An approach to mathematical induction — starting from the early stages of teaching mathematics*

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Abstract. *It is well-known that the method of proof by mathematical induction often creates serious problems to students who encounter these proofs in secondary schools. It is Therefore important to prepare the students for understanding and the application of the method of mathematical induction. For this purpose, our students should develop the abilities to notice, compare, generalize, conjecture by analogy and conjecture by inductive reasoning. On the other hand, the examples showing that conjectures suggested by inductive reasoning could be incorrect, should be pointed out at early stages. In that way, we show the students the need of proving the conjectures as well as finding the methods for proving. In this paper we mention a number of such examples through which the students at early stages of teaching mathematics are induced to inductive reasoning, but we also mention examples which illustrate how cautious one has to be when accepting hypotheses obtained by inductive reasoning.*

Key words: *inductive reasoning, mathematical induction*

Sažetak. **Pristup metodi matematičke indukcije od početne nastave matematike**. *Poznato je da metoda dokazivanja primjenom matematičke indukcije često dovodi do ozbiljnih poteškoća kod učenika koji se s tim pojmom prvi puta susreću u osnovnoj ili srednjoj školi. Zato je potrebno pripremiti učenike za razumijevanje i primjenu metode matematičke indukcije. Zato valja kod učenika razviti sposobnost uočavanja, uspoređivanja, uopćavanja i postavljanja hipoteze induktivnim zaključivanjem. S druge strane, valja dovoljno rano ukazati na primjere iz kojih je vidljivo da hipoteze postavljene induktivnim zaključivanjem mogu biti netočne. Tako uvjeravamo učenika u potrbu dokaza hipoteze, te nalaženja metoda za dokazivanje. U ovom članku navodimo niz upravo takvih primjera kroz koje učenike početne nastave matematike navodimo na induktivno zaključivanje, ali i primjere koji ukazuju na potrebu opreza prilikom prihvaćanja hipoteza dobivenih induktivnim zaključivanjem.*

Ključne riječi: *induktivno zaključivanje, matematička indukcija*

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1. Determining regularities — exercises in inductive reasoning

In several simple, but illustrative examples, through exercises in inductive reasoning students should become able to independently find out regularities.

Example 1 *Let us consider the sums of three consecutive integers*

$$\begin{aligned} 1+2+3 &= 6 \\ 2+3+4 &= 9 \\ 3+4+5 &= 12 \\ &\vdots \end{aligned}$$

We notice that these sums are multiples of 3, and conjecture that this is valid for any three consecutive natural numbers.

Example 2 *Let us consider the first several sums of consecutive natural numbers, starting with* 1. *To every number associate a column with the corresponding number of squares. We arrange these columns next to each other and in this way to each sum we associate a 'zigzagged triangle'.*

$$\begin{aligned} 1 &= 1 \\ 1+2 &= 3 \\ 1+2+3 &= 6 \\ 1+2+3+4 &= 10 \end{aligned}$$

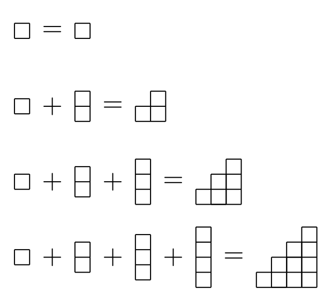

Figure 1.

The question is whether we can, without calculations, make an educated guess about the result (sum) of the next or any other sum of consecutive natural numbers.
One way is to draw corresponding 'zigzagged triangle' and to count squares within it. We would like to find a more economical procedure. The teacher will propose to put two congruent 'zigzagged triangles' together to obtain a rectangle. For the case $1+2+3+4$ *the corresponding rectangle is of the area* $4 \cdot 5 = 20$ *(see Fig. 2.)*

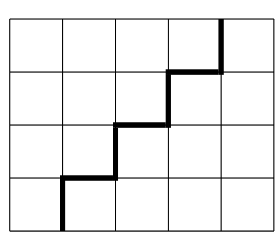

Figure 2.

> *What do the lengths of rectangle edges 'say' about the left-hand side of the equality?*

Number 4 *is the number of summands, and number* 5 *is its successor. The sum* $1+2+3+4$ *makes one half of the area of the corresponding rectangle.*

The teacher should advise the students to check this conjecture on previous examples. Once the students have checked the correctness of such sum calculation by direct addition of, say $1+2+3+4+5$*, they should confirm the inductively foreseen conjecture on an example like this:*

$$1+2+3+4+5+6+7+8+9 = (9 \cdot 10) : 2 = 45.$$

In this way the students are convinced that the equality

$$1+2+3+\cdots+n = \frac{1}{2}n(n+1).$$

always holds. Of course, they are not aware that for now this is only a hypothesis.

Example 3 *Let each half of a domino tile be called a 'field'. How many domino tiles are there in the game of domino in which each field bears at most* 6 *dots?*

For the beginning, let us consider the following task: How many domino tiles are there in the game of domino in which each field bears at most 3 *dots?*
Students can come to a conclusion by counting dominoes in the game of domino in which tiles have fields with at most 3 *dots. But, students should be incited to find the solution by inductive reasoning so that the task and then its solution could be generalized. For that purpose we shall arrange the dominoes in the following way:*

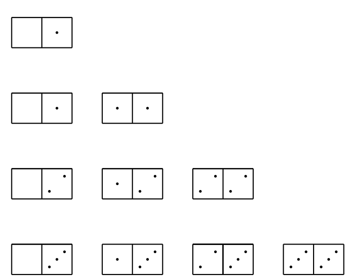

Figure 3.

By simple counting we find that the described domino has altogether 10 *domino tiles. The analysis of the total number of fields, i.e. domino tiles, is important. The fields differ in the number of dots. There are fields without dots, fields with one, two and three dots. So, there are* 4 *different types of fields in the game. We find out that every field appears* 5 *times. Therefore, the considered game has the total of* $4 \cdot 5 = 20$ *fields (halves of domino tiles), which means* $20 : 2 = 10$ *domino tiles. By adding the tiles in rows (see Fig. 3.), we obtain:*

$$1+2+3+4 = 10.$$

Now we repeat our reasoning with the game of domino in which each field bears at most 4 *dots. For the number of domino tiles we write*

$$1+2+3+4+5 = (5 \cdot 6) : 2 = 15.$$

We notice that the number of field types is by 1 greater than the maximal number of dots allowed in each field, and the total number of fields of the same type is by 2 greater than the maximal number of dots allowed in each field. By multiplying these two numbers we obtain the number of fields in the game. If this number is divided by 2, we obtain the number of domino tiles in that game. This hypothesis can be checked by determining the number of domino tiles in the game in which a field bears at most 5, 6 or 7 dots. This example gives another justification for our previous conjecture that

$$1+2+3+\cdots+n=\big(n\cdot(n+1)\big):2.$$

Example 4 Let us add the first several consecutive odd natural numbers.
We successively increase the number of summands by one. The question is whether we can, without calculations, conjecture what is the sum of an arbitrary number of consecutive odd natural numbers (starting with 1)?

$$\begin{aligned}&1=1\\&1+3=4\\&1+3+5=9\end{aligned}$$

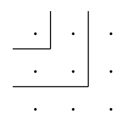

Figure 4.

If the students do not see the connection between the given sum on the left-hand side and the calculated sum on the right-hand side, the teacher might suggest that the sum on the right-hand side be expressed as the product of two equal factors:

$$\begin{aligned}&1=1\cdot 1\\&1+3=2\cdot 2\\&1+3+5=3\cdot 3\end{aligned}$$

By analyzing the problem written in such a form we find that the factor on the right-hand side says something about the left-hand side! It will be a discovery for the students to see that the factor on the right-hand side equals the number of summands on the left-hand side of the equality. The question naturally arises whether this holds in general. The conjecture is tested in the following example:

$$1+3+5+7+9+11=6\cdot 6.$$

The question remains how to determine the number of summands without actually counting them. Analyzing the solved examples, it is important that the students come up with the fact that the number of summands equals one half of the last summand increased by one. The teacher should advise students to check their conjecture against the solved examples. To check whether the students understood the principle or not, try the following example:

$$1+2+3+\cdots+15+17=?$$

If the students arrive to the solution by halving the number $17+1$, i.e. if they write something like

$$1+3+\cdots+17=9\cdot 9=81,$$

they did acquire the principle.

Now it takes just a step to write down the conjecture:

$$1+3+5+\cdots+2n-1=n^2.$$

Example 5 *Let us add the first several consecutive even natural numbers. We successively increase the number of summands by one. Can we surmise the sum of an arbitrary number of consecutive even natural numbers (starting with* 2*)?*

$$2 = 2, \quad 2 + 4 = 6, \quad 2 + 4 + 6 = 12.$$

Analyzing this problem we find that the result can be written as a product of two consecutive numbers.

$$2 = 1 \cdot 2, \quad 2 + 4 = 2 \cdot 3, \quad 2 + 4 + 6 = 3 \cdot 4.$$

It is important for the students to see that the smaller of the two factors is the number of summands. This can be checked with the help of the following example: $2+4+6+8+10+12+14$*. By counting the number of summands, students see that there are* 7 *of them. The successor of* 7 *is* 8*, so the result should be* $7 \cdot 8 = 56$*. The result is verified by calculation.*

It remains to determine the number of summands without actually counting them. If the students cannot find the answer themselves, the teacher should draw their attention to the last summand. Students should discover that the number of summands in the sum equals one half of the last summand. In that way students inductively come to the conjecture:

$$2 + 4 + 6 + \cdots + 2n = n \cdot (n + 1).$$

2. Inductive reasoning sometimes yields false conjectures

The next three examples illustrate the possibility of getting false conjectures.

Example 6 *Let us consider the sums of two consecutive natural numbers.*

$$1 + 2 = 3, \quad 2 + 3 = 5, \quad 3 + 4 = 7.$$

Notice that these sums are prime numbers. One could rashly conjecture that the sum of any two consecutive natural numbers is a prime number. This is of course not true, as shown by the following example:

$$4 + 5 = 9.$$

Example 7 *Let us find the connection between the number of chosen points on a circle and the maximal number of disc regions determined by the line segments connecting these points (see Fig. 5.).*

Figure 5.

Choose one point on the first circle. The number of corresponding disc regions is one. The line segment joining two points on the second circle, divides the disc into two regions. The results for depicted cases in Figure 5 are given in the following table:

number of chosen points	1	2	3	4	5	6
maximal number of disc regions	1	2	4	8	16	?

What is the maximal number of corresponding disc regions for 6 *chosen points?*

Students used to come up to conclusions by inductive reasoning will see the following regularity:
One point leaves the disc complete.
For 2 *points, the number of regions is* 2.
For 3 *points, the number of regions is* $2 \cdot 2 = 4$.
For 4 *points, the number of regions is* $2 \cdot 2 \cdot 2 = 8$.
For 5 *points, the number of regions is* $2 \cdot 2 \cdot 2 \cdot 2 = 16$.

Based on this, one could easily conjecture that for 6 points one would obtain at most $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$ *disc regions. But, it can easily be checked that for* 6 *points the maximal number of disc regions is* 31, *and not* 32.

The next, striking example, illustrates this point even better.

Example 8 *Consider the quadratic polynomial* $f(x) = x^2 - x + 41$. *After calculating the values* $f(x)$ *at, say* $x = 1, 2, \ldots, 10$, *is there anything reasonable that one can say about the values which the polynomial* f *assumes as* x *runs over the positive integers?*

At the first glance, looking at the values $f(x)$ *for the first ten positive integers*

x	1	2	3	4	5	6	7	8	9	10
$f(x)$	41	43	47	53	61	71	83	97	113	131

one notices that these are all odd numbers. A more thorough inspection reveals that all these are prime numbers. Is it possibly true that for every positive integer x *the value of* $f(x) = x^2 - x + 41$ *is a prime number? Testing this conjecture for ever larger integer* x, *one becomes more and more inclined to believe it is true. But, finally, for* $x = 41$, *the value* $f(x) = f(41) = 41 \cdot 41$ *— not a prime number. Hence the conjecture was false, and all we can say about these values, is that they are odd numbers. Which still remains to be rigorously proved!*

The three previous examples show that hypotheses based on inductive reasoning might prove to be incorrect. Conjectures have to be proved in order to be accepted as correct mathematical statements, which is not always possible at this level of education. The awareness of the need for proving the inductive hypotheses, at this level of education, will do.

3. Every conjecture has to be proved

The proofs of hypotheses or mathematical statements which depend on natural number n, often use the *method of mathematical induction*. This method consists of two parts.

The first part consists in verifying the statement for some initial value n_0, usually $n_0 = 1$. This is the so called *induction basis*. The second part is the so called *induction step*: Assuming the hypothesis to be true for $n = k$, the so called *induction assumption*, prove the validity of the statement for $n = k + 1$. This two parts taken together, prove the hypothesis to be true for all natural numbers $n \geq n_0$.

Example 9 *n circles of arbitrary radii and centers are drawn in the plane (see Fig. 6.a). Prove that the 'map' determined by these circles can be colored with two colors in such a way that any two neighboring regions are colored by different colors. (Regions are said to be* neighboring *if their boundaries contain a common arc.) (Fig. 6.b)*

Figure 6.a
A map

Figure 6.b
The same map properly colored

The proof is done using mathematical induction on the number of circles. For $n = 1$ we have a disc in the plane. If we color the disc black, and the rest remains white, we have the map colored as requested. In the induction step let us assume that we can properly color any map determined by k circles. Let K be an arbitrary map determined by $k + 1$ circles (Fig. 7.a). Choose a circle, call it c_0, and denote by K' the map obtained from K by removing the circle c_0. The map K' is determined by k circles, thus, according to the inductive assumption, it can be properly colored (Fig. 7.b). Adding the circle c_0 to the map K', gives back our original map K, although not yet properly colored (Fig. 7.c). But a simple argument shows that by switching the colors inside c_0, the map K becomes properly colored (Fig. 7.d).

This does the induction step and completes the proof.

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Figure 7.a
Map K with $k+1$ circles. Circle c_0 is emphasized.

Figure 7.b
Properly colored map K' with k circles

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Figure 7.c
Properly colored map K' with circle c_0 added

Figure 7.d
Colors inside c_0 are switched, giving properly colored map K

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