*

$$|T_1| + |T_2| - |T_3| - |T_1|\,|T_2|\,|T_3| = \mathbf{0},$$

*where $i \neq j, i \neq k, j \neq k$, and $i, j, k = 1, 2, 3$.*

**Theorem 2.** *Let $T_1$ and $T_i \in \mathbb{C}_n$ be involutory matrices and $T_j \in \mathbb{C}_n$ a nonzero singular tripotent matrix that all mutually commute, and let $T$ be a linear combination of the form* (6) *with $c_1, c_2, c_3 \in \mathbb{C} \setminus \{0\}$. Then, $T$ is tripotent if and only if*

*(a′)* $(c_1 + |c_i|, |c_j|) \in \{(1, -1), (-1, 1), (0, 1), (0, -1), (-1, 2), (1, -2)\}$,

$$T_1 = |T_i| \neq \pm T_j, \text{ and } T_1\,|T_j| = T_j^2,$$

*(b′)* $(|c_1| + c_3, |c_2| + 2c_3) \in \{(1, -1), (-1, 1), (0, 1), (0, -1), (-1, 2), (1, -2)\}$ *and*

$$|T_1| + 2\,|T_2| = T_3,$$

$(c')$ $(|c_1|+\frac{1}{2}c_3, |c_2|+\frac{1}{2}c_3) \in \left\{ \begin{array}{l} (0,0),(-\frac{1}{2},\frac{1}{2}),(\frac{1}{2},-\frac{1}{2}),(\frac{1}{2},\frac{1}{2}),(0,1), \\ (1,0),(-\frac{1}{2},-\frac{1}{2}),(-1,0),(0,-1) \end{array} \right\}$ *and*

$$\frac{1}{2}|T_1| + \frac{1}{2}|T_2| = T_3,$$

$(d')$ $(|c_1|, |c_i|, |c_j|) \in \left\{ \begin{array}{l} (-\frac{1}{2},-1,\frac{3}{2}),(\frac{1}{2},1,-\frac{3}{2}),(-\frac{1}{2},1,-\frac{1}{2}),(\frac{1}{2},-1,\frac{1}{2}), \\ (\frac{1}{2},1,-\frac{1}{2}),(-\frac{1}{2},-1,\frac{1}{2}),(\frac{1}{2},2,-\frac{3}{2}),(-\frac{1}{2},-2,\frac{3}{2}), \\ (1,1,-1),(-1,-1,1),(1,2,-2),(-1,-2,2) \end{array} \right\}$ *and*

$$-2|T_1| - 5|T_i| + 2|T_j| + (3|T_j| - |T_1|)T_i^2 + 3|T_1||T_i||T_j| = \mathbf{0},$$

$(e')$ $(c_1 + |c_i|, c_j) \in \{(0,1),(0,-1)\}$ *and* $T_1 = |T_i| \neq T_j$;

$(f')$ $(|c_1|, |c_i|, |c_j|) \in \left\{ \begin{array}{l} (\frac{1}{4},-\frac{1}{2},\frac{1}{4}),(-\frac{1}{4},\frac{1}{2},-\frac{1}{4}),(-\frac{1}{4},-\frac{1}{2},\frac{3}{4}),(\frac{1}{4},\frac{1}{2},-\frac{3}{4}), \\ (\frac{3}{4},-\frac{1}{2},-\frac{1}{4}),(-\frac{3}{4},\frac{1}{2},\frac{1}{4}),(\frac{1}{4},\frac{1}{2},\frac{1}{4}),(-\frac{1}{4},-\frac{1}{2},\frac{1}{4}), \\ (-\frac{1}{4},\frac{1}{2},\frac{3}{4}),(\frac{1}{4},-\frac{1}{2},-\frac{3}{4}),(\frac{3}{4},\frac{1}{2},-\frac{1}{4}),(-\frac{3}{4},-\frac{1}{2},\frac{1}{4}), \\ (-\frac{1}{2},1,\frac{1}{2}),(\frac{1}{2},-1,-\frac{1}{2}),(\frac{1}{2},1,-\frac{1}{2}),(-\frac{1}{2},-1,\frac{1}{2}) \end{array} \right\}$ *and*

$$-|T_1| + |T_i| - |T_j| + (|T_1| + |T_j|)T_i^2 + |T_1||T_i||T_j| = \mathbf{0},$$

$(g')$ $(|c_1|, |c_i|, |c_j|) \in \left\{ \begin{array}{l} (\frac{3}{2},-\frac{1}{2},-1),(-\frac{3}{2},\frac{1}{2},1),(-\frac{1}{2},-\frac{1}{2},1),(\frac{1}{2},\frac{1}{2},-1), \\ (-\frac{1}{2},\frac{1}{2},1),(\frac{1}{2},-\frac{1}{2},-1),(-\frac{3}{2},\frac{1}{2},2),(\frac{3}{2},-\frac{1}{2},-2), \\ (-1,1,1),(1,-1,-1),(-2,1,2),(2,-1,-2) \end{array} \right\}$ *and*

$$-|T_1| + |T_i| + 3|T_j| + (|T_i| - 2|T_1|)T_j^2 - 2|T_1||T_i||T| = \mathbf{0},$$

$(h')$ $(|c_1|, |c_i|, |c_j|) \in \left\{ \begin{array}{l} (\frac{1}{2},-1,\frac{1}{2}),(-\frac{1}{2},1,-\frac{1}{2}),(-\frac{1}{2},1,\frac{1}{2}),(\frac{1}{2},-1,-\frac{1}{2}), \\ (\frac{1}{2},1,-\frac{1}{2}),(-\frac{1}{2},-1,\frac{1}{2}),(-\frac{1}{2},2,-\frac{1}{2}),(\frac{1}{2},-2,\frac{1}{2}) \end{array} \right\}$ *and*

$$-3|T_i| + 2(|T_1| + |T_j|)T_i^2 - |T_1||T_i||T_j| = \mathbf{0},$$

*where* $i \neq j$ *and* $i, j = 2, 3$.

**Theorem 3.** *Let* $T_1 \in \mathbb{C}_n$ *be an involutory matrix and* $T_2$ *and* $T_3 \in \mathbb{C}_n$ *two nonzero singular tripotent matrices that all mutually commute. Let* $T$ *be a linear combination of the form* (6) *with* $c_1, c_2, c_3 \in \mathbb{C} \setminus \{0\}$. *Then,* $T$ *is tripotent if and only if*

$(a'')$ $(|c_1|, |c_2| + c_3) \in \{(1,-1),(-1,1),(1,0),(-1,0),(-1,2),(1,-2)\}$,

$$|T_2| = T_3 \neq \pm T_1, \text{ and } |T_1||T_2| = T_2^2,$$

$(b'')$ $(|c_1| + c_3, |c_2| + c_3) \in \left\{ \begin{array}{l} (0,0),(0,1),(0,-1),(1,-1),(1,0), \\ (-1,2),(-1,1),(1,-2),(-1,0) \end{array} \right\}$ *and*

$$|T_1| + |T_2| = T_3,$$

$(c'')$ $(c_1, c_2 + |c_3|) \in \{(1,0), (-1,0)\}$ *and*

$$T_2 = |T_3| \neq \pm T_1,$$

$(d'')$ $$(|c_1|, |c_i|, |c_j|) \in \left\{ \begin{array}{l} (1,-\frac{1}{2},-\frac{1}{2}), (-1,\frac{1}{2},\frac{1}{2}), (-1,-\frac{1}{2},\frac{3}{2}), (1,\frac{1}{2},-\frac{3}{2}), \\ (1,\frac{1}{2},-\frac{1}{2}), (-1,-\frac{1}{2},\frac{1}{2}), (-1,\frac{1}{2},\frac{3}{2}), (1,-\frac{1}{2},-\frac{3}{2}), \\ (1,1,-1), (-1,-1,1), (-1,1,1), (1,-1,-1) \end{array} \right\} \textit{ and}$$

$$3|T_i| - 17|T_j| + (2|T_1| - 3|T_j|)T_i^2 + (18|T_1| + 9|T_i|)T_j^2 - 12|T_1||T_i||T_j| = \mathbf{0},$$

$(e'')$ $$(|c_1|, |c_i|, |c_j|) \in \left\{ \begin{array}{l} (2,-1,-1), (-2,1,1), (-1,1,1), (1,-1,-1), \\ (-2,1,2), (2,-1,-2), (-2,2,1), (2,-2,-1), \\ (-3,2,2), (3,-2,-2) \end{array} \right\} \textit{ and}$$

$$-2|T_1| + 5(|T_i| + |T_j|) + (2|T_j| - 3|T_1|)T_i^2 + (2|T_i| - 3|T_1|)T_j^2 - 6|T_1||T_i||T_j| = \mathbf{0},$$

$(f'')$ $$(|c_1|, |c_i|, |c_j|) \in \left\{ \begin{array}{l} (-1,-1,2), (1,1,-2), (-2,-1,3), (2,1,-3), \\ (1,1,-1), (-1,-1,1), (2,1,-2), (-2,-1,2), \\ (2,2,-3), (-2,-2,3), (1,2,-2), (-1,-2,2), \\ (3,2,-4), (-3,-2,4) \end{array} \right\} \textit{ and}$$

$$|T_1| + 2|T_i| - 10|T_j| + (|T_1| - \frac{3}{2}|T_j|)T_i^2 + (9|T_1| - \frac{9}{2}|T_i|)T_j^2 - 6|T_1||T_i||T_j| = \mathbf{0},$$

$(g'')$ $$(|c_1|, |c_i|, |c_j|) \in \left\{ \begin{array}{l} (-1,-1,2), (1,1,-2), (1,1,-1), (-1,-1,1), \\ (-1,1,1), (1,-1,-1), (1,2,-2), (-1,-2,2) \end{array} \right\} \textit{ and}$$

$$|T_i| - 4|T_j| + (|T_1| - 2|T_j|)T_i^2 + (4|T_1| + 4|T_i|)T_j^2 - 4|T_1||T_i||T_j| = \mathbf{0},$$

$(h'')$ $$(|c_1|, |c_i|, |c_j|) \in \left\{ \begin{array}{l} (-\frac{1}{3},\frac{2}{3},-\frac{1}{3}), (\frac{1}{3},-\frac{2}{3},\frac{1}{3}), (-\frac{1}{3},-\frac{1}{3},\frac{2}{3}), (\frac{1}{3},\frac{1}{3},-\frac{2}{3}), \\ (-\frac{2}{3},\frac{1}{3},\frac{1}{3}), (\frac{2}{3},-\frac{1}{3},-\frac{1}{3}), (\frac{1}{3},\frac{1}{3},\frac{1}{3}), (-\frac{1}{3},-\frac{1}{3},-\frac{1}{3}), \\ (\frac{2}{3},-\frac{1}{3},\frac{2}{3}), (-\frac{2}{3},\frac{1}{3},-\frac{2}{3}), (-\frac{1}{3},\frac{2}{3},\frac{2}{3}), (\frac{1}{3},-\frac{2}{3},-\frac{2}{3}), \\ (\frac{1}{3},-\frac{2}{3},\frac{4}{3}), (-\frac{1}{3},\frac{2}{3},-\frac{4}{3}), (\frac{2}{3},\frac{2}{3},-\frac{1}{3}), (-\frac{2}{3},-\frac{2}{3},\frac{1}{3}), \\ (\frac{1}{3},\frac{4}{3},-\frac{2}{3}), (-\frac{1}{3},-\frac{4}{3},\frac{2}{3}) \end{array} \right\} \textit{ and}$$

$$-8|T_1| + 4|T_i| - 5|T_j| + (12|T_1| + 12|T_j|)T_i^2 + (3|T_1| - 6|T_j|)T_i^2 - 12|T_1||T_i||T_j| = \mathbf{0},$$

$(i'')$ $$(|c_1|, |c_i|, |c_j|) \in \left\{ \begin{array}{l} (1,-2,1), (-1,2,-1), (1,-3,2), (-1,3,-2), \\ (1,-1,1), (-1,1-,1), (-1,3,-1), (1,-3,1), \\ (1,-2,2), (-1,2,-2), (-1,4,-2), (1,-4,2) \end{array} \right\} \textit{ and}$$

$$11|T_i| - |T_j| - (9|T_1| - 9|T_j|)T_i^2 + (3|T_1| - |T_i|)T_j^2 - 6|T_1||T_i||T_j| = \mathbf{0},$$

$(j'')$ $$(|c_1|, |c_i|, |c_j|) \in \left\{ \begin{array}{l} (-\frac{1}{2},\frac{1}{2},\frac{1}{2}), (\frac{1}{2},-\frac{1}{2},-\frac{1}{2}), (-\frac{1}{2},\frac{3}{2},\frac{1}{2}), (\frac{1}{2},-\frac{3}{2},-\frac{1}{2}), \\ (-\frac{1}{2},-\frac{1}{2},\frac{1}{2}), (\frac{1}{2},\frac{1}{2},-\frac{1}{2}), (\frac{1}{2},-\frac{1}{2},\frac{1}{2}), (-\frac{1}{2},\frac{1}{2},-\frac{1}{2}), \\ (\frac{1}{2},\frac{1}{2},\frac{1}{2}), (-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}), (\frac{1}{2},-\frac{3}{2},\frac{1}{2}), (-\frac{1}{2},\frac{3}{2},-\frac{1}{2}) \end{array} \right\} \textit{ and}$$

$$|T_1| + (|T_j| - |T_1|)T_i^2 + (|T_i| - |T_1|)T_j^2 - 2|T_1||T_i||T_j| = \mathbf{0},$$

$(k'')$ $(|c_1|, |c_i|, |c_j|) \in \left\{ \begin{array}{l} (1,-1,-1), (-1,1,1), (-1,2,1), (1,-2,-1), \\ (-1,1,2), (1,-1,-2), (-1,2,2), (1,-2,-2) \end{array} \right\}$ *and*

$$|T_i| + |T_j| + (|T_j| - |T_1|)T_i^2 + (|T_i| - |T_1|)T_j^2 - 2\,|T_1|\,|T_i|\,|T_j| = \mathbf{0},$$

*where* $i \neq j$ *and* $i, j = 2, 3$.

**Proof**. Sufficiency of the conditions in all items of Theorems 1–3 follows by direct verification of criteria (4)–(7).

In the proof of necessity, we consider all of the linear systems listed in Table 3, except linear systems 5)–9), 36), 39), 42)–45), 122), and 129), since either of the matrices $T_2$ or $T_3$ is a zero matrix or both of them are zero matrices in these cases.

Solving linear system 1) gives $c_1+c_2+c_3 \in \{1,-1,0\}$, and $T_1 = T_2 = T_3$. Similar results are obtained from linear systems 2), 3), and 4). Combining all of these results leads to item $(a)$ of Theorem 1.

Solving linear system 10) gives

$$(c_1 + c_2, c_3) \in \left\{ (\frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2}), (0,1), (0,-1) \right\}$$

and $T_1 = T_2 \neq \pm T_3$. Similar results are obtained from linear systems 11), 12), 18), 19), and 25). Combining all of these results leads to item $(b)$ of Theorem 1.

Solving linear system 46) gives

$$(c_1, c_2, c_3) \in \left\{ \begin{array}{l} (-\frac{1}{2}, -\frac{1}{2}, 1), (\frac{1}{2}, \frac{1}{2}, -1), (\frac{1}{2}, 1, -\frac{1}{2}), (-\frac{1}{2}, -1, \frac{1}{2}) \\ (1, \frac{1}{2}, -\frac{1}{2}), (-1, -\frac{1}{2}, \frac{1}{2}), (1,1,-1), (-1,-1,1) \end{array} \right\},$$

and putting the result $(\frac{1}{2}, \frac{1}{2}, -1)$ in equation (8) gives $T_1 + T_2 - T_3 - T_1T_2T_3 = \mathbf{0}$. Similar results are obtained from systems 47), 53), and 74). Combining all of these results leads to item $(c)$ of Theorem 1.

Solving linear system 13) gives

$$(c_1 + c_2, c_3) \in \{(1,-1), (-1,1), (0,1), (0,-1), (-1,2), (1,-2)\},$$

$T_1 = T_2 \neq \pm T_3$, and $T_1T_3 = T_3^2$. Similar results are obtained from linear systems 15), 21), 23), 26), 29), 32), and 33). Combining all of these results leads to item $(a')$ of Theorem 2.

Solving linear system 14) gives

$$(c_1 - c_3, c_2 + 2c_3) \in \{(1,-1), (-1,1), (0,1), (0,-1), (-1,2), (1,-2)\}$$

and $-T_1 + 2T_2 = T_3$. Similar results are obtained from linear systems 20), 27), and 31). Combining all of these results leads to item $(b')$ of Theorem 2.

Solving the linear system 16) gives

$$(c_1 + \frac{1}{2}c_3, c_2 + \frac{1}{2}c_3) \in \left\{ \begin{array}{l} (0,0), (-\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, \frac{1}{2}), (0,1), \\ (1,0), (-\frac{1}{2}, -\frac{1}{2}), (-1,0), (0,-1) \end{array} \right\}$$

and $\frac{1}{2}T_1 + \frac{1}{2}T_2 = T_3$. Similar results are obtained from the linear systems 22), 28), and 34). Combining all of these results leads to the item $(c')$ of Theorem 2.

Solving linear system 48) gives

$$(c_1, c_2, c_3) \in \left\{ \begin{array}{l} (-\frac{1}{2}, -1, \frac{3}{2}), (\frac{1}{2}, 1, -\frac{3}{2}), (-\frac{1}{2}, 1, -\frac{1}{2}), (\frac{1}{2}, -1, \frac{1}{2}), \\ (\frac{1}{2}, 1, -\frac{1}{2}), (-\frac{1}{2}, -1, \frac{1}{2}), (\frac{1}{2}, 2, -\frac{3}{2}), (-\frac{1}{2}, -2, \frac{3}{2}), \\ (1, 1, -1), (-1, -1, 1), (1, 2, -2), (-1, -2, 2) \end{array} \right\},$$

and putting the result $(-\frac{1}{2}, -1, \frac{3}{2})$ in the equation (10) gives $-2T_1 - 5T_2 + 2T_3 + (3T_3 - T_1)T_2^2 + 3T_1T_2T_3 = \mathbf{0}$. Similar results are obtained from linear systems 49), 50), 51), 95), 96), 97), and 98). Combining all of these results leads to e item $(d')$ of Theorem 2.

Solving linear system 54) gives $(c_1+c_3, c_2) \in \{(0, 1), (0, -1)\}$ and $T_1 = T_3 \neq \pm T_2$. Similar results are obtained from linear systems 61), 78), and 81). Combining all of these results leads to item $(e')$ of Theorem 2.

Solving linear system 55) gives

$$(c_1, c_2, c_3) \in \left\{ \begin{array}{l} (\frac{1}{4}, -\frac{1}{2}, \frac{1}{4}), (-\frac{1}{4}, \frac{1}{2}, -\frac{1}{4}), (-\frac{1}{4}, -\frac{1}{2}, \frac{3}{4}), (\frac{1}{4}, \frac{1}{2}, -\frac{3}{4}), \\ (\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}), (-\frac{3}{4}, \frac{1}{2}, \frac{1}{4}), (\frac{1}{4}, \frac{1}{2}, \frac{1}{4}), (-\frac{1}{4}, -\frac{1}{2}, \frac{1}{4}), \\ (-\frac{1}{4}, \frac{1}{2}, \frac{3}{4}), (\frac{1}{4}, -\frac{1}{2}, -\frac{3}{4}), (\frac{3}{4}, \frac{1}{2}, -\frac{1}{4}), (-\frac{3}{4}, -\frac{1}{2}, \frac{1}{4}), \\ (-\frac{1}{2}, 1, \frac{1}{2}), (\frac{1}{2}, -1, -\frac{1}{2}), (\frac{1}{2}, 1, -\frac{1}{2}), (-\frac{1}{2}, -1, \frac{1}{2}) \end{array} \right\},$$

and putting the result $(\frac{1}{4}, -\frac{1}{2}, \frac{1}{4})$ in equation (10) gives $-T_1 + T_2 - T_3 + (T_1 + T_3)T_2^2 + T_1T_2T_3 = \mathbf{0}$. Similar results are obtained from linear systems 62), 77), and 80). Combining all of these results leads to the item $(f')$ of Theorem 2.

Solving linear system 56) gives

$$(c_1, c_2, c_3) \in \left\{ \begin{array}{l} (\frac{3}{2}, -\frac{1}{2}, -1), (-\frac{3}{2}, \frac{1}{2}, 1), (-\frac{1}{2}, -\frac{1}{2}, 1), (\frac{1}{2}, \frac{1}{2}, -1), \\ (-\frac{1}{2}, \frac{1}{2}, 1), (\frac{1}{2}, -\frac{1}{2}, -1), (-\frac{3}{2}, \frac{1}{2}, 2), (\frac{3}{2}, -\frac{1}{2}, -2), \\ (-1, 1, 1), (1, -1, -1), (-2, 1, 2), (2, -1, -2) \end{array} \right\},$$

and putting the result $(-2, 1, 2)$ in equation (9) gives $-T_1+T_2+3T_3+(T_2-2T_1)T_3^2 - 2T_1T_2T_3 = \mathbf{0}$. Similar results are obtained from linear systems 57), 59), 60), 75), 76), 82), and 83). Combining all of these results leads to item $(g')$ of Theorem 2.

Solving linear system 64) gives

$$(c_1, c_2, c_3) \in \left\{ \begin{array}{l} (\frac{1}{2}, -1, \frac{1}{2}), (-\frac{1}{2}, 1, -\frac{1}{2}), (-\frac{1}{2}, 1, \frac{1}{2}), (\frac{1}{2}, -1, -\frac{1}{2}), \\ (\frac{1}{2}, 1, -\frac{1}{2}), (-\frac{1}{2}, -1, \frac{1}{2}), (-\frac{1}{2}, 2, -\frac{1}{2}), (\frac{1}{2}, -2, \frac{1}{2}) \end{array} \right\},$$

and putting the result $(\frac{1}{2}, -2, \frac{1}{2})$ in equation (10) gives $-3T_2 + 2(T_1 + T_3)T_2^2 - T_1T_2T_3 = \mathbf{0}$. Similar results are obtained from linear systems 71), 85), 92), 100), 107), 110), and 117). Combining all of these results leads to item $(h')$ of Theorem 2.

Solving linear system 17) gives

$$(c_1, c_2 + c_3) \in \{(1, -1), (-1, 1), (1, 0), (-1, 0), (-1, 2), (1, -2)\}$$

$T_2 = T_3 \neq \pm T_1$, and $T_1T_2 = T_2^2$. Similar results are obtained from linear systems 24), 30), and 35). Combining all of these results leads to item $(a'')$ of Theorem 3.

Solving linear system 37) gives

$$(c_1+c_3, c_2-c_3) \in \left\{ \begin{array}{l} (0,0),(0,1),(0,-1),(1,-1),(1,0), \\ (-1,2),(-1,1),(1,-2),(-1,0) \end{array} \right\}$$

and $T_1 - T_2 = T_3$. Similar results are obtained from linear systems 38), 40), and 41). Combining all of these results leads to item $(b'')$ of Theorem 3.

Solving linear system 52) gives $(c_1, c_2+c_3) \in \{(1,0),(-1,0)\}$ and $T_2 = T_3 \neq \pm T_1$. Similar results are obtained from linear system 99). Combining all of these results leads to item $(c'')$ of Theorem 3.

Solving linear system 58) gives

$$(c_1,c_2,c_3) \in \left\{ \begin{array}{l} (1,-\frac{1}{2},-\frac{1}{2}),(-1,\frac{1}{2},\frac{1}{2}),(-1,-\frac{1}{2},\frac{3}{2}),(1,\frac{1}{2},-\frac{3}{2}), \\ (1,\frac{1}{2},-\frac{1}{2}),(-1,-\frac{1}{2},\frac{1}{2}),(-1,\frac{1}{2},\frac{3}{2}),(1,-\frac{1}{2},-\frac{3}{2}), \\ (1,1,-1),(-1,-1,1),(-1,1,1),(1,-1,-1) \end{array} \right\},$$

and putting the result $(1,\frac{1}{2},-\frac{3}{2})$ in equation (11) gives $3T_2 - 17T_3 + (2T_1 - 3T_3)T_2^2 + (18T_1 + 9T_2)T_3^2 - 12T_1T_2T_3 = \mathbf{0}$. Similar results are obtained from linear systems 63), 79), and 84). Combining all of these results leads to item $(d'')$ of Theorem 3.

Solving linear system 65) gives

$$(c_1,c_2,c_3) \in \left\{ \begin{array}{l} (2,-1,-1),(-2,1,1),(-1,1,1),(1,-1,-1), \\ (-2,1,2),(2,-1,-2),(-2,2,1),(2,-2,-1), \\ (-3,2,2),(3,-2,-2) \end{array} \right\},$$

and putting the result $(-3,2,2)$ in equation (11) gives $-2T_1 + 5(T_2+T_3) + (2T_3 - 3T_1)T_2^2 + (2T_2 - 3T_1)T_3^2 - 6T_1T_2T_3 = \mathbf{0}$. Similar results are obtained from linear systems 90), 102), and 114). Combining all of these results leads to item $(e'')$ of Theorem 3.

Solving linear system 66) gives

$$(c_1,c_2,c_3) \in \left\{ \begin{array}{l} (-1,-1,2),(1,1,-2),(-2,-1,3),(2,1,-3), \\ (1,1,-1),(-1,-1,1),(2,1,-2),(-2,-1,2), \\ (2,2,-3),(-2,-2,3),(1,2,-2),(-1,-2,2), \\ (3,2,-4),(-3,-2,4) \end{array} \right\},$$

and putting the result $(2,1,-3)$ in equation (11) gives $T_1 + 2T_2 - 10T_3 + (T_1 - \frac{3}{2}T_3)T_2^2 + (9T_1 - \frac{9}{2}T_2)T_3^2 - 6T_1T_2T_3 = \mathbf{0}$. Similar results are obtained from linear systems 68), 87), 89), 101), 105), 111), and 115). Combining all of these results leads to item $(f'')$ of Theorem 3.

Solving linear system 67) gives

$$(c_1,c_2,c_3) \in \left\{ \begin{array}{l} (-1,-1,2),(1,1,-2),(1,1,-1),(-1,-1,1), \\ (-1,1,1),(1,-1,-1),(1,2,-2),(-1,-2,2) \end{array} \right\},$$

and putting the result $(1,1,-2)$ in equation (11) gives $T_2 - 4T_3 + (T_1 - 2T_3)T_2^2 + (4T_1 + 4T_2)T_3^2 - 4T_1T_2T_3 = \mathbf{0}$. Similar results are obtained from linear systems 72), 91), 94), 103), 109), 116), and 118). Combining all of these results leads to item $(g'')$ of Theorem 3.

Solving linear system 69) gives

$$(c_1, c_2, c_3) \in \left\{ \begin{array}{l} (-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}), (\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}), (-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}), (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}), \\ (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}), \\ (\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}), (-\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}), (-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}), (\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}), \\ (\frac{1}{3}, -\frac{2}{3}, \frac{4}{3}), (-\frac{1}{3}, \frac{2}{3}, -\frac{4}{3}), (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}), (-\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}), \\ (\frac{1}{3}, \frac{4}{3}, -\frac{2}{3}), (-\frac{1}{3}, -\frac{4}{3}, \frac{2}{3}) \end{array} \right\},$$

and putting the result $(\frac{1}{3}, -\frac{2}{3}, \frac{1}{3})$ in equation (11) gives $-8T_1 + 4T_2 - 5T_3 + (12T_1 + 12T_3)T_2^2 + (3T_1 - 6T_3)T_2^2 - 12T_1T_2T_3 = \mathbf{0}$. Similar results are obtained from linear systems 86), 104), and 112). Combining all of these results leads to item $(h'')$ of Theorem 3.

Solving linear system 70) gives

$$(c_1, c_2, c_3) \in \left\{ \begin{array}{l} (1, -2, 1), (-1, 2, -1), (1, -3, 2), (-1, 3, -2), \\ (1, -1, 1), (-1, 1-, 1), (-1, 3, -1), (1, -3, 1), \\ (1, -2, 2), (-1, 2, -2), (-1, 4, -2), (1, -4, 2) \end{array} \right\},$$

and putting the result $(-1, 3, -1)$ in equation (11) gives $11T_2 - T_3 - (9T_1 - 9T_3)T_2^2 + (3T_1 - T_2)T_3^2 - 6T_1T_2T_3 = \mathbf{0}$. Similar results are obtained from linear systems 73), 88), 93), 106), 108), 113), and 119). Combining all of these results leads to item $(i'')$ of Theorem 3.

Solving linear system 120) gives

$$(c_1, c_2, c_3) \in \left\{ \begin{array}{l} (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, \frac{3}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}), \\ (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}), \\ (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, -\frac{3}{2}, \frac{1}{2}), (-\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}) \end{array} \right\},$$

and putting the result $(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ in equation (11) gives $T_1 + (T_3 - T_1)T_2^2 + (T_2 - T_1)T_3^2 - 2T_1T_2T_3 = \mathbf{0}$. Similar results are obtained from linear systems 121), 123), and 126). Combining all of these results leads to item $(j'')$ of Theorem 3.

Solving linear system 124) gives

$$(c_1, c_2, c_3) \in \left\{ \begin{array}{l} (1, -1, -1), (-1, 1, 1), (-1, 2, 1), (1, -2, -1), \\ (-1, 1, 2), (1, -1, -2), (-1, 2, 2), (1, -2, -2) \end{array} \right\},$$

and putting the result $(-1, 1, 1)$ in equation (11) gives $T_2 + T_3 + (T_3 - T_1)T_2^2 + (T_2 - T_1)T_3^2 - 2T_1T_2T_3 = \mathbf{0}$. Similar results are obtained from linear systems 125), 127), and 128). Combining all of these results leads to item $(k'')$ of Theorem 3.

Hence, the proofs of Theorem 1, Theorem 2, and Theorem 3 are completed. □

## 4. Conclusion

The striking feature of this method is that it is very systematic. Therefore, anyone who uses this method for characterizing linear combinations of some special types of matrices requires less intuition than one who uses the methods proposed in the papers in the literature. To show this, let us reconsider, for example, the problem of idempotency of linear combinations of two idempotent matrices handled in [2]. Let $P_1$ and $P_2$ be nonzero idempotent matrices that commute, and let $\Pi_1$ and $\Pi_2$ be their corresponding diagonal matrices, respectively. A linear combination of the form $c_1P_1 + c_2P_2$ is idempotent if and only if

$$(c_1^2 - c_1)P_1 + 2c_1c_2P_1P_2 + (c_2^2 - c_2)P_2 = \mathbf{0}, \tag{13}$$

where $c_1$ and $c_2$ are nonzero complex scalars. It is known that $\sigma(P_1) \subseteq \{1, 0\}$ and $\sigma(P_2) \subseteq \{1, 0\}$, [16, Proposition 5.5.21]. Apparently, there are $2 \cdot 2 = 4$ possible left-hand sides. However, note that one of these four left-hand sides is $0c_1 + 0c_2$. Hence, there are three left-hand sides. Nevertheless, assuming that the corresponding blocks of the left-hand side $0c_1 + 0c_2$ exists in the corresponding diagonal matrices of linear systems makes us obtain the results for more general matrices. In the framework of the previous explanations, we have the following results:

1) $\left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right) \Rightarrow c_1+c_2 \in \{1, 0\}$ and $\Pi_1 = \Pi_2 = I_{r_1} \oplus \mathbf{0}_{r_2} \Rightarrow c_1+c_2 \in \{1, 0\}$ and $P_1 = P_2$;

2) $\left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right) \Rightarrow P_2 = \mathbf{0}$;

3) $\left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right) \Rightarrow P_1 = \mathbf{0}$;

4) $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{smallmatrix}\right) \Rightarrow (c_1, c_2) \in \{(1, -1)\}$, $\Pi_1 = I_{s_1} \oplus I_{s_2} \oplus \mathbf{0}_{s_3}$, and $\Pi_2 = I_{s_1} \oplus \mathbf{0}_{s_2} \oplus \mathbf{0}_{s_3} \Rightarrow (c_1, c_2) \in \{(1, -1)\}$ and $P_1P_2 = P_2$;

5) $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{smallmatrix}\right) \Rightarrow (c_1, c_2) \in \{(-1, 1)\}$, $\Pi_1 = I_{s_1} \oplus \mathbf{0}_{s_2} \oplus \mathbf{0}_{s_3}$, and $\Pi_2 = I_{s_1} \oplus I_{s_2} \oplus \mathbf{0}_{s_3} \Rightarrow (c_1, c_2) \in \{(-1, 1)\}$ and $P_1P_2 = P_1$;

6) $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{smallmatrix}\right) \Rightarrow (c_1, c_2) \in \{(1, 1)\}$, $\Pi_1 = I_{s_1} \oplus \mathbf{0}_{s_2} \oplus \mathbf{0}_{s_3}$, and $\Pi_2 = \mathbf{0}_{s_1} \oplus I_{s_2} \oplus \mathbf{0}_{s_3} \Rightarrow (c_1, c_2) \in \{(1, 1)\}$ and $P_1P_2 = \mathbf{0}$,

where $r_1 + r_2 = n$ and $s_1 + s_2 + s_3 = n$. The results $P_1P_2 = P_2$, $P_1P_2 = P_1$, and $P_1P_2 = \mathbf{0}$ can also be obtained by putting the results $(c_1, c_2) = (1, -1)$, $(c_1, c_2) = (-1, 1)$, and $(c_1, c_2) = (1, 1)$ in equation (13), respectively. Please, note that the results obtained from cases 4), 5), and 6) correspond to the results (ii), (iii), and (i) of the theorem in [2], respectively.

Many open problems, such as quadripotency of linear combinations of two quadripotent matrices that commute, or tripotency of linear combinations of three tripotent matrices that mutually commute, can be easily solved by using this method. The first problem is a generalization of the problems handled in [3, 9], and the number of linear systems that needs to be solved for the characterization is $\binom{15}{1} + \binom{15}{2} = 120$. The second problem is a generalization of the problems handled in this note and in [36], and the number of linear systems that needs to be solved for the characterization is $\binom{13}{1} + \binom{13}{2} + \binom{13}{3} = 376$.

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