Trigonometric Functions in the \( m \)-plane

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\textbf{ABSTRACT} \\
In this paper, we define the trigonometric functions in the plane with the \( m \)-metric. And then we give two properties about these trigonometric functions, one of which states the area formula for a triangle in the \( m \)-plane in terms of the \( m \)-metric. \\
\textbf{Key words:} Taxicab metric, Chinese checker metric, alpha metric, \( m \)-metric, \( m \)-trigonometry \\
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1 Introduction

The \textit{taxicab metric} was given in a family of metrics of the real plane by Minkowski [16]. And the taxicab geometry introduced by Menger [15], and developed by Krause [14]. Later, Chen [7] developed the \textit{Chinese checker metric}, and Tian [19] gave a family of metrics, \( \alpha \)-metric for \( \alpha \in [0, \pi/4] \), which includes the taxicab and Chinese checker metrics as special cases, and Çolakoğlu [8] extended the \( \alpha \)-metric for \( \alpha \in [0, \pi/2] \). Afterwards, Bayar, Ekmeği and Akça [5] presented a generalization of \( \alpha \)-metric: the \textit{generalized absolute value metric}. Finally, Çolakoğlu and Kaya [10] gave a generalization for all these metrics: \( m \)-metric (or \( m \)-generalized absolute value metric).

During the recent years, trigonometry on the plane geometries based on these metrics have been studied. See [1], [2], [3], [4], [5], [6], [12], [17] and [18] for some of studies. In this paper, we study on trigonometry in the plane with the generalized \( m \)-metric. First, we give definitions of trigonometric functions for the \( m \)-metric, which also generalize the definitions given before, and then give two properties about these trigonometric functions, one of which states a formula to calculate the area of any triangle in the \( m \)-plane, being an alternative to the one given in [13]. This study also provides a facility for further subjects as cosine theorem, norm and inner-product in terms of the \( m \)-metric.

Let \( P_1 = (x_1, y_1) \) and \( P_2 = (x_2, y_2) \) be two points in \( \mathbb{R}^2 \). For each real numbers \( a \) and \( b \), such that \( a \geq b \geq 0 \neq a \), the function \( d_m: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow [0, \infty) \) defined by
\[
d_m(P_1, P_2) = (a \Delta_{AB} + b \delta_{AB})/\sqrt{1+m^2} \tag{1}
\]
where
\[
\Delta_{AB} = \max\{|(x_1 - x_2) + m(y_1 - y_2)|, |m(x_1 - x_2) - (y_1 - y_2)|\}
\]
and
\[
\delta_{AB} = \min\{|(x_1 - x_2) + m(y_1 - y_2)|, |m(x_1 - x_2) - (y_1 - y_2)|\},
\]
is called the \textit{m-distance function} in \( \mathbb{R}^2 \), and the real number \( d_m(P_1, P_2) \) is called the \textit{m-distance} between points \( P_1 \) and \( P_2 \).

Cartesian coordinate plane endowed with the \( m \)-metric forms a metric space, \( \mathbb{R}^2_m \) or \( (\mathbb{R}^2, d_m) \), and it is constructed by simply replacing the well-known Euclidean distance function
\[
d_E(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \tag{2}
\]
by the \( m \)-distance function \( d_m \) in \( \mathbb{R}^2 \) (see [10]). In all that follows, we use \( a' = a/\sqrt{1+m^2} \) and \( b' = b/\sqrt{1+m^2} \) to shorten phrases.
2 Trigonometric Functions

We know that if \( P = (x, y) \) is a point on the Euclidean unit circle, then \( x = \cos \theta \) and \( y = \sin \theta \), where \( \theta \) is the angle with the positive \( x \)-axis as the initial side and the radial line passing through the point \( P \) as the terminal side. One can determine the standard definitions of the trigonometric functions using the unit \( m \)-circle in \( \mathbb{R}^2_m \), in the same way one determines their Euclidean analogues. The unit \( m \)-circle (see Figure 1) is the set of points \( (x, y) \), which satisfies the equation

\[
a' \max \{|x+my|, |mx-y|\} + b' \min \{|x+my|, |mx-y|\} = 1.
\]

(3)

Figure 1: Graph of unit \( m \)-circles

So, for the point \( P = (x, y) \) on the \( m \)-unit circle, let us determine sine and cosine functions in \( \mathbb{R}^2_m \) as \( x = \cos_m \theta \) and \( y = \sin_m \theta \), where \( \theta \) is the angle with the positive \( x \)-axis as the initial side and the radial line passing through the point \( P \) as the terminal side. Clearly, tangent and cotangent functions do not depend on the metric, since the slope of the radial line passing through the point \( (x, y) \) does not change. Thus, we have

\[
\tan_m \theta = \frac{\sin_m \theta}{\cos_m \theta} = \tan \theta \quad \text{and} \quad \cot_m \theta = \frac{\cos_m \theta}{\sin_m \theta} = \cot \theta.
\]

Obviously, the equation of the line joining \( (x, y) \) and \( (0, 0) \) is \( y = (\tan \theta)x \). Solving the system

\[
\begin{align*}
y &= (\tan \theta)x \\
n' \max \{|x+my|, |mx-y|\} + b' \min \{|x+my|, |mx-y|\} = 1
\end{align*}
\]

one gets sine and cosine functions in \( \mathbb{R}^2_m \):

\[
\cos_m \theta = \frac{\cos \theta}{a' \max \{X,Y\} + b' \min \{X,Y\}},
\]

(4)

\[
\sin_m \theta = \frac{\sin \theta}{a' \max \{X,Y\} + b' \min \{X,Y\}},
\]

(5)

where \( X = |\cos \theta + m \sin \theta| \), \( Y = |m \cos \theta - \sin \theta| \).

We can also determine secant and cosecant functions as in Euclidean plane: \( \csc_m \theta = \frac{1}{\sin_m \theta} \) and \( \sec_m \theta = \frac{1}{\cos_m \theta} \). For some values of \( a, b \) and \( m \), graphs of \( y = \sin_m x \) and \( y = \cos_m x \) are given in Figure 2, Figure 3, Figure 4 and Figure 5, for \(-2\pi < x < 2\pi\).
In \( \mathbb{R}^2_m \), the trigonometric identities differ from their Euclidean analogues in most cases. Some of the identities of these functions are like their Euclidean counterparts:
\[
\begin{align*}
\cos_m\left(\frac{\pi}{2} + \theta\right) &= -\sin_m \theta, \\
\sin_m\left(\frac{\pi}{2} + \theta\right) &= \cos_m \theta \\
\cos_m(\pi + \theta) &= -\cos_m \theta, \\
\sin_m(\pi + \theta) &= -\sin_m \theta \\
\cos_m(2\pi + \theta) &= \cos_m \theta, \\
\sin_m(2\pi + \theta) &= \sin_m \theta.
\end{align*}
\]

It is well known that the Pythagorean identity is the relation between sine and cosine functions: \( \sin^2 \theta + \cos^2 \theta = 1 \). In terms of the generalized \( m \)-metric, we get the following equation:
\[
\begin{align*}
\alpha' \max \{ |\cos_m \theta + m \sin_m \theta|, |m \cos_m \theta - \sin_m \theta| \} &+ b' \min \{ |\cos_m \theta + m \sin_m \theta|, |m \cos_m \theta - \sin_m \theta| \} = 1. 
\end{align*}
\]

One can also get the following equations easily:
\[
\begin{align*}
\alpha' \max \{ |\cot \theta + m|, |m \cot \theta - 1| \} &+ b' \min \{ |\cot \theta + m|, |m \cot \theta - 1| \} = |\csc_m \theta|.
\end{align*}
\]

Using the sum and difference formulas for tangent function one gets also the following equations:
\[
\begin{align*}
\tan_m(u + v) &= \frac{\tan_m u + \tan_m v}{1 - \tan_m u \tan_m v} \\
\cot_m(u + v) &= \frac{1 + \cot_m u \cot_m v}{\cot_m u + \cot_m v} \\
\sin_m(u + v) &= \sin_m u \cos_m v + \sin_m v \cos_m u \\
\cos_m(u + v) &= \cos_m u \cos_m v \pm \sin_m u \sin_m v.
\end{align*}
\]

3 Trigonometric Functions with Reference Angle

Unlike the Euclidean case, there is a non-uniform increment in the arc length as the angle \( \theta \) is incremented by a fix amount, in \( \mathbb{R}^2_m \). So, it is necessary to develop the trigonometric functions for any angle \( \theta \) using the reference angle \( \alpha \) of \( \theta \) (see [18]).

**Definition 1** Let \( \theta \) be an angle with the reference angle \( \alpha \) which is the angle between \( \theta \) and the positive direction of the x-axis in \( m \)-unit circle. Then the cosine and sine functions of the angle \( \theta \) with the reference angle \( \alpha \), \( \cos_m \alpha \) and \( \sin_m \alpha \) are defined by
\[
\begin{align*}
\cos_m \theta &= \cos_m(\alpha + \theta) \cos_m \alpha + \sin_m(\alpha + \theta) \sin_m \alpha \\
\sin_m \theta &= \sin_m(\alpha + \theta) \cos_m \alpha - \cos_m(\alpha + \theta) \sin_m \alpha.
\end{align*}
\]

In this definition, the angles of \( \alpha \) and \( (\alpha + \theta) \) are in standard position. So, the values of \( \cos_m(\alpha + \theta) \), \( \sin_m(\alpha + \theta) \), \( \cos_m \alpha \) and \( \sin_m \alpha \) are calculated by using equations (4) and (5). If \( \alpha = 0 \), then
\[
\begin{align*}
m \cos \theta &= \frac{m \cos \theta}{\alpha' \max \{ 1, |m| \} + b' \min \{ 1, |m| \}} \\
m \sin \theta &= \frac{m \sin \theta}{\alpha' \max \{ 1, |m| \} + b' \min \{ 1, |m| \}}.
\end{align*}
\]

The general definitions of other trigonometric functions for the angles which are not in standard position can be given similarly: \( \sin_m \theta = \frac{\sin \theta}{m \sin \theta} = \tan \theta, \cos_m \theta = \frac{\cos \theta}{m \cos \theta} = \cot \theta, \sec_m \theta = \frac{1}{m \sin \theta} \) and \( \csc_m \theta = \frac{1}{m \cos \theta} \). Consequently, the general definitions of trigonometric functions can be given by defining angles with the reference angle in plane with the generalized \( m \)-metric. It is well known that all rotations and translations preserve the Euclidean distance. In \( \mathbb{R}^2_m \), all translations and the rotations of the angle \( \theta \in \{ \frac{\pi}{2}, \frac{3\pi}{2}, 2\pi \} \) when \( b/a \neq 1/\sqrt{2} - 1 \) and also the rotations of the angle \( \theta \in \{ \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \} \) when \( b/a = 1/\sqrt{2} - 1 \) preserve \( m \)-distance (see [10]). The change of the length of a line segment by a rotation can be given by the following theorem:

**Theorem 1** Let any two points be \( A \) and \( B \) in \( \mathbb{R}^2_m \), and let the line segment \( AB \) be not parallel to the x-axis and the angle \( \alpha \) between the line segment \( AB \) and the positive direction of x-axis. If \( A'B' \) is the image of \( AB \) under the rotation with the angle \( \theta \), then
\[
d_m(A', B') = d_m(A, B) \sqrt{\frac{\cos^2_m(\alpha + \theta) + \sin^2_m(\alpha + \theta)}{\cos^2_m(\alpha + \theta) + \sin^2_m(\alpha + \theta)}}
\]

**Proof.** Since all translations preserve the \( m \)-distance, the line segment \( AB \) can be translated to the line segment \( OX \) such that \( O \) is the origin. Let the line segment \( OX' \) be the image of \( OX \) under rotation with the angle \( \theta \), and let \( d_m(A, B) = d_m(O, X') = k' \) and \( d_m(O, X') = k' \). If \( \alpha \) is the reference angle of \( \theta \), then \( X = (k \cos_m \alpha, k \sin_m \alpha) \) and \( X' = (k' \cos_m(\alpha + \theta), k' \sin_m(\alpha + \theta)) \). Since \( d_E(O, X) = d_m(O, X') \), one gets
\[
k' \sqrt{\cos^2_m(\alpha + \theta) + \sin^2_m(\alpha + \theta)} = k \sqrt{\cos^2_m \alpha + \sin^2_m \alpha}
\]
and
\[
d_m(A', B') = d_m(A, B) \sqrt{\frac{\cos^2_m(\alpha + \theta) + \sin^2_m(\alpha + \theta)}{\cos^2_m(\alpha + \theta) + \sin^2_m(\alpha + \theta)}}.
\]

The following corollary shows how one can find the generalized \( m \)-length of a line segment, after a rotation with an angle \( \theta \) in standard position:
Corollary 1 If the line segment AB is parallel to the x-axis, then
\[ d_m(A', B') = \frac{d_m(A, B)}{a' \max\{1, |m|\} + b' \min\{1, |m|\}} \sqrt{\cos_m^2 \theta + \sin_m^2 \theta} \]

(15)

Proof. Since \( \alpha = 0 \), proof is obvious. \( \square \)

In [13], an area formula for a triangle is given in the plane with the generalized \( m \)-metric (see also [11]). In the following theorem, the area of a triangle is given by using the trigonometric functions in \( \mathbb{R}^2_m \).

Theorem 2 Let ABC be any triangle in \( \mathbb{R}^2_m \), and let \( \theta \) be the angle between the line segments AC and BC. Then the area \( \mathcal{A} \) of the triangle ABC can be given by the following formula:
\[ \mathcal{A} = \frac{1}{2} d_m(A, C)d_m(B, C) \sin \theta. \]

(16)

Proof. Let \( d_m(A, C) = k \) and \( d_m(B, C) = k' \). We can take the vertex C as the origin, and \( A = (k \cos_m \alpha, k \sin_m \alpha) \) and \( B = (k' \cos_m (\alpha + \theta), k' \sin_m (\alpha + \theta)) \), without loss of generality. Thus, we have \( d_E(A, C) = k \sqrt{\cos_m^2 \alpha + \sin_m^2 \alpha} \) and \( d_E(B, C) = k' \sqrt{\cos_m^2 (\alpha + \theta) + \sin_m^2 (\alpha + \theta)} \). Also, it is easy to show that if \( \gamma \) is in standard position, then \( \cos_m \gamma = \cos \gamma \sqrt{\cos_m^2 \gamma + \sin_m^2 \gamma} \) and \( \sin_m \gamma = \sin \gamma \sqrt{\cos_m^2 \gamma + \sin_m^2 \gamma} \). Thus, one gets the equation
\[ \sin \theta = \sin \gamma \sqrt{\cos_m^2 \alpha + \sin_m^2 \alpha} \sqrt{\cos_m^2 (\alpha + \theta) + \sin_m^2 (\alpha + \theta)}. \]

(17)

If we use the values of \( d_E(A, C) \), \( d_E(B, C) \) and \( \sin \theta \) in the formula \( \mathcal{A} = \frac{1}{2} d_E(A, C)d_E(B, C) \sin \theta \), we get the area formula in the plane with the generalized \( m \)-metric:
\[ \mathcal{A} = \frac{1}{2} d_m(A, C)d_m(B, C) \sin \theta. \]

(16)

References

