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Radii of starlikeness and convexity of Wright functions*

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Abstract. In this paper, our aim is to find the radii of starlikeness and convexity of the normalized Wright functions for three different kinds of normalization. The key tools in the proof of our main results are the Mittag-Leffler expansion for Wright function and properties of real zeros of the Wright function and its derivative. In addition, by using the Euler-Rayleigh inequalities we obtain some tight lower and upper bounds for the radii of starlikeness and convexity of order zero for the normalized Wright functions. The main results of the paper are natural extensions of some known results on classical Bessel functions of the first kind. Some open problems are also proposed, which may be of interest for further research.
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1. Introduction

Special functions are indispensable in many branches of mathematics and applied mathematics. Geometric properties of some special functions were recently examined by many authors (see [1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 20, 23, 24, 25, 26, 27]). However, its origins can be traced to Brown [12] (see also [13, 14]), Kreyszig and Todd [19] and Wilf [28]. Recently, there has been a vivid interest in geometric properties of special functions such as Bessel, Struve, Lommel functions of the first kind and regular Coulomb wave functions. The first author and his collaborators examined in detail the determination of the radii of starlikeness and convexity of some normalized forms of these special functions, see [1, 2, 3, 4, 5, 7, 8, 9, 10, 11] and the references therein. Moreover, one of the most important things which we have learned in these studies is that the radii of univalence, starlikeness and convexity

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are obtained as solutions of some transcendental equations and the obtained radii satisfy some interesting inequalities. In addition, in view of these studies, we know that the radii of univalence of some normalized Bessel, Struve, Lommel and regular Coulomb wave functions coincide with the radii of starlikeness of these functions. The positive zeros of Bessel, Struve, Lommel functions of the first kind and regular Coulomb wave functions and the Laguerre-Pólya class of real entire functions played an important role in these papers. Motivated by the above series of papers on geometric properties of special functions, our aim in this paper is to present some similar results for the normalized forms of the Wright function which has important applications in different areas of mathematics. In this paper, we are mainly focused on the determination of the radii of starlikeness and convexity of normalized Wright functions. Furthermore, our aim is also to give some lower and upper bounds for the radii of starlikeness and convexity of order zero by using some Euler-Rayleigh inequalities for the smallest positive zero of some transcendental equations (for more details on such kind of inequalities we refer to [18]). The paper is arranged as follows: the rest of this section is devoted to some basic definitions needed for the proof of our main results. Section 2 is divided into four subsections: the first subsection is dedicated to the radii of starlikeness of normalized Wright functions. Also, at the end of this subsection, lower and upper bounds for radii of starlikeness of order zero are given. The second subsection contains the study of the radii of convexity of normalized Wright functions, and lower and upper bounds for the radii of convexity of order zero for some normalized Wright functions are given in its last part. The third subsection contains some particular cases of the main results in terms of classical Bessel functions of the first kind. In the fourth subsection, some open problems are stated, which may be of interest for further research.

Before we start with the presentation of results, we would like to state some basic definitions. For $r > 0$ by $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ we denote the open disk of radius r centered at the origin. Let $f : \mathbb{D}_r \to \mathbb{C}$ be the function defined by

$$f(z) = z + \sum_{n\geq 2} a_n z^n, \tag{1}$$

where r is less than or equal to the radius of convergence of the above power series. Let $\mathcal{A}$ be the class of analytic functions of the form (1), that is, normalized by the conditions $f(0) = f'(0) - 1 = 0$. The function f, defined by (1), is called starlike in $\mathbb{D}_r$ if f is univalent in $\mathbb{D}_r$, and the image domain $f(\mathbb{D}_r)$ is a starlike domain in $\mathbb{C}$ with respect to the origin (see [16] for more details). Analytically, the function f is starlike in $\mathbb{D}_r$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad \text{for all} \ \ z \in \mathbb{D}_r.$$

For $\alpha \in [0, 1)$, we say that the function f is starlike of order α in $\mathbb{D}_r$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad \text{for all} \ \ z \in \mathbb{D}_r.$$

The real number

$$r^{\star}_{\alpha}(f) = \sup\left\{r > 0 \,\middle|\, \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \text{ for all } z \in \mathbb{D}_r\right\}$$

is called the radius of starlikeness of order α of the function f. Note that $r^\star(f) = r_0^\star(f)$ is in fact the largest radius such that the image region $f(\mathbb{D}_{r^\star(f)})$ is a starlike domain with respect to the origin.

The function f, defined by (1), is convex in the disk $\mathbb{D}_r$ if f is univalent in $\mathbb{D}_r$, and the image domain $f(\mathbb{D}_r)$ is a convex domain in $\mathbb{C}$. Analytically, the function f is convex in $\mathbb{D}_r$ if and only if

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > 0 \quad \text{for all} \quad z \in \mathbb{D}_r.$$

For $\alpha \in [0,1)$, we say that the function f is convex of order α in $\mathbb{D}_r$ if and only if

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha \quad \text{for all} \quad z \in \mathbb{D}_r.$$

The radius of convexity of order α of the function f is defined by the real number

$$r_\alpha^c(f) = \sup\left\{ r>0 \,\middle|\, \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha \text{ for all } z \in \mathbb{D}_r \right\}.$$

Note that $r^c(f) = r_0^c(f)$ is the largest radius such that the image region $f(\mathbb{D}_{r^c(f)})$ is a convex domain.

Finally, we recall that a real entire function q belongs to the Laguerre-Pólya class $\mathcal{LP}$ if it can be represented in the form

$$q(x) = cx^m e^{-ax^2+bx} \prod_{n\geq 1}\left(1+\frac{x}{x_n}\right)e^{-\frac{x}{x_n}},$$

with $c, b, x_n \in \mathbb{R}, a \geq 0, m \in \mathbb{N}\cup\{0\}$ and $\sum_{n\geq 1} {x_n}^{-2} < \infty$. We note that the class $\mathcal{LP}$ consists of entire functions which are uniform limits on the compact sets of the complex plane of polynomials with only real zeros. For more details on the class $\mathcal{LP}$ we refer to [15, p. 703] and the references therein.

2. The radii of starlikeness and convexity of normalized Wright functions

In this section, we will investigate the generalized Bessel function

$$\phi(\rho,\beta,z) = \sum_{n\geq 0}\frac{z^n}{n!\Gamma(n\rho+\beta)},$$

where $\rho > -1$ and $z, \beta \in \mathbb{C}$, which is named after E.M. Wright. This function was introduced by Wright for $\rho > 0$ in connection with his investigations on the asymptotic theory of partitions [29]; for further details see also [17]. Furthermore, it is important to mention that the Wright function is an entire function of z for $\rho > -1$; consequently, as we will see in some parts of our paper, some properties of the general theory of entire functions can be applied.

The following lemma, which we believe is of independent interest, plays an important role in the proof of our main results.

Lemma 1. *If $\rho > 0$ and $\beta > 0$, then the function $z \mapsto \lambda_{\rho,\beta}(z) = \phi(\rho, \beta, -z^2)$ has infinitely many zeros which are all real. Denoting by $\lambda_{\rho,\beta,n}$ the nth positive zero of $\phi(\rho, \beta, -z^2)$, under the same conditions the Weierstrassian decomposition*

$$\Gamma(\beta)\phi(\rho, \beta, -z^2) = \prod_{n\geq 1}\left(1 - \frac{z^2}{\lambda^2_{\rho,\beta,n}}\right)$$

is valid, and this product is uniformly convergent on compact subsets of the complex plane. Moreover, if we denote by $\zeta'_{\rho,\beta,n}$ the nth positive zero of $\Psi'_{\rho,\beta}$, where $\Psi_{\rho,\beta}(z) = z^{\beta}\lambda_{\rho,\beta}(z)$, then the positive zeros of $\lambda_{\rho,\beta}$ (or the positive real zeros of the function $\Psi_{\rho,\beta}$) are interlaced with those of $\Psi'_{\rho,\beta}$. In other words, the zeros satisfy the chain of inequalities

$$\zeta'_{\rho,\beta,1} < \lambda_{\rho,\beta,1} < \zeta'_{\rho,\beta,2} < \lambda_{\rho,\beta,2} < \ldots.$$

Proof. The proof of the reality of the zeros is given in [6] by using two somehow similar approaches. Now, since the growth order of the entire function $\phi(\rho, \beta, \cdot)$ is $(\rho + 1)^{-1}$ (see [17]), which is a non-integer number and lies in $(0, 1)$, it follows that indeed the Wright function has infinitely many zeros. Since the Wright function is entire, its infinite product clearly exists, and in view of the Hadamard theorem on growth order of the entire function, it follows that its canonical representation is exactly what we have in Lemma 1. Using the infinite product representation we get that

$$\frac{\Psi'_{\rho,\beta}(z)}{\Psi_{\rho,\beta}(z)} = \frac{\beta}{z} + \frac{\lambda'_{\rho,\beta}(z)}{\lambda_{\rho,\beta}(z)} = \frac{\beta}{z} + \sum_{n\geq 1}\frac{2z}{z^2 - \lambda^2_{\rho,\beta,n}}. \tag{2}$$

Differentiating both sides of (2) we have

$$\frac{d}{dz}\left(\frac{\Psi'_{\rho,\beta}(z)}{\Psi_{\rho,\beta}(z)}\right) = -\frac{\beta}{z^2} - 2\sum_{n\geq 1}\frac{z^2 + \lambda^2_{\rho,\beta,n}}{(z^2 - \lambda^2_{\rho,\beta,n})^2}, \quad z \neq \lambda_{\rho,\beta,n}.$$

Since the expression on the right-hand side is real and negative for z real and $\rho, \beta > 0$, the quotient $\Psi'_{\rho,\beta}/\Psi_{\rho,\beta}$ is a strictly decreasing function from $+\infty$ to $-\infty$ as z increases through real values over the open interval $(\lambda_{\rho,\beta,n}, \lambda_{\rho,\beta,n+1})$, $n \in \mathbb{N}$. Hence, the function $\Psi'_{\rho,\beta}$ vanishes just once between two consecutive zeros of the function $\lambda_{\rho,\beta}$. □

Observe that the function $z \mapsto \phi(\rho, \beta, -z^2)$ does not belong to $\mathcal{A}$, and thus first we perform some natural normalization. We define three functions originating from $\phi(\rho, \beta, \cdot)$:

$$f_{\rho,\beta}(z) = \left(z^{\beta}\Gamma(\beta)\phi(\rho, \beta, -z^2)\right)^{\frac{1}{\beta}},$$
$$g_{\rho,\beta}(z) = z\Gamma(\beta)\phi(\rho, \beta, -z^2),$$
$$h_{\rho,\beta}(z) = z\Gamma(\beta)\phi(\rho, \beta, -z).$$

Obviously these functions belong to the class $\mathcal{A}$. Of course, there exist infinitely many other normalizations; the main motivation to consider the above ones is the fact that their particular cases in terms of Bessel functions appear in the literature or are similar to the studied normalization in the literature.

2.1. The radii of starlikeness of order α of functions $f_{\rho,\beta}$, $g_{\rho,\beta}$ and $h_{\rho,\beta}$

In this subsection, our aim is to present some results for the radii of starlikeness of normalized Wright functions $f_{\rho,\beta}$, $g_{\rho,\beta}$ and $h_{\rho,\beta}$. We will see that the radii of starlikeness of order α of normalized Wright functions are actually solutions of some transcendental equations. Moreover, we will also find lower and upper bounds for the radii of starlikeness of order zero.

Our first main result is the following theorem.

Theorem 1. *Let $\rho > 0, \beta > 0$ and $\alpha \in [0, 1)$.*

a. *The radius of starlikeness of order α of $f_{\rho,\beta}$ is $r^{\star}_{\alpha}(f_{\rho,\beta}) = x_{\rho,\beta,1}$, where $x_{\rho,\beta,1}$ is the smallest positive zero of the transcendental equation*

$$r\lambda'_{\rho,\beta}(r) - (\alpha - 1)\beta\lambda_{\rho,\beta}(r) = 0.$$

b. *The radius of starlikeness of order α of $g_{\rho,\beta}$ is $r^{\star}_{\alpha}(g_{\rho,\beta}) = y_{\rho,\beta,1}$, where $y_{\rho,\beta,1}$ is the smallest positive zero of the transcendental equation*

$$r\lambda'_{\rho,\beta}(r) - (\alpha - 1)\lambda_{\rho,\beta}(r) = 0.$$

c. *The radius of starlikeness of order α of $h_{\rho,\beta}$ is $r^{\star}_{\alpha}(h_{\rho,\beta}) = z_{\rho,\beta,1}$, where $z_{\rho,\beta,1}$ is the smallest positive zero of the transcendental equation*

$$\sqrt{r}\lambda'_{\rho,\beta}(\sqrt{r}) - 2(\alpha - 1)\lambda_{\rho,\beta}(\sqrt{r}) = 0.$$

Proof. We need to show that the inequalities

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \alpha,\ \operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) \geq \alpha \quad \text{and} \quad \operatorname{Re}\left(\frac{zh'(z)}{h(z)}\right) \geq \alpha \tag{3}$$

hold for $z \in \mathbb{D}_{x_{\rho,\beta,1}}(f_{\rho,\beta})$, $z \in \mathbb{D}_{y_{\rho,\beta,1}}(g_{\rho,\beta})$ and $z \in \mathbb{D}_{z_{\rho,\beta,1}}(h_{\rho,\beta})$, respectively, and each of the above inequalities does not hold in any larger disk. By definition we get

$$\begin{aligned}
f_{\rho,\beta}(z) &= \left(z^{\beta}\Gamma(\beta)\lambda_{\rho,\beta}(z)\right)^{\frac{1}{\beta}},\\
g_{\rho,\beta}(z) &= z\Gamma(\beta)\lambda_{\rho,\beta}(z),\\
h_{\rho,\beta}(z) &= z\Gamma(\beta)\lambda_{\rho,\beta}(\sqrt{z}).
\end{aligned}$$

The logarithmic derivation yields

$$\begin{aligned}
\frac{zf'_{\rho,\beta}(z)}{f_{\rho,\beta}(z)} &= 1 + \frac{1}{\beta}\left(\frac{z\lambda'_{\rho,\beta}(z)}{\lambda_{\rho,\beta}(z)}\right) = 1 - \frac{1}{\beta}\sum_{n\geq 1}\frac{2z^2}{\lambda^2_{\rho,\beta,n} - z^2},\\
\frac{zg'_{\rho,\beta}(z)}{g_{\rho,\beta}(z)} &= 1 + \left(\frac{z\lambda'_{\rho,\beta}(z)}{\lambda_{\rho,\beta}(z)}\right) = 1 - \sum_{n\geq 1}\frac{2z^2}{\lambda^2_{\rho,\beta,n} - z^2},
\end{aligned}$$

$$\frac{zh'_{\rho,\beta}(z)}{h_{\rho,\beta}(z)} = 1 + \frac{1}{2}\left(\sqrt{z}\frac{\lambda'_{\rho,\beta}(\sqrt{z})}{\lambda_{\rho,\beta}(\sqrt{z})}\right) = 1 - \sum_{n\geq 1}\frac{z}{\lambda^2_{\rho,\beta,n} - z}.$$

It is known [4] that if $z \in \mathbb{C}$ and $\theta \in \mathbb{R}$ are such that $\theta > |z|$, then

$$\frac{|z|}{\theta - |z|} \geq \operatorname{Re}\left(\frac{z}{\theta - z}\right). \tag{4}$$

Then the inequality

$$\frac{|z|^2}{\lambda^2_{\rho,\beta,n} - |z|^2} \geq \operatorname{Re}\left(\frac{z^2}{\lambda^2_{\rho,\beta,n} - z^2}\right)$$

is valid for every $\rho > 0$, $\beta > 0$, $n \in \mathbb{N}$ and $|z| < \lambda_{\rho,\beta,n}$. Therefore,

$$\begin{aligned}
\operatorname{Re}\left(\frac{zf'_{\rho,\beta}(z)}{f_{\rho,\beta}(z)}\right) &= 1 - \frac{1}{\beta}\operatorname{Re}\left(\sum_{n\geq 1}\frac{2z^2}{\lambda^2_{\rho,\beta,n} - z^2}\right) \\
&\geq 1 - \frac{1}{\beta}\sum_{n\geq 1}\frac{2|z|^2}{\lambda^2_{\rho,\beta,n} - |z|^2} = \frac{|z|\,f'_{\rho,\beta}(|z|)}{f_{\rho,\beta}(|z|)}, \\
\operatorname{Re}\left(\frac{zg'_{\rho,\beta}(z)}{g_{\rho,\beta}(z)}\right) &= 1 - \operatorname{Re}\left(\sum_{n\geq 1}\frac{2z^2}{\lambda^2_{\rho,\beta,n} - z^2}\right) \\
&\geq 1 - \sum_{n\geq 1}\frac{2|z|^2}{\lambda^2_{\rho,\beta,n} - |z|^2} = \frac{|z|\,g'_{\rho,\beta}(|z|)}{g_{\rho,\beta}(|z|)},
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Re}\left(\frac{zh'_{\rho,\beta}(z)}{h_{\rho,\beta}(z)}\right) &= 1 - \operatorname{Re}\left(\sum_{n\geq 1}\frac{z}{\lambda^2_{\rho,\beta,n} - z}\right) \\
&\geq 1 - \sum_{n\geq 1}\frac{|z|}{\lambda^2_{\rho,\beta,n} - |z|} = \frac{|z|\,h'_{\rho,\beta}(|z|)}{h_{\rho,\beta}(|z|)},
\end{aligned}$$

where equalities are attained only when $z = |z| = r$. The latter inequalities and the minimum principle for harmonic functions imply that the corresponding inequalities in (3) hold if and only if $|z| < x_{\rho,\beta,1}$, $|z| < y_{\rho,\beta,1}$ and $|z| < z_{\rho,\beta,1}$, respectively, where $x_{\rho,\beta,1}$, $y_{\rho,\beta,1}$ and $z_{\rho,\beta,1}$ are the smallest positive roots of the equations

$$\frac{rf'_{\rho,\beta}(r)}{f_{\rho,\beta}(r)} = \alpha,\ \ \frac{rg'_{\rho,\beta}(r)}{g_{\rho,\beta}(r)} = \alpha \quad \text{and} \quad \frac{rh'_{\rho,\beta}(r)}{h_{\rho,\beta}(r)} = \alpha,$$

which are equivalent to

$$r\lambda'_{\rho,\beta}(r) - (\alpha - 1)\beta\lambda_{\rho,\beta}(r) = 0, \quad r\lambda'_{\rho,\beta}(r) - (\alpha - 1)\lambda_{\rho,\beta}(r) = 0$$

and

$$\sqrt{r}\lambda'_{\rho,\beta}(\sqrt{r}) - (\alpha - 1)\lambda_{\rho,\beta}(\sqrt{r}) = 0.$$

In other words, we proved that

$$\inf_{z\in\mathbb{D}_r} \operatorname{Re}\left(\frac{zf'_{\rho,\beta}(z)}{f_{\rho,\beta}(z)}\right) = \frac{rf'_{\rho,\beta}(r)}{f_{\rho,\beta}(r)} = F_{\rho,\beta}(r),$$
$$\inf_{z\in\mathbb{D}_r} \operatorname{Re}\left(\frac{zg'_{\rho,\beta}(z)}{g_{\rho,\beta}(z)}\right) = \frac{rg'_{\rho,\beta}(r)}{g_{\rho,\beta}(r)} = G_{\rho,\beta}(r)$$

and

$$\inf_{z\in\mathbb{D}_r} \operatorname{Re}\left(\frac{zh'_{\rho,\beta}(z)}{h_{\rho,\beta}(z)}\right) = \frac{rh'_{\rho,\beta}(r)}{h_{\rho,\beta}(r)} = H_{\rho,\beta}(r).$$

Since the real functions $F_{\rho,\beta}, G_{\rho,\beta}, H_{\rho,\beta} : (0,\lambda_{\rho,\beta,1}) \longrightarrow \mathbb{R}$ are decreasing, and take the limits

$$\lim_{r\searrow 0} F_{\rho,\beta}(r) = \lim_{r\searrow 0} G_{\rho,\beta}(r) = \lim_{r\searrow 0} H_{\rho,\beta}(r) = 1$$

and

$$\lim_{r\nearrow\lambda_{\rho,\beta,1}} F_{\rho,\beta}(r) = \lim_{r\nearrow\lambda_{\rho,\beta,1}} G_{\rho,\beta}(r) = \lim_{r\nearrow\lambda_{\rho,\beta,1}} H_{\rho,\beta}(r) = -\infty,$$

it follows that the inequalities in (3) indeed hold for $z \in \mathbb{D}_{x_{\rho,\beta,1}}$, $z \in \mathbb{D}_{y_{\rho,\beta,1}}$ and $z \in \mathbb{D}_{z_{\rho,\beta,1}}$, respectively. □

The following theorems provide some tight lower and upper bounds for the radii of starlikeness of the functions considered in the above theorems. In these theorems, for simplicity, we use the notation

$$\Delta_{a,b}(\rho,\beta) = a\Gamma(\beta)\Gamma(2\rho+\beta) - b\Gamma^2(\rho+\beta),$$

and mention that the positivity of this expression for $a > b > 0$ and $\rho,\beta > 0$ is guaranteed by the log-convexity of the Euler gamma function.

Theorem 2. *For $\rho,\beta > 0$, the radius of starlikeness $r^\star(f_{\rho,\beta})$ satisfies*

$$\sqrt{\frac{\Gamma(\rho+\beta)}{(\beta+2)\Gamma(\beta)}} < r^\star(f_{\rho,\beta}) < \sqrt{\frac{\beta(\beta+2)\Gamma(\rho+\beta)\Gamma(2\rho+\beta)}{\Delta_{(\beta+2)^2,\beta+4}(\rho,\beta)}},$$
$$\sqrt[4]{\frac{\beta\Gamma^2(\rho+\beta)\Gamma(2\rho+\beta)}{\Gamma(\beta)\Delta_{(\beta+2)^2,\beta+4}(\rho,\beta)}} < r^\star(f_{\rho,\beta}) < \sqrt{\frac{2\beta\Gamma(\rho+\beta)\Gamma(3\rho+\beta)\Delta_{(\beta+2)^2,\beta+4}(\rho,\beta)}{\beta(\beta+6)\Gamma^3(\rho+\beta)\Gamma(2\rho+\beta)+\Xi_{\rho,\beta}}},$$

where

$$\Xi_{\rho,\beta} = (\beta+2)^2\Gamma(\beta)\Gamma(3\rho+\beta)\Delta_{2(\beta+2),\beta+4}(\rho,\beta).$$

Proof. The radius of starlikeness of the normalized Wright function $f_{\rho,\beta}$ corresponds to the radius of starlikeness of the function $\Psi_{\rho,\beta}(z) = z^\beta\lambda_{\rho,\beta}(z)$. The infinite series representations of the function $\Psi'_{\rho,\beta}$ and its derivative read as follows:

$$\Upsilon_{\rho,\beta}(z) = \Psi'_{\rho,\beta}(z) = \sum_{n\geq 0} \frac{(-1)^n(2n+\beta)}{n!\Gamma(n\rho+\beta)} z^{2n+\beta-1}, \tag{5}$$

$$\Upsilon'_{\rho,\beta}(z)=\sum_{n\geq 0}\frac{(-1)^n(2n+\beta)(2n+\beta-1)}{n!\Gamma(n\rho+\beta)}z^{2n+\beta-2}. \tag{6}$$

In view of Lemma 1, the function $z\mapsto z^{1-\beta}\Upsilon_{\rho,\beta}(z)$ belongs to the Laguerre-Pólya class $\mathcal{LP}$. Hence, the zeros of the function $\Upsilon_{\rho,\beta}$ are all real. Suppose that $\iota_{\rho,\beta,n}$'s are the positive zeros of the function $\Upsilon_{\rho,\beta}$. The expression $\Upsilon_{\rho,\beta}(z)$ can be written as

$$\Gamma(\beta)\Upsilon_{\rho,\beta}(z)=\beta z^{\beta-1}\prod_{n\geq 1}\left(1-\frac{z^2}{\iota^2_{\rho,\beta,n}}\right). \tag{7}$$

By the logarithmic derivation of both sides of (7), for $|z|<\iota_{\rho,\beta,1}$we obtain

$$\frac{z\Upsilon'_{\rho,\beta}(z)}{\Upsilon_{\rho,\beta}(z)}-(\beta-1)=-2\sum_{n\geq 1}\frac{z^2}{\iota^2_{\rho,\beta,n}-z^2}=-2\sum_{n\geq 1}\sum_{k\geq 0}\frac{z^{2k+2}}{\iota^{2k+2}_{\rho,\beta,n}}=-2\sum_{k\geq 0}\chi_{k+1}z^{2k+2}, \tag{8}$$

where $\chi_k=\sum_{n\geq 1}\iota^{-2k}_{\rho,\beta,n}$. Thus, by using relations (5), (6) and (8) we get

$$\frac{z\Upsilon'_{\rho,\beta}(z)}{\Upsilon_{\rho,\beta}(z)}=\sum_{n\geq 0}\xi_n z^{2n}\Big/\sum_{n\geq 0}\nu_n z^{2n}, \tag{9}$$

where

$$\xi_n=(-1)^n\frac{(2n+\beta)(2n+\beta-1)}{n!\Gamma(n\rho+\beta)}\quad\text{and}\quad \nu_n=(-1)^n\frac{(2n+\beta)}{n!\Gamma(n\rho+\beta)}.$$

By comparing the coefficients of (8) and (9) we have

$$\begin{cases}(\beta-1)\nu_0=\xi_0\\ (\beta-1)\nu_1-2\chi_1\nu_0=\xi_1\\ (\beta-1)\nu_2-2\chi_1\nu_1-2\chi_2\nu_0=\xi_2\\ (\beta-1)\nu_3-2\chi_1\nu_2-2\chi_2\nu_1-2\chi_3\nu_0=\xi_3\end{cases},$$

which implies that

$$\chi_1=\frac{(\beta+2)\Gamma(\beta)}{\Gamma(\rho+\beta)},\ \chi_2=\frac{(\beta+2)^2}{\beta}\frac{\Gamma^2(\beta)}{\Gamma^2(\rho+\beta)}-\frac{\beta+4}{\beta}\frac{\Gamma(\beta)}{\Gamma(2\rho+\beta)}$$

and

$$\chi_3=\frac{(\beta+2)^3}{\beta^2}\frac{\Gamma^3(\beta)}{\Gamma^3(\rho+\beta)}-\frac{(\beta+2)^2(\beta+4)\Gamma^2(\beta)}{2\beta^2\Gamma(\rho+\beta)\Gamma(2\rho+\beta)}+\frac{\beta+6}{2\beta}\frac{\Gamma(\beta)}{\Gamma(3\rho+\beta)}.$$

By using the Euler-Rayleigh inequalities $\chi_k^{-1/k}<\iota^2_{\rho,\beta,1}<\frac{\chi_k}{\chi_{k+1}}$, $k\in\{1,2\}$, we get the inequalities of the theorem. □

Theorem 3. *For $\rho,\beta>0$, the radius of starlikeness $r^\star(g_{\rho,\beta})$ satisfies*

$$\sqrt{\frac{\Gamma(\rho+\beta)}{3\Gamma(\beta)}}<r^\star(g_{\rho,\beta})<\sqrt{\frac{3\Gamma(\rho+\beta)\Gamma(2\rho+\beta)}{\Delta_{9,5}(\rho,\beta)}},$$

$$\sqrt[4]{\frac{\Gamma^2(\rho+\beta)\Gamma(2\rho+\beta)}{\Gamma(\beta)\Delta_{9,5}(\rho,\beta)}} < r^\star(g_{\rho,\beta}) < \sqrt{\frac{2\Gamma(\rho+\beta)\Gamma(3\rho+\beta)\Delta_{9,5}(\rho,\beta)}{9\Gamma(\beta)\Gamma(3\rho+\beta)\Delta_{6,5}(\rho,\beta)+7\Gamma^3(\rho+\beta)\Gamma(2\rho+\beta)}}.$$

Proof. For $\alpha = 0$, in view of the second part of Theorem 1, we have that the radius of starlikeness of order zero is the smallest positive root of the equation $(z\lambda_{\rho,\beta}(z))' = 0$. Therefore, we shall study the first positive zero of

$$\psi_{\rho,\beta}(z) = (z\lambda_{\rho,\beta}(z))' = \sum_{n\geq 0} \frac{(-1)^n(2n+1)}{n!\Gamma(n\rho+\beta)} z^{2n}. \tag{10}$$

We know that the function $\lambda_{\rho,\beta}$ belongs to the Laguerre-Pólya class of entire functions $\mathcal{LP}$, which is closed under differentiation. Therefore, we get that the function $\psi_{\rho,\beta}$ also belongs to the Laguerre-Pólya class. Hence, the zeros of the function $\psi_{\rho,\beta}$ are all real. Suppose that $\gamma_{\rho,\beta,n}$'s are the positive zeros of the function $\psi_{\rho,\beta}$. Then, the function $\psi_{\rho,\beta}$ has the infinite product representation as follows:

$$\Gamma(\beta)\psi_{\rho,\beta}(z) = \prod_{n\geq 1}\left(1-\frac{z^2}{\gamma^2_{\rho,\beta,n}}\right), \tag{11}$$

since its growth order corresponds to the growth order of the Wright function itself. If we take the logarithmic derivative of both sides of (11), then for $|z| < \gamma_{\rho,\beta,1}$ we get

$$\begin{aligned}\frac{\psi'_{\rho,\beta}(z)}{\psi_{\rho,\beta}(z)} &= \sum_{n\geq 1}\frac{2z}{z^2-\gamma^2_{\rho,\beta,n}} = -2\sum_{n\geq 1}\sum_{k\geq 0}\frac{z^{2k+1}}{\gamma^{2k+2}_{\rho,\beta,n}} \\ &= -2\sum_{k\geq 0}\sum_{n\geq 1}\frac{z^{2k+1}}{\gamma^{2k+2}_{\rho,\beta,n}} = -2\sum_{k\geq 0}\delta_{k+1}z^{2k+1},\end{aligned} \tag{12}$$

where $\delta_k = \sum_{n\geq 1}\gamma^{-2k}_{\rho,\beta,n}$. Moreover, in view of (10), we have

$$\frac{\psi'_{\rho,\beta}(z)}{\psi_{\rho,\beta}(z)} = -2\sum_{n\geq 0} a_n z^{2n+1} \Bigg/ \sum_{n\geq 0} b_n z^{2n}, \tag{13}$$

where

$$a_n = \frac{(-1)^n(2n+3)}{n!\Gamma((n+1)\rho+\beta)} \quad \text{and} \quad b_n = \frac{(-1)^n(2n+1)}{n!\Gamma(n\rho+\beta)}.$$

Comparing the coefficients of (12) and (13) we obtain

$$\delta_1 b_0 = a_0,\ \delta_2 b_0 + \delta_1 b_1 = a_1,\ \delta_3 b_0 + \delta_2 b_1 + \delta_1 b_2 = a_2,$$

which yields the following Rayleigh sums

$$\delta_1 = \frac{3\Gamma(\beta)}{\Gamma(\rho+\beta)},\ \delta_2 = \frac{9\Gamma^2(\beta)}{\Gamma^2(\rho+\beta)} - \frac{5\Gamma(\beta)}{\Gamma(2\rho+\beta)}$$

and

$$\delta_3=\frac{27\Gamma^3(\beta)}{\Gamma^3(\rho+\beta)}-\frac{45}{2}\frac{\Gamma^2(\beta)}{\Gamma(\rho+\beta)\Gamma(2\rho+\beta)}+\frac{7}{2}\frac{\Gamma(\beta)}{\Gamma(3\rho+\beta)}.$$

By using Euler-Rayleigh inequalities $\delta_k^{-\frac{1}{k}}<\gamma^2_{\rho,\beta,1}<\frac{\delta_k}{\delta_{k+1}}$, $k\in\{1,2\}$, we obtain

$$\sqrt{\frac{\Gamma(\rho+\beta)}{3\Gamma(\beta)}}<r^\star(g_{\rho,\beta})<\sqrt{\frac{3\Gamma(\rho+\beta)\Gamma(2\rho+\beta)}{\Delta_{9,5}(\rho,\beta)}},$$

$$\sqrt[4]{\frac{\Gamma^2(\rho+\beta)\Gamma(2\rho+\beta)}{\Gamma(\beta)\Delta_{9,5}(\rho,\beta)}}<r^\star(g_{\rho,\beta})$$

$$<\sqrt{\frac{2\Gamma(\rho+\beta)\Gamma(3\rho+\beta)\Delta_{9,5}(\rho,\beta)}{9\Gamma(\beta)\Gamma(3\rho+\beta)\Delta_{6,5}(\rho,\beta)+7\Gamma^3(\rho+\beta)\Gamma(2\rho+\beta)}}.$$

□

Theorem 4. *For $\rho,\beta>0$, the radius of starlikeness $r^\star(h_{\rho,\beta})$ satisfies*

$$\frac{\Gamma(\rho+\beta)}{2\Gamma(\beta)}<r^\star(h_{\rho,\beta})<\frac{2\Gamma(\rho+\beta)\Gamma(2\rho+\beta)}{\Delta_{4,3}(\rho,\beta)},$$

$$\sqrt{\frac{\Gamma^2(\rho+\beta)\Gamma(2\rho+\beta)}{\Gamma(\beta)\Delta_{4,3}(\rho,\beta)}}<r^\star(h_{\rho,\beta})$$

$$<\frac{\Gamma(\rho+\beta)\Gamma(3\rho+\beta)\Delta_{4,3}(\rho,\beta)}{\Gamma(\beta)\Gamma(3\rho+\beta)\Delta_{8,9}(\rho,\beta)+2\Gamma^3(\rho+\beta)\Gamma(2\rho+\beta)}.$$

Proof. If we take $\alpha=0$ in the third part of Theorem 1, then we conclude that the radius of starlikeness of the function $h_{\rho,\beta}$ is actually the smallest positive root of the transcendental equation $(z\lambda_{\rho,\beta}(\sqrt{z}))'=0$. Therefore, it is of interest to study the first positive zero of

$$\Omega_{\rho,\beta}(z)=(z\lambda_{\rho,\beta}(\sqrt{z}))'=\sum_{n\geq0}\frac{(-1)^n(n+1)}{n!\Gamma(n\rho+\beta)}z^n. \tag{14}$$

In view of Lemma 1 and because of the fact that $\mathcal{LP}$ is closed under differentiation, the function $\Omega_{\rho,\beta}$ also belongs to the Laguerre-Pólya class. Assume that $\sigma_{\rho,\beta,n}$ are the positive zeros of the function $\Omega_{\rho,\beta}$. Thus, due to the Hadamard factorization theorem, the expression $\Omega_{\rho,\beta}(z)$ can be written as

$$\Gamma(\beta)\Omega_{\rho,\beta}(z)=\prod_{n\geq1}\left(1-\frac{z}{\alpha_{\rho,\beta,n}}\right). \tag{15}$$

By taking the logarithmic derivative of both sides of (15) we have

$$\frac{\Omega'_{\rho,\beta}(z)}{\Omega_{\rho,\beta}(z)}=-\sum_{k\geq0}\eta_{k+1}z^k,\qquad |z|<\sigma_{\rho,\beta,1}, \tag{16}$$

where $\eta_k = \sum_{n\geq 1} \sigma_{\rho,\beta,n}^{-k}$. Also, by taking the derivative of (14) we get

$$\frac{\Omega'_{\rho,\beta}(z)}{\Omega_{\rho,\beta}(z)} = -\sum_{n\geq 0} c_n z^n \Bigg/ \sum_{n\geq 0} d_n z^n, \tag{17}$$

where

$$c_n = \frac{(-1)^n(n+2)}{n!\Gamma((n+1)\rho+\beta)} \qquad d_n = \frac{(-1)^n(n+1)}{n!\Gamma(n\rho+\beta)}.$$

Comparing the coefficients of (16) and (17) we get the following Rayleigh sums

$$\eta_1 = \frac{2\Gamma(\beta)}{\Gamma(\rho+\beta)},\ \eta_2 = \frac{4\Gamma^2(\beta)}{\Gamma^2(\rho+\beta)} - \frac{3\Gamma(\beta)}{\Gamma(2\rho+\beta)}$$

and

$$\eta_3 = \frac{8\Gamma^3(\beta)}{\Gamma^3(\rho+\beta)} + \frac{2\Gamma(\beta)}{\Gamma(3\rho+\beta)} - \frac{9\Gamma^2(\beta)}{\Gamma(\rho+\beta)\Gamma(2\rho+\beta)},$$

and by using the Euler-Rayleigh inequalities $\eta_k^{-1/k} < \sigma_{\rho,\beta,1} < \frac{\eta_k}{\eta_{k+1}}$, for $k \in \{1,2\}$ we get the inequalities of the theorem. □

2.2. The radii of convexity of order α of functions $f_{\rho,\beta}$, $g_{\rho,\beta}$ and $h_{\rho,\beta}$

In this subsection, we present the radii of convexity of order α for functions $f_{\rho,\beta}$, $g_{\rho,\beta}$ and $h_{\rho,\beta}$. In addition, we find tight lower and upper bounds for the radii of convexity of order zero for the functions $g_{\rho,\beta}$ and $h_{\rho,\beta}$.

Theorem 5. *Let $\rho > 0$, $\beta > 0$ and $\alpha \in [0,1)$.*

a. *The radius of convexity of order α of $f_{\rho,\beta}$ is the smallest positive root of*

$$1 + \frac{r\Psi''_{\rho,\beta}(r)}{\Psi'_{\rho,\beta}(r)} + \left(\frac{1}{\beta} - 1\right)\frac{r\Psi'_{\rho,\beta}(r)}{\Psi_{\rho,\beta}(r)} = \alpha,$$

where $\Psi_{\rho,\beta}(z) = z^\beta \lambda_{\rho,\beta}(z)$.

b. *The radius of convexity of order α of $g_{\rho,\beta}$ is the smallest positive root of*

$$1 + \frac{rg''_{\rho,\beta}(r)}{g'_{\rho,\beta}(r)} = \alpha.$$

c. *The radius of convexity of order α of $h_{\rho,\beta}$ is the smallest positive root of*

$$1 + \frac{rh''_{\rho,\beta}(r)}{h'_{\rho,\beta}(r)} = \alpha.$$

Proof. **a.** Observe that

$$1+\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}=1+\frac{z\Psi''_{\rho,\beta}(z)}{\Psi'_{\rho,\beta}(z)}+\left(\frac{1}{\beta}-1\right)\frac{z\Psi'_{\rho,\beta}(z)}{\Psi_{\rho,\beta}(z)}.$$

Now, we consider the following infinite product representations

$$\Gamma(\beta)\Psi_{\rho,\beta}(z)=z^{\beta}\prod_{n\geq1}\left(1-\frac{z^2}{\zeta^2_{\rho,\beta,n}}\right),\ \Gamma(\beta)\Psi'_{\rho,\beta}(z)=z^{\beta-1}\prod_{n\geq1}\left(1-\frac{z^2}{\zeta'^2_{\rho,\beta,n}}\right),$$

where $\zeta_{\rho,\beta,n}$ and $\zeta'_{\rho,\beta,n}$ are the nth positive roots of $\Psi_{\rho,\beta}$ and $\Psi'_{\rho,\beta}$, respectively. Note that $\zeta_{\rho,\beta,n}$ is in fact equal to $\lambda_{\rho,\beta,n}$; however since the zeros of $\lambda'_{\rho,\beta}$ and $\Psi'_{\rho,\beta}$ do not coincide, we use different notations for the zeros of the derivatives, and hence also for the zeros of $\Psi_{\rho,\beta}$. The logarithmic differentiation on both sides of the above relations yields

$$\frac{z\Psi'_{\rho,\beta}(z)}{\Psi_{\rho,\beta}(z)}=\beta-\sum_{n\geq1}\frac{2z^2}{\zeta^2_{\rho,\beta,n}-z^2},\qquad \frac{z\Psi''_{\rho,\beta}(z)}{\Psi'_{\rho,\beta}(z)}=\beta-1-\sum_{n\geq1}\frac{2z^2}{\zeta'^2_{\rho,\beta,n}-z^2},$$

which implies that

$$1+\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}=1-\left(\frac{1}{\beta}-1\right)\sum_{n\geq1}\frac{2z^2}{\zeta^2_{\rho,\beta,n}-z^2}-\sum_{n\geq1}\frac{2z^2}{\zeta'^2_{\rho,\beta,n}-z^2}.$$

By using inequality (4) for $\beta\in(0,1]$ we obtain that

$$\operatorname{Re}\left(1+\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}\right)\geq1-\left(\frac{1}{\beta}-1\right)\sum_{n\geq1}\frac{2r^2}{\zeta^2_{\rho,\beta,n}-r^2}-\sum_{n\geq1}\frac{2r^2}{\zeta'^2_{\rho,\beta,n}-r^2},\tag{18}$$

where $|z|=r$. Moreover, in view of [7, Lemma 2.1], that is,

$$\alpha\operatorname{Re}\left(\frac{z}{a-z}\right)-\operatorname{Re}\left(\frac{z}{b-z}\right)\geq\alpha\frac{|z|}{a-|z|}-\frac{|z|}{b-|z|},$$

where $a>b>0$, $z\in\mathbb{C}$ such that $|z|<b$, we obtain that (18) is also valid when $\beta>1$ for all $z\in\mathbb{D}_{\zeta'_{\rho,\beta,1}}$. Here we used tacitly that the zeros of $\zeta_{\rho,\beta,n}$ and $\zeta'_{\rho,\beta,n}$ interlace according to Lemma 1, that is, we have $\zeta'_{\rho,\beta,1}<\zeta_{\rho,\beta,1}$. Now, the above deduced inequalities imply for $r\in(0,\zeta'_{\rho,\beta,1})$

$$\inf_{z\in\mathbb{D}_r}\left\{\operatorname{Re}\left(1+\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}\right)\right\}=1+r\frac{f''_{\rho,\beta}(r)}{f'_{\rho,\beta}(r)}.$$

On the other hand, the function $u_{\rho,\beta}:(0,\zeta'_{\rho,\beta,1})\to\mathbb{R}$, defined by

$$u_{\rho,\beta}(r)=1+\frac{rf''_{\rho,\beta}(r)}{f'_{\rho,\beta}(r)},$$

is strictly decreasing when $\beta \in (0,1]$. Moreover, it is also strictly decreasing when $\beta > 1$ since

$$\begin{aligned} u'_{\rho,\beta}(r) &= -\left(\frac{1}{\beta}-1\right)\sum_{n\geq 1}\frac{4r\zeta^2_{\rho,\beta,n}}{(\zeta^2_{\rho,\beta,n}-r^2)^2}-\sum_{n\geq 1}\frac{4r\zeta'^2_{\rho,\beta,n}}{(\zeta'^2_{\rho,\beta,n}-r^2)^2} \\ &< \sum_{n\geq 1}\frac{4r\zeta^2_{\rho,\beta,n}}{(\zeta^2_{\rho,\beta,n}-r^2)^2}-\sum_{n\geq 1}\frac{4r\zeta'^2_{\rho,\beta,n}}{(\zeta'^2_{\rho,\beta,n}-r^2)^2} < 0 \end{aligned}$$

for $r \in (0, \zeta'_{\rho,\beta,1})$. Here we used the interlacing property of the zeros stated in Lemma 1. Observe also that $\lim_{r\searrow 0} u_{\rho,\beta}(r) = 1$ and $\lim_{r\nearrow \zeta'_{\rho,\beta,1}} u_{\rho,\beta}(r) = -\infty$, which means that for $z \in \mathbb{D}_{r_1}$ we get

$$\operatorname{Re}\left(1+\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}\right) > \alpha$$

if and only if r_1 is the unique root of

$$1+\frac{rf''_{\rho,\beta}(r)}{f'_{\rho,\beta}(r)} = \alpha$$

situated in $(0, \zeta'_{\rho,\beta,1})$.

b. Since $g_{\rho,\beta} \in \mathcal{LP}$, it follows that $g'_{\rho,\beta} \in \mathcal{LP}$, and since their growth orders (which coincide according to the theory of entire functions) are equal to $(\rho+1)^{-1}$, via the Hadamard theorem we get the Weierstrassian canonical representation

$$g'_{\rho,\beta}(z) = \prod_{n\geq 1}\left(1-\frac{z^2}{\vartheta^2_{\rho,\beta,n}}\right).$$

The logarithmic derivation of both sides yields

$$1+\frac{zg''_{\rho,\beta}(z)}{g'_{\rho,\beta}(z)} = 1-\sum_{n\geq 1}\frac{2z^2}{\vartheta^2_{\rho,\beta,n}-z^2}.$$

Application of inequality (4) implies that

$$\operatorname{Re}\left(1+\frac{zg''_{\rho,\beta}(z)}{g'_{\rho,\beta}(z)}\right) \geq 1-\sum_{n\geq 1}\frac{2r^2}{\vartheta^2_{\rho,\beta,n}-r^2},$$

where $|z| = r$. Thus, for $r \in (0, \vartheta_{\rho,\beta,1})$, we get

$$\inf_{z\in\mathbb{D}_r}\left\{\operatorname{Re}\left(1+\frac{zg''_{\rho,\beta}(z)}{g'_{\rho,\beta}(z)}\right)\right\} = 1-\sum_{n\geq 1}\frac{2r^2}{\vartheta^2_{\rho,\beta,n}-r^2} = 1+\frac{rg''_{\rho,\beta}(r)}{g'_{\rho,\beta}(r)}.$$

The function $v_{\rho,\beta} : (0, \vartheta_{\rho,\beta,1}) \to \mathbb{R}$ defined by

$$\lim_{r\searrow 0} v_{\rho,\beta}(r) = 1, \qquad \lim_{r\nearrow \vartheta_{\rho,\beta,1}} v_{\rho,\beta}(r) = -\infty.$$

Consequently, the equation

$$1+\frac{rg''_{\rho,\beta}(r)}{g'_{\rho,\beta}(r)}=\alpha$$

has a unique root r_2 in $(0,\vartheta_{\rho,\beta,1})$. In other words, we have

$$\operatorname{Re}\left(1+\frac{zg''_{\rho,\beta}(z)}{g'_{\rho,\beta}(z)}\right)>\alpha,\ z\in\mathbb{D}_{r_2}\quad\text{and}\quad\inf_{z\in\mathbb{D}_{r_2}}\left\{\operatorname{Re}\left(1+\frac{zg''_{\rho,\beta}(z)}{g'_{\rho,\beta}(z)}\right)\right\}=\alpha.$$

c. By using again the fact that the zeros of the Wright function $\lambda_{\rho,\beta}$ are all real and in view of the Hadamard theorem, we obtain

$$h'_{\rho,\beta}(z)=\prod_{n\geq1}\left(1-\frac{z}{\tau_{\rho,\beta,n}}\right),$$

which implies that

$$1+\frac{zh''_{\nu}(z)}{h'_{\nu}(z)}=1-\sum_{n\geq1}\frac{z}{\tau_{\rho,\beta,n}-z}.$$

Let $r\in(0,\tau_{\rho,\beta,1})$ be a fixed number. The minimum principle for harmonic functions and inequality (4) imply that for $z\in\mathbb{D}_r$ we have

$$\begin{aligned}\operatorname{Re}\left(1+\frac{zh''_{\rho,\beta}(z)}{h'_{\rho,\beta}(z)}\right)&=\operatorname{Re}\left(1-\sum_{n\geq1}\frac{z}{\tau_{\rho,\beta,n}-z}\right)\geq\min_{|z|=r}\operatorname{Re}\left(1-\sum_{n\geq1}\frac{z}{\tau_{\rho,\beta,n}-z}\right)\\&=\min_{|z|=r}\left(1-\sum_{n\geq1}\operatorname{Re}\frac{z}{\tau_{\rho,\beta,n}-z}\right)\\&\geq1-\sum_{n\geq1}\frac{r}{\tau_{\rho,\beta,n}-r}=1+\frac{rh''_{\nu}(r)}{h'_{\nu}(r)}.\end{aligned}$$

Consequently, it follows that

$$\inf_{z\in\mathbb{D}_r}\left\{\operatorname{Re}\left(1+\frac{zh''_{\rho,\beta}(z)}{h'_{\rho,\beta}(z)}\right)\right\}=1+\frac{rh''_{\rho,\beta}(r)}{h'_{\rho,\beta}(r)}.$$

Now, let r_3 be the smallest positive root of the equation

$$1+\frac{rh''_{\rho,\beta}(r)}{h'_{\rho,\beta}(r)}=\alpha\tag{19}$$

For $z\in\mathbb{D}_{r_3}$, we have

$$\operatorname{Re}\left(1+\frac{zh''_{\rho,\beta}(z)}{h'_{\rho,\beta}(z)}\right)>\alpha.$$

In order to finish the proof, we need to show that equation (19) has a unique root in $(0,\tau_{\rho,\beta,1})$. But, equation (19) is equivalent to

$$w_{\nu}(r)=1-\alpha-\sum_{n\geq1}\frac{r}{\tau_{\rho,\beta,n}-r}=0,$$

and we have

$$\lim_{r\searrow 0} w_\nu(r) = 1-\alpha > 0, \qquad \lim_{r\nearrow \tau_{\rho,\beta,1}} w_\nu(r) = -\infty.$$

Now, since the function w_ν is strictly decreasing on $(0, \tau_{\rho,\beta,1})$, it follows that the equation $w_\nu(r) = 0$ has a unique root. □

Now, we present some lower and upper bounds for the radii of convexity of the functions $g_{\rho,\beta}$ and $h_{\rho,\beta}$ by using the corresponding Euler-Rayleigh inequalities.

Theorem 6. *For $\rho, \beta > 0$, the radius of convexity $r^c(g_{\rho,\beta})$ of the function $g_{\rho,\beta}$ is the smallest positive root of the equation $(zg'_{\rho,\beta}(z))' = 0$ and it satisfies the following inequalities*

$$\sqrt{\frac{\Gamma(\rho+\beta)}{9\Gamma(\beta)}} < r^c(g_{\rho,\beta}) < \sqrt{\frac{9\Gamma(\rho+\beta)\Gamma(2\rho+\beta)}{\Delta_{81,25}(\rho,\beta)}},$$

$$\sqrt[4]{\frac{\Gamma^2(\rho+\beta)\Gamma(2\rho+\beta)}{\Gamma(\beta)\Delta_{81,25}(\rho,\beta)}} < r^c(g_{\rho,\beta})$$
$$< \sqrt{\frac{2\Gamma(\rho+\beta)\Gamma(3\rho+\beta)\Delta_{81,25}(\rho,\beta)}{\Gamma(\beta)\Gamma(3\rho+\beta)\Delta_{1458,675}(\rho,\beta) + 49\Gamma^3(\rho+\beta)\Gamma(2\rho+\beta)}}.$$

Proof. By using the infinite series representations of the Wright function and its derivative we obtain

$$\Theta_{\rho,\beta}(z) = (zg'_{\rho,\beta})' = 1 + \sum_{n\geq 1} \frac{(-1)^n (2n+1)^2 \Gamma(\beta)}{n!\Gamma(n\rho+\beta)} z^{2n}.$$

We know that the function $g_{\rho,\beta}$ belongs to the Laguerre-Pólya class and $\mathcal{LP}$ is closed under differentiation. Thus, the function $\Theta_{\rho,\beta}$ also belongs to the Laguerre-Pólya class and hence its zeros are all real. Assume that $\varsigma_{\rho,\beta,n}$ are the positive zeros of the function $\Theta_{\rho,\beta}$. The function $\Theta_{\rho,\beta}$ can be written as follows

$$\Theta_{\rho,\beta}(z) = \prod_{n\geq 1} \left(1 - \frac{z^2}{\varsigma^2_{\rho,\beta,n}}\right),$$

which for $|z| < \varsigma_{\rho,\beta,1}$ yields

$$\frac{\Theta'_{\rho,\beta}(z)}{\Theta_{\rho,\beta}(z)} = -2\sum_{n\geq 1} \frac{z}{\varsigma^2_{\rho,\beta,n} - z^2} = -2\sum_{k\geq 0}\sum_{n\geq 1} \frac{z^{2k+1}}{\varsigma^{2k+2}_{\rho,\beta,n}} = -2\sum_{k\geq 0} \kappa_{k+1} z^{2k+1}, \tag{20}$$

where $\kappa_k = \sum_{n\geq 1} \varsigma^{-2k}_{\rho,\beta,n}$. On the other hand, we have

$$\frac{\Theta'_{\rho,\beta}(z)}{\Theta_{\rho,\beta}(z)} = -2\sum_{n\geq 0} q_n z^{2n+1} \bigg/ \sum_{n\geq 0} r_n z^{2n}, \tag{21}$$

where

$$q_n = \frac{(-1)^n(2n+3)^2\Gamma(\beta)}{n!\Gamma((n+1)\rho+\beta)} \qquad r_n = \frac{(-1)^n(2n+1)^2\Gamma(\beta)}{n!\Gamma(n\rho+\beta)}.$$

By comparing the coefficients of (20) and (21) we obtain

$$\kappa_1 = \frac{9\Gamma(\beta)}{\Gamma(\rho+\beta)}, \quad \kappa_2 = \frac{81\Gamma^2(\beta)}{\Gamma^2(\rho+\beta)} - \frac{25\Gamma(\beta)}{\Gamma(2\rho+\beta)}$$

and

$$\kappa_3 = \frac{729\Gamma^3(\beta)}{\Gamma^3(\rho+\beta)} + \frac{49}{2}\frac{\Gamma(\beta)}{\Gamma(3\rho+\beta)} - \frac{675}{2}\frac{\Gamma^2(\beta)}{\Gamma(\rho+\beta)\Gamma(2\rho+\beta)}.$$

By using the Euler-Rayleigh inequalities $\kappa_k^{-1/k} < \varsigma_{\rho,\beta,1}^2 < \frac{\kappa_k}{\kappa_{k+1}}$ for $k \in \{1,2\}$ we obtain the inequalities of the theorem. □

Theorem 7. *For $\rho,\beta > 0$, the radius of convexity $r^c(h_{\rho,\beta})$ of the function $h_{\rho,\beta}$ is the smallest positive root of the equation $(zh'_{\rho,\beta}(z))' = 0$ and it satisfies the following inequalities*

$$\frac{\Gamma(\rho+\beta)}{4\Gamma(\beta)} < r^c(h_{\rho,\beta}) < \frac{4\Gamma(\rho+\beta)\Gamma(2\rho+\beta)}{\Delta_{16,9}(\rho,\beta)},$$

$$\sqrt{\frac{\Gamma^2(\rho+\beta)\Gamma(2\rho+\beta)}{\Gamma(\beta)\Delta_{16,9}(\rho,\beta)}} < r^c(h_{\rho,\beta})$$
$$< \frac{\Gamma(\rho+\beta)\Gamma(3\rho+\beta)\Delta_{16,9}(\rho,\beta)}{8\Gamma^3(\rho+\beta)\Gamma(2\rho+\beta) + 2\Gamma(\beta)\Gamma(3\rho+\beta)\Delta_{32,27}(\rho,\beta)}.$$

Proof. By definition we have

$$\omega_{\rho,\beta}(z) = (zh'_{\rho,\beta}(z))' = 1 + \sum_{n\geq 1} \frac{(-1)^n(n+1)^2\Gamma(\beta)}{n!\Gamma(n\rho+\beta)} z^n. \tag{22}$$

Moreover, we know that $h_{\rho,\beta}$ belongs to the Laguerre-Pólya class $\mathcal{LP}$, and consequently the function $\omega_{\rho,\beta}$ also belongs to the Laguerre-Pólya class. In other words, the zeros of the function $\omega_{\rho,\beta}$ are all real. Assume that $\varrho_{\rho,\beta,n}$ are the positive zeros of the function $\omega_{\rho,\beta}$. In this case, the function $\omega_{\rho,\beta}$ has the infinite product representation as follows:

$$\omega_{\rho,\beta}(z) = \prod_{n\geq 1}\left(1 - \frac{z}{\varrho_{\rho,\beta,n}}\right). \tag{23}$$

By taking the logarithmic derivative of both sides of (23) for $|z| < \varrho_{\rho,\beta,1}$ we have

$$\frac{\omega'_{\rho,\beta}(z)}{\omega_{\rho,\beta}(z)} = -\sum_{k\geq 0}\mu_{k+1}z^k, \tag{24}$$

where $\mu_k = \sum_{n\geq 1} \varrho_{\rho,\beta,n}^{-k}$. In addition, by using the derivative of the infinite sum representation of (22) we obtain

$$\frac{\omega'_{\rho,\beta}(z)}{\omega_{\rho,\beta}(z)} = -\sum_{n\geq 0} t_n z^n \Big/ \sum_{n\geq 0} s_n z^n, \tag{25}$$

where

$$t_n = \frac{(-1)^n (n+2)^2 \Gamma(\beta)}{n!\Gamma((n+1)\rho+\beta)} \qquad s_n = \frac{(-1)^n (n+1)^2 \Gamma(\beta)}{n!\Gamma(n\rho+\beta)}.$$

By comparing the coefficients of (24) and (25) we get

$$\mu_1 = \frac{4\Gamma(\beta)}{\Gamma(\rho+\beta)}, \qquad \mu_2 = \frac{16\Gamma^2(\beta)}{\Gamma^2(\rho+\beta)} - \frac{9\Gamma(\beta)}{\Gamma(2\rho+\beta)}$$

and

$$\mu_3 = \frac{64\Gamma^3(\beta)}{\Gamma^3(\rho+\beta)} + \frac{8\Gamma(\beta)}{\Gamma(3\rho+\beta)} - \frac{54\Gamma^2(\beta)}{\Gamma(\rho+\beta)\Gamma(2\rho+\beta)}.$$

By considering the Euler-Rayleigh inequalities $\mu_k^{-1/k} < \varrho_{\rho,\beta,1} < \frac{\mu_k}{\mu_{k+1}}$, $k \in \{1,2\}$, we have the inequalities of the theorem. □

2.3. Some particular cases of the main results

It is important to mention that the Wright function is actually a generalization of a transformation of the Bessel function of the first kind. Namely, we have the relation

$$\lambda_{1,1+\nu}(z) = \phi(1, 1+\nu, -z^2) = z^{-\nu} J_\nu(2z),$$

where J_ν stands for the Bessel function of the first kind and order ν. Taking this into account, it is clear that Theorem 1, in particular when $\rho = 1$ and $\beta = \nu+1$, reduces to some interesting results, and one of them naturally complements the results from [4, Theorem 1]. The result on $f_{1,\nu+1}$ is new and complements [4, Theorem 1]; however, the results on $g_{1,\nu+1}$ and $h_{1,\nu+1}$ are not new, they were proved in [4, Theorem 1]. Thus the last two parts of Theorem 1 are natural generalizations of parts **b** and **c** of [4, Theorem 1].

Corollary 1. *Let $\nu > -1$ and $\alpha \in [0,1)$.*

a. *The radius of starlikeness of order α of $f_{1,\nu+1}(z) = (\Gamma(\nu+1) z J_\nu(2z))^{\frac{1}{\nu+1}}$ is the smallest positive root of the equation*

$$2zJ'_\nu(2z) + (1-\alpha(\nu+1))J_\nu(2z) = 0.$$

b. *The radius of starlikeness of order α of $g_{1,\nu+1}(z) = \Gamma(\nu+1) z^{1-\nu} J_\nu(2z)$ is the smallest positive root of the equation*

$$2zJ'_\nu(2z) + (1-\alpha-\nu)J_\nu(2z) = 0.$$

c. *The radius of starlikeness of order α of $h_{1,\nu+1}(z)=\Gamma(\nu+1)z^{1-\frac{\nu}{2}}J_\nu(2\sqrt{z})$ is the smallest positive root of the equation*

$$2\sqrt{z}J_\nu'(2\sqrt{z})+(2-2\alpha-\nu)J_\nu(2\sqrt{z})=0.$$

By choosing the values $\rho=1$ and $\beta=\nu+1$ in Theorem 2 we obtain the following corollary.

Corollary 2. *If $\nu>-1$, then we have*

$$\sqrt{\frac{\nu+1}{\nu+3}}<r^\star(f_{1,\nu+1})<(\nu+1)\sqrt{\frac{(\nu+2)(\nu+3)}{\nu^3+7\nu^2+15\nu+13}},$$
$$\sqrt[4]{\frac{(\nu+1)^3(\nu+2)}{\nu^3+7\nu^2+15\nu+13}}<r^\star(f_{1,\nu+1})$$
$$<(\nu+1)\sqrt{\frac{2(\nu+3)(\nu^3+7\nu^2+15\nu+13)}{\nu^5+15\nu^4+80\nu^3+222\nu^2+319\nu+196}}.$$

Now, by using the relation between the Wright function and the Bessel function of the first kind we can see that our main results, which are given in Theorem 3 and Theorem 4 when we take $\rho=1$ and $\beta=\nu+1$, correspond to the results in [2, Theorem 1] and [2, Theorem 2].

Corollary 3. *If $\nu>-1$, then we have*

$$\sqrt{\frac{\nu+1}{3}}<r^\star(g_{1,\nu+1})<\sqrt{\frac{3(\nu+1)(\nu+2)}{4\nu+13}},$$
$$\sqrt[4]{\frac{(\nu+1)^2(\nu+2)}{4\nu+13}}<r^\star(g_{1,\nu+1})<\sqrt{\frac{(\nu+1)(\nu+3)(4\nu+13)}{2(4\nu^2+26\nu+49)}}.$$

Consider the function $z\mapsto\varphi_\nu(z)=2^\nu\Gamma(\nu+1)z^{1-\nu}J_\nu(z)$, which is a normalized Bessel function of the first kind, considered in [2, Theorem 1]. Since $\varphi_\nu(2z)=2g_{1,\nu+1}(z)$, we obtain that the above inequalities coincide with the inequalities of [2, Theorem 1].

Corollary 4. *If $\nu>-1$, then we have*

$$\frac{\nu+1}{2}<r^\star(h_{1,\nu+1})<\frac{2(\nu+1)(\nu+2)}{\nu+5},$$
$$\frac{(\nu+1)\sqrt{\nu+2}}{\sqrt{\nu+5}}<r^\star(h_{1,\nu+1})<\frac{(\nu+1)(\nu+3)(\nu+5)}{\nu^2+8\nu+23}.$$

By considering that $\phi_\nu(4z)=4h_{1,\nu+1}(z)$, where $\Phi_\nu(z)=2^\nu\Gamma(\nu+1)z^{1-\frac{\nu}{2}}J_\nu(\sqrt{z})$, we can see that the above inequalities correspond to the results of [2, Theorem 2].

Finally, we mention that if we take $\rho=1$, $\beta=\nu+1$ with $\nu>-1$ in Theorem 6 and Theorem 7, we can see that the following inequalities correspond to the results which are given in [1, Theorem 6] and [1, Theorem 7], respectively.

Corollary 5. *If $\nu > -1$, then we have*

$$\frac{\sqrt{\nu+1}}{3} < r^c(g_{1,\nu+1}) < 3\sqrt{\frac{(\nu+1)(\nu+2)}{56\nu+137}},$$
$$\sqrt[4]{\frac{(\nu+1)^2(\nu+2)}{56\nu+137}} < r^c(g_{1,\nu+1}) < \sqrt{\frac{(\nu+1)(\nu+3)(56\nu+137)}{2(208\nu^2+1172\nu+1693)}}.$$

Corollary 6. *If $\nu > -1$, then we have*

$$\frac{\nu+1}{4} < r^c(h_{1,\nu+1}) < \frac{4(\nu+1)(\nu+2)}{7\nu+23},$$
$$\sqrt{\frac{(\nu+1)^2(\nu+2)}{7\nu+23}} < r^c(h_{1,\nu+1}) < \frac{(\nu+1)(\nu+3)(7\nu+23)}{2(9\nu^2+60\nu+115)}.$$

2.4. Problems for further research

It is interesting to see how far the properties of Bessel functions of the first kind may be extended to apply to the Wright function. In this paper, we can see that those properties of Bessel functions which come from the fact that they are entire can be extended to the Wright function without a major difficulty. However, we would like to see whether other properties of Bessel functions of the first kind can be extended to Wright functions or not. Here is a short list of possible open questions/problems, which are worth studying:

1. *What can we say about the monotonicity of the zeros $\lambda_{\rho,\beta,n}$ with respect to β (or ρ)?* The answer to this question would ensure that it would be possible to obtain necessary and sufficient conditions on the parameters ρ and β such that the normalized forms of the Wright function belong to a certain class of univalent functions, like starlike, convex or spirallike. Such kind of results would improve the existing results in the literature (see [20, 21, 22, 23, 24, 25]).

2. *Is it possible to express the derivative of the zeros $\lambda_{\rho,\beta,n}$ with respect to β (or ρ) for fixed n?* In [7], the Watson formulae for the derivative of the zeros of the Bessel function of the first kind and its derivative played an important role in obtaining necessary conditions for the order of normalized Bessel functions of the first kind such that these functions belong to the class of convex functions.

3. *Is it possible to use continued fractions to obtain the order of starlikeness and convexity of normalized Wright functions?*

Each of the above problems seems to be difficult to solve because the Wright function is not a solution of a second order homogeneous linear differential equation (as the Bessel function) and although its power series structure is similar to that of Bessel functions, it seems that its properties are more difficult to be studied.

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