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Stability with respect to part of the variables of nonlinear Caputo fractional differential equations

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Abstract. In this paper, stability with respect to part of the variables of nonlinear Caputo fractional differential equations is studied. Sufficient conditions of stability, uniform stability, Mittag Leffler stability and asymptotic uniform stability of this type are obtained within the method of Lyapunov-like functions.

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1. Introduction

The study of fractional order systems [8, 3] has gained importance in recent years. Based on the concept of integration or differentiation of fractional order, several dynamic systems are better described with fractional order model [6], for example, in electromagnetic systems [5], dielectric polarization [17], economy [10] and image processing [13].

Because of the development of science and complex engineering systems, previous research has documented the use of fractional calculus in many issues of control theory, such as stability [4, 11, 14, 18]. Indeed, in [4], authors described an uniform stability for fractional order systems using general quadratic Lyapunov functions. In [11], Yan Li et al. presented the Mittag-Leffler stability of fractional order nonlinear dynamic systems. Furthermore, stability analysis of Hilfer fractional differential systems is shown in [14]. On the other hand, in [18], the authors described the asymptotical stability of nonlinear fractional differential system with Caputo derivative.

In this way, considerable attention has been paid to the concept of stability with respect to part of the system's states [12, 15]. Such concept for integer-order systems was originally introduced by [16]. From then on, stability with respect to part of the variables (SPV) analysis for integer-order systems has gained lots of attention [1, 7, 12, 15]. For instance, in [1] authors presented an approach to SPV in systems with impulse effect, and introduced sufficient conditions based on the Lyapunov function to guarantee their main results.

However, to the best of our knowledge, no paper in the literature has tackled the problem of SPV analysis for fractional order systems. By this fact, the main

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contribution of this paper is to study SPV of nonautonomous systems in the sense of Caputo fractional derivative.

The remainder of this paper is organized as follows. In Section 2, necessary notations and preliminaries are given. Sufficient conditions for stability, uniform stability, asymptotic uniform stability and Mittag-Leffler stability with respect to part of the variables of fractional nonautonomous systems are presented in Section 3. In Section 4, two illustrative examples are given.

2. Preliminaries

In this section, some notations and preliminary results are introduced.

Definition 1 (see [3]). *Given an interval $[a,b]$ of $\mathbb{R}$, the Riemann-Liouville fractional integral of a function $x \in L^1([a,b])$ of order $\alpha > 0$ is defined by*

$$I_a^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} x(\tau) d\tau, \quad t \in [a,b],$$

where Γ is the Gamma function.

For $\alpha = 0$, $I_a^0 := I$, the identity operator.

Definition 2 (see [3]). *Given an interval $[a,b]$ of $\mathbb{R}$, the Caputo fractional derivative of a function x of order $\alpha > 0$ is defined by*

$${}^C D_{a,t}^{\alpha} x(t) = I_a^{m-\alpha} x^{(m)}(t), \quad t \in [a,b],$$

where $0 < m-1 < \alpha \leq m$.

When $0 < \alpha < 1$, then the Caputo fractional derivative of order α of an absolutely continuous function x on $[a,b]$ reduces to

$${}^C D_{t_0,t}^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t (t-\tau)^{-\alpha} x'(\tau) d\tau, \quad t \in [a,b]. \tag{1}$$

Lemma 1 (see [4]). *Let $\alpha \in (0,1)$ and let $P \in R^{n \times n}$ be a constant, square, symmetric and positive definite matrix. Then the following relationship holds*

$$\frac{1}{2}\, {}^C D_{t_0,t}^{\alpha} (x^T(t) P x(t)) \leq x^T(t) P\, {}^C D_{t_0,t}^{\alpha} x(t), \ t \geq t_0.$$

Similarly to the exponential function used in the solutions of integer-order differential systems, the Mittag-Leffler function is frequently used in the solutions of fractional-order differential systems.

Definition 3 (see [8]). *The Mittag-Leffler function with two parameters is defined as*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\alpha+\beta)},$$

where $\alpha > 0$, $\beta > 0$, $z \in C$.

When $\beta = 1$, we have $E_{\alpha}(z) = E_{\alpha,1}(z)$; furthermore, $E_{1,1}(z) = e^z$.

We consider the nonhomogeneous linear fractional differential equation with the Caputo fractional derivative

$$\begin{aligned} {}^{C}D^{\alpha}_{t_0,t}x(t) &= \lambda x + h(t),\ t \geq t_0 \\ x(t_0) &= x_0. \end{aligned} \tag{2}$$

Problem (2) was studied by Kilbas et al. [8] (see pp. 295, (5.2.83)), and its solution has the form

$$x(t; t_0, x_0) = x_0 E_\alpha(\lambda(t-t_0)^\alpha) + \int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) h(s) ds, \tag{3}$$

provided that the integral on the right-hand side of (3) is convergent.

3. Main results

In this section, several sufficient conditions on stability with respect to part of the variables of nonlinear Caputo fractional differential equations are given.

Consider the system of fractional differential equations with a Caputo derivative for $\alpha \in (0, 1)$

$$ {}^{C}D^{\alpha}_{t_0,t}x(t) = f(t, x),\ t \geq t_0, \tag{4}$$

$$x \in R^n,\ x = (y, z),\ y \in R^m,\ z \in R^p,\ m > 0,$$

with initial condition $x(t_0) = x_0 = (y_0, z_0)$, where $\alpha \in (0, 1)$ and $f \in C(R_+ \times R^n, R^n)$.

Suppose that f is smooth enough to guarantee the existence of a global solution $x(t) = x(t; t_0, x_0)$ of system (4) for each initial condition (t_0, x_0). Some sufficient conditions for the existence and uniqueness of solutions for fractional differential equations are given in [2, 9].

Assume that the origin $x = 0$ is an equilibrium point of fractional-order system (4); that is, $f(t, 0) = 0,\ \forall t \geq 0$.

Definition 4. *The equilibrium point $x = 0$ of fractional-order system (4) is said to be*

(*i*) *Stable with respect to y, if for every $\epsilon > 0$ and $t_0 \in R_+$ there exists $\delta := \delta(\epsilon, t_0)$ such that for any $x_0 \in R^n$, the inequality $\|x_0\| < \delta$ implies $\|y(t; t_0, x_0)\| < \epsilon$ for $t \geq t_0$.*

(*ii*) *Uniformly stable with respect to y, if it is stable and δ depends only on ϵ.*

(*iii*) *Uniformly attractive with respect to y, if there exists $\beta > 0$ such that for every $\epsilon > 0$ there exists $T := T(\epsilon) > 0$ such that for any $t_0 \in R_+$, $x_0 \in R^n$ with $\|x_0\| < \beta$ the inequality $\|y(t; t_0, x_0)\| < \epsilon$ holds for $t \geq t_0 + T$.*

(*iv*) *Globally uniformly attractive with respect to y if (iii) is satisfied for any $\beta > 0$.*

(*v*) *Uniformly asymptotically stable with respect to y, if it is uniformly stable with respect to y and uniformly attractive with respect to y.*

(*vi*) *Globally uniformly asymptotically stable with respect to* y, *if it is uniformly stable with respect to* y *and globally uniformly attractive with respect to* y.

(*vii*) *Uniformly Mittag-Leffler stable with respect to* y, *if each solution of system (4) satisfies:*

$$\|y(t;t_0,x_0)\| \leq \Big[m\big(x_0\big)E_\alpha\big(-\lambda(t-t_0)^\alpha\big)\Big]^b, \ \forall t \geq t_0, \tag{5}$$

with $b > 0$, $\lambda > 0$, $m(0) = 0$, $m(x) \geq 0$ *and* m *is locally Lipschitz.*

Definition 5. *A continuous function* $\psi : R_+ \longrightarrow R_+$ *is said to belong to class* $\mathcal{K}$ *if it is strictly increasing and* $\psi(0) = 0$. *It is to belong to class* $\mathcal{K}_\infty$ *if in addition* $\lim\limits_{s\longrightarrow+\infty} \psi(s) = +\infty$.

Within the method of Lyapunov-like functions, we present the following results.

Theorem 1. *Consider system (4) and assume that there exist a continuously differentiable function* $V : R_+ \times R^n \longrightarrow R$ *and class* $\mathcal{K}$ *function* α_1 *satisfying*

$$\alpha_1(\|y\|) \leq V(t,x), \ V(t,0) = 0, \quad \forall t \geq 0, \ \forall x \in R^n, \tag{6}$$

$$^C D^\alpha_{t_0,t} V(t, x(t;t_0,x_0)) \leq 0, \quad \forall t \geq t_0, \ \forall t_0 \geq 0; \tag{7}$$

then $x = 0$ *is stable with respect to* y.

Moreover, if for some $\alpha_2 \in \mathcal{K}$:

$$V(t,x) \leq \alpha_2(\|x\|), \quad \forall t \geq 0, \ \forall x \in R^n; \tag{8}$$

then, $x = 0$ *is uniformly stable with respect to* y.

Proof. It follows from (7) that there exists a nonnegative function $h(t)$ satisfying

$$^C D^\alpha_{t_0,t} V(t, x(t;t_0,x_0)) = -h(t), \ \forall t \geq t_0. \tag{9}$$

It follows from (3) that for $t \geq t_0$,

$$\begin{aligned} V(t,x(t;t_0,x_0)) &= V(t_0,x_0) - \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} h(s)ds \\ &\leq V(t_0,x_0). \end{aligned} \tag{10}$$

Using (6) and (10) we obtain

$$\alpha_1(\|y(t;t_0,x_0)\|) \leq V(t_0,x_0), \ \forall t \geq t_0, \tag{11}$$

Let $\epsilon > 0$. Since $V(t_0,0) = 0$ and V is continuous, then there exists $\delta := \delta(\epsilon,t_0)$ such that:

$$\|x_0\| < \delta \implies V(t_0,x_0) < \alpha_1(\epsilon). \tag{12}$$

Hence, by (11) and (12) we have:

$$\|x_0\| < \delta \implies \|y(t;t_0,x_0)\| < \epsilon, \ \forall \ t \geq t_0. \tag{13}$$

Therefore, $x = 0$ is stable with respect to y.

Let us now show uniform stability of $x = 0$ with respect to y.

Let $\epsilon > 0$, there exists $\delta := \delta(\epsilon) > 0$ such that $\alpha_2(\delta) < \alpha_1(\epsilon)$.

Let $x_0 \in R^n$ such that $\|x_0\| < \delta$; then using (8) and (11) we obtain:

$$\begin{aligned}\alpha_1(\|y(t;t_0,x_0)\|) &\leq V(t_0,x_0)\\ &\leq \alpha_2(\|x_0\|) < \alpha_2(\delta) < \alpha_1(\epsilon).\end{aligned}$$

Since $\alpha_1 \in \mathcal{K}$, then

$$\|y(t;t_0,x_0)\| < \epsilon,\ \forall t \geq t_0.$$

Hence, $x = 0$ is uniformly stable with respect to y. □

Theorem 2. *Consider system (4) and assume that there exist a continuously differentiable function* $V : R_+ \times R^n \longrightarrow R$, $k \in \{m, m+1, ...n\}$, $c > 0$ *and class* $\mathcal{K}$ *functions* α_i, $(i = 1, 2)$ *satisfying*

$$\alpha_1(\|y\|) \leq V(t,x) \leq \alpha_2(\|w\|), \quad \forall t \geq 0,\ \forall x \in R^n, \tag{14}$$

$$^{C}D^{\alpha}_{t_0,t}V(t,x(t;t_0,x_0)) \leq -c\alpha_2(\|w(t;t_0,x_0)\|), \quad \forall t \geq t_0,\ \forall t_0 \geq 0, \tag{15}$$

where, $w = (x_1, x_2, ..., x_k)$.

Then, $x = 0$ *is uniformly asymptotically stable with respect to* y.

Moreover, if $\alpha_i \in \mathcal{K}_\infty$, $(i = 1, 2)$, *then* $x = 0$ *is globally uniformly asymptotically stable with respect to* y.

Proof. From Theorem 1 we have that $x = 0$ is uniformly stable with respect to y.

Let $r_1 = \lim\limits_{s \longrightarrow +\infty} \alpha_1(s)$ and $r \in \big(0, r_1\big)$.

It follows from (14) and (15) that

$$^{C}D^{\alpha}_{t_0,t}V(t,x(t;t_0,x_0)) \leq -cV(t,x(t;t_0,x_0)). \tag{16}$$

There exists a nonnegative function $h(t)$ satisfying:

$$^{C}D^{\alpha}_{t_0,t}V(t,x(t;t_0,x_0)) = -cV(t,x(t;t_0,x_0)) - h(t). \tag{17}$$

Since $E_{\alpha,\alpha}\Big(-c(t-s)^\alpha\Big)$ and $h(t)$ are nonnegative functions, it follows from (3) that for $t \geq t_0$,

$$V(t,x(t;t_0,x_0)) \leq V(t_0,x_0)E_\alpha\big(-c(t-t_0)^\alpha\big),\ \forall t \geq t_0. \tag{18}$$

Hence by (14) we have for $t \geq t_0$:

$$\begin{aligned}\alpha_1(\|y(t;t_0,x_0)\|) \leq V(t,x(t;t_0,x_0)) &\leq V(t_0,x_0)E_\alpha\big(-c(t-t_0)^\alpha\big)\\ &\leq \alpha_2(\|w_0\|)E_\alpha\big(-c(t-t_0)^\alpha\big)\\ &\leq \alpha_2(\|x_0\|)E_\alpha\big(-c(t-t_0)^\alpha\big).\end{aligned} \tag{19}$$

Let $x_0 \in R^n$ such that $\|x_0\| < \alpha_2^{-1}(r)$. It follows from (19) that

$$\|y(t;t_0,x_0)\| \leq \alpha_1^{-1}\Big(rE_\alpha\big(-c(t-t_0)^\alpha\big)\Big),\ \forall t \geq t_0. \tag{20}$$

Let $\epsilon > 0$. We have

$$\lim_{s\to+\infty} E_\alpha\big(-cs^\alpha\big) = 0. \tag{21}$$

Then there exists $T := T(\epsilon)$ such that

$$E_\alpha\Big(-c(t-t_0)^\alpha\Big)\Big) < \frac{\alpha_1(\epsilon)}{r},\ \forall t - t_0 \geq T. \tag{22}$$

From (20) and (22), it follows that

$$\|y(t;t_0,x_0)\| \leq \epsilon,\ \forall t \geq t_0 + T.$$

This inequality shows that $x = 0$ is uniformly attractive with respect to y.

Hence, $x = 0$ is uniformly asymptotically stable with respect to y.

Let us consider now the case where $\alpha_i \in \mathcal{K}_\infty$, $(i = 1, 2)$.

It follows from (19) that

$$\|y(t;t_0,x_0)\| \leq \alpha_1^{-1}\Big(\alpha_2(\|x_0\|)E_\alpha\big(-c(t-t_0)^\alpha\big)\Big),\ \forall t \geq t_0. \tag{23}$$

Let $\epsilon > 0$, $\beta > 0$ and $x_0 \in R^n$ such that $\|x_0\| < \beta$.

Then by (23) we have:

$$\|y(t;t_0,x_0)\| \leq \alpha_1^{-1}\Big(\alpha_2(\beta)E_\alpha\big(-c(t-t_0)^\alpha\big)\Big),\ \forall t \geq t_0. \tag{24}$$

It follows from (21) that there exists $T := T(\epsilon, \beta)$ such that

$$E_\alpha\Big(-c(t-t_0)^\alpha\Big)\Big) < \frac{\alpha_1(\epsilon)}{\alpha_2(\beta)},\ \forall t - t_0 \geq T. \tag{25}$$

From (24) and (25), it follows that

$$\|y(t;t_0,x_0)\| \leq \epsilon,\ \forall t \geq t_0 + T.$$

This inequality shows that $x = 0$ is globally uniformly attractive with respect to y.

Hence, $x = 0$ is globally uniformly asymptotically stable with respect to y. □

Theorem 3. *Consider system (4) and assume that there exist a continuously differentiable function $V : R_+ \times R^n \longrightarrow R$ and $k \in \{m, m+1, ...n\}$ such that*

$$c_1\|y\|^a \leq V(t,x) \leq c_2\|w\|^a,\quad \forall t \geq 0,\ \forall x \in R^n, \tag{26}$$

$$^C D^\alpha_{t_0,t} V(t, x(t;t_0,x_0)) \leq -c_3\|w(t;t_0,x_0)\|^a,\quad \forall t \geq t_0,\ \forall t_0 \geq 0, \tag{27}$$

where $w = (x_1, x_2, ..., x_k) \in R^k$, $a \geq 1$, c_1, c_2 and c_3 are positive constants.

Then $x = 0$ is uniformly Mittag-Leffler stable with respect to y.

Proof. Consider the functions $\alpha_1(s) = c_1 s^a$, $\alpha_2(s) = c_2 s^a$ and the constant $c = \frac{c_3}{c_2}$.

We have $c_3 s^a = \frac{c_3}{c_2}\alpha_2(s)$. Then the assumptions of Theorem 2 are satisfied and from (23) it follows that:

$$\|y(t;t_0,x_0)\| \leq \Big(\frac{c_2}{c_1}\|x_0\|^a E_\alpha\big(-\frac{c_3}{c_2}(t-t_0)^\alpha\big)\Big)^{\frac{1}{a}},\ \forall t \geq t_0. \tag{28}$$

Hence, $x = 0$ is uniformly Mittag-Leffler stable with respect to y. □

4. Two illustrative examples

The following two illustrative examples are provided to show the usefulness of the stability with respect to part of variables notion.

Example 1. *Consider the following fractional order system*

$$\begin{aligned} {}^C D^{\alpha}_{t_0,t} x_1 &= -x_1 + \sin(x_3)x_1, \\ {}^C D^{\alpha}_{t_0,t} x_2 &= -x_2 + e^{-t}\cos(x_1)x_2 \\ {}^C D^{\alpha}_{t_0,t} x_3 &= x_3, \end{aligned} \tag{29}$$

where $0 < \alpha < 1$ *and* $x(t) = \big(x_1(t), x_2(t), x_3(t)\big) \in \mathbb{R}^3$.

Consider the Lyapunov-like function: $V(t,x) = \dfrac{x_1^2 + x_2^2}{2}$.

By Lemma 1 we have

$$\begin{aligned} &{}^C D^{\alpha}_{t_0,t} V(t, x(t;t_0,x_0)) \\ &\quad \le x_1(t;t_0,x_0)\,{}^C D^{\alpha}_{t_0,t} x_1(t;t_0,x_0) + x_2(t;t_0,x_0)\,{}^C D^{\alpha}_{t_0,t} x_2(t;t_0,x_0) \\ &\quad \le -x_1^2(t;t_0,x_0) + \sin(x_3(t;t_0,x_0))x_1^2(t;t_0,x_0) - x_2^2(t;t_0,x_0) \\ &\qquad + x_2^2(t;t_0,x_0)e^{-t}\cos(x_1(t;t_0,x_0)) \\ &\quad \le 0. \end{aligned} \tag{30}$$

Then, the assumptions of Theorem 1 *are satisfied.*

Hence, $x = 0$ *is uniformly stable with respect to* (x_1, x_2).

Remark 1. *We have* $x_3(t;t_0,x_0) = x_{30}E_{\alpha}\big((t-t_0)^{\alpha}\big)$, *where* $x_{30} = x_3(t_0;t_0,x_0)$; *then* $x = 0$ *is unstable.*

Example 2. *Consider the following fractional-order system*

$$\begin{aligned} {}^C D^{\alpha}_{t_0,t} x_1 &= -2x_1 + \tfrac{\sin(x_3)}{1+t^2}x_1, \\ {}^C D^{\alpha}_{t_0,t} x_2 &= -2x_2 + \cos(x_1)x_2 \\ {}^C D^{\alpha}_{t_0,t} x_3 &= x_3, \end{aligned} \tag{31}$$

where $0 < \alpha < 1$ *and* $x(t) = \big(x_1(t), x_2(t), x_3(t)\big) \in \mathbb{R}^3$.

Consider the Lyapunov-like function: $V(t,x) = \dfrac{x_1^2 + x_2^2}{2}$.

By Lemma 1 we have

$$\begin{aligned} &{}^C D^{\alpha}_{t_0,t} V(t, x(t;t_0,x_0)) \\ &\quad \le x_1(t;t_0,x_0)\,{}^C D^{\alpha}_{t_0,t} x_1(t;t_0,x_0) + x_2(t;t_0,x_0)\,{}^C D^{\alpha}_{t_0,t} x_2(t;t_0,x_0) \\ &\quad \le -2x_1(t;t_0,x_0) + \frac{\sin(x_3(t;t_0,x_0))}{1+t^2}x_1^2(t;t_0,x_0) - 2x_2^2(t;t_0,x_0) \\ &\qquad + x_2^2(t;t_0,x_0)\cos(x_1(t;t_0,x_0)) \\ &\quad \le -(x_1^2(t;t_0,x_0) + x_2^2(t;t_0,x_0)). \end{aligned} \tag{32}$$

Then, the assumptions of Theorem 3 *are satisfied.*

Hence, $x = 0$ *is uniformly Mittag-Leffler stable with respect to* (x_1, x_2).

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