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Non-standard Aspects of Fibonacci Type Sequences

Dedicated to Hellmuth Stachel on the occasion of his 75th birthday

Non-standard Aspects of Fibonacci Type Sequences

ABSTRACT

Fibonacci sequence and the limit of the quotient of adjacent Fibonacci numbers, namely the Golden Mean, belong to general knowledge of almost anybody, not only of mathematicians and geometers. There were several attempts to generalize these fundamental concepts which also found applications in art and architecture, as e.g. number series and quadratic equations leading to the so-called "Metallic means" by V. DE SPINADEL [8] or the cubic "plastic number" by VAN DER LAAN [5] resp. the "cubi ratio" by L. ROSENBUSCH [7]. The mentioned generalisations consider series of integers or real numbers. "Non-standard aspects" now mean generalisations with respect to a given number field or ring as well as visualisations of the resulting geometric objects. Another aspect concerns Fibonacci type resp. Padovan type combinations of given start objects. Here it turns out that the concept "Golden Mean" or "van der Laan Mean" also makes sense for vectors, matrices and mappings.

Key words: generalized Fibonacci sequence, Golden Mean, non-Euclidean geometry, number field, ring

MSC2010: 51Mxx

Nestandardni pristupi nizovima Fibonaccijevog tipa

SAŽETAK

Fibonaccijev niz i zlatni rez, limes kvocijenata susjednih Fibonaccijevih brojeva, su pojmovi poznati ne samo matematičarima i geometričarima, već gotovo svima. Oni svoju primjenu nalaze u umjetnosti i arhitekturi. Poznato je nekoliko pokušaja poopćenja ovih pojmova, kao što su nizovi brojeva i kvadratne jednadžbe koje rezultiraju takozvanim "metalnim rezovima" V. DE SPINADEL [8], ili kubni "plastični broj" VAN DER LAANA [5], odnosno "kubni omjer" L. ROSENBUSCHA [7]. Spomenuta se poopćenja odnose na nizove cijelih ili realnih brojeva. "Nestandardnim pristupima" ovdje smatramo poopćenja u odnosu na dano polje ili prsten brojeva, kao i na vizualizaciju dobivenih geometrijskih objekata. Idući se pristup odnosi na Fibonaccijev, odnosno Padovanov tip kombinacija danih početnih objekata. Pokazuje se da pojam zlatnog reza ili van der Laanovog reza ima smisla promatrati i za vektore, matrice i preslikavanja.

Ključne riječi: popćeni Fibonaccijev niz, zlatni rez, neeuklidska geometrija, polje brojeva, prsten

1 Introduction

There exists already a huge set of references concerning the Fibonacci sequence and the Golden Mean and some generalisations of these concepts, most of them only repeating results of former works, such that it seems to be hopeless to add something essentially new to that topic. We shall start with basic facts about the Fibonacci sequence and the limit of adjacent Fibonacci numbers, namely the Golden Mean. Subsequently, we shall have a short look at the main traces of existing generalisations of these basic properties and concepts before adding another generalisation type. Finally, it will turn out that these proposed additional points of view are very natural and they are unifying consequences of what is already known.

2 Basic facts about Fibonacci sequences and the Golden Mean

The Fibonacci sequence is based on the recursive definition

$$F_{n+1} = F_n + F_{n-1}, (n \in \mathbb{N}), \quad F_1 = 1, \quad F_0 = 0. \tag{1}$$

Therewith, one receives the standard representation $\{\mathbf{0}, \mathbf{1}, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots\}$.

The limit of adjacent Fibonacci numbers for $n \to \infty$ is, therefore,

$$\phi := \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{F_n + F_{n-1}}{F_n} = 1 + \lim_{n\to\infty} \frac{F_{n-1}}{F_n} = 1 + \frac{1}{\phi}, \tag{2}$$

and ϕ solves the quadratic equation

$$x^2 - x - 1 = 0. \tag{3}$$

Remark 1 *This limit is independent of the two start elements, i.e. 0 and 1 can be replaced by any numbers $a_0, a_1 \in \mathbb{R}$ (not both 0). By recursion we receive $a_n = F_n a_1 + F_{n-1} a_0$, and obviously we get*

$$\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = \lim_{n\to\infty} \frac{\left(\frac{F_n}{F_{n-1}} a_1 + a_0\right) F_{n-1}}{\left(\frac{F_{n-1}}{F_{n-2}} a_1 + a_0\right) F_{n-2}} = \frac{\phi a_1 + a_0}{\phi a_1 + a_0}\phi = \phi.$$

In the following we will interpret a_{n+1} as a special linear combination, say a "Fibonacci combination", of the pair of start elements a_0, a_1. To present an example we remind to the "Lucas numbers", which use the start elements $\mathbf{2}, \mathbf{1}$ (see e.g. [4]).

Remark 2 *The Golden Mean value ϕ is the well-known result of the periodic continuous fraction*

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}} = 1{,}618\ldots \tag{4}$$

Remark 3 *The classical Greek point of view intersects a given segment $[A, B]$ (of length 1) into two segments M (the maior) and m (the minor) such that*

$$m : M = M : 1 = 1 - M : M$$

Therewith, it follows the quadratic condition

$$M^2 + M - 1 = 0, \tag{3$'$}$$

which has the positive solution $M = 0{,}618\cdots = \frac{1}{\phi}$. This value is also considered as the "Golden Mean" and it suits to the line of numbers with $A := 0$, $B := 1$, $M := 0{,}618\ldots$, while the affine ratio $R(M, A, B)$ is the negative of this value.

Remark 4 *It is possible to directly calculate the n^{th} Fibonacci number F_n as*

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^{-n}}{2^n\sqrt{5}}. \tag{5}$$

There is an obvious connection of F_n to the Pascal triangle, (Fig. 1 and [15]) that leads to an expression for F_n also as a sum of binomial coefficients as

$$F_n = \sum_{p+q=n-2} \binom{p}{q}.$$

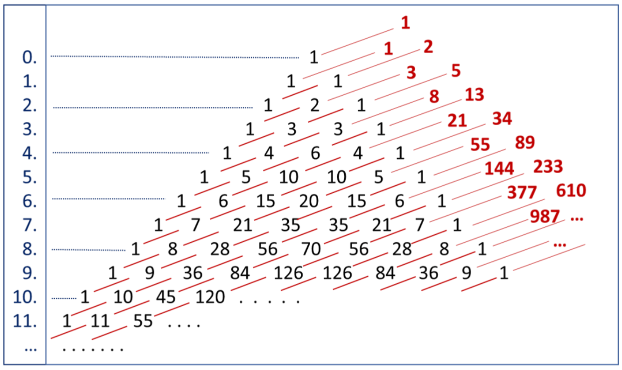

Figure 1: *The Fibonacci numbers F_n can be received as transversal sums of binomial coefficients as indicated in the Pascal triangle.*

Remark 5 *The Fibonacci sequence $\{\mathbf{0}, \mathbf{1}, 1, 2, 3, 5, 8, \ldots\}$ can be interpreted as an approximation of the geometric sequence $\{1, \phi, \phi^2, \phi^3, \phi^4, \ldots\}$. By the way, Lucas numbers are an even better approximation of this geometric sequence (c.f. [15]).*

We conclude this chapter noticing that the Golden Mean can be connected

(i) with the Fibonacci sequence,

(ii) with the quadratic equation (3), and

(iii) with the periodic continued fraction (4).

Generalisations of the Golden Mean are based on each of these mentioned connections.

3 Some known generalisations of the Fibonacci sequence and the Golden Mean

a) Generalisations of the Fibonacci sequence

As an example we mention the *Padovan sequence*, which leads to van der Laan's *plastic number* as a 3D-generalisation of the Golden Mean:
By replacing (1) by the rule

$$P_{n+1} = P_{n-1} + P_{n-2} \tag{6}$$

and using the start elements $\mathbf{0}, \mathbf{1}, \mathbf{1}$ we get the "Padovan standard sequence" $\{\mathbf{0}, \mathbf{1}, \mathbf{1}, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, \ldots\}$. The limit of the ratio of consecutive Padovan numbers $P_{n+1} : P_n$ is the (positive) solution ψ of the cubic equation

$$x^3 - x - 1 = 0. \tag{7}$$

This solution $\psi = 1{,}324717958\ldots$ is called "van der Laan's plastic number", as he used it as a 3D-replacement for the Golden Mean in his architectural design (see [5], [6], [10], and [16]).

Remark 6 *Using arbitrary start values a_0, a_1, a_2 (not all $a_i = 0$) we get the recursion formula*

$$a_{n+1} = P_{n-3}a_0 + P_{n-1}a_1 + P_{n-2}a_2, \tag{8}$$

and again the limit of the quotient of consecutive numbers a_{n+1}, a_n is van der Laan's plastic number ψ. As an example one could mention the "Perrin sequence", which uses the start elements $\mathbf{3}, \mathbf{0}, \mathbf{2}$ *instead of* $\mathbf{0}, \mathbf{1}, \mathbf{1}$.

Remark 7 *The Padovan sequence can also be connected to the Pascal triangle by choosing a special "skew" direction for summing up binomial coefficients, similar to the Fibonacci case Fig. 1, c.f. [16]. This gives rise to a procedure to construct other maybe interesting sequences derived from the Pascal triangle.*

Remark 8 *To calculate the polynomial (7) one can use the linear mapping, which maps the vector $(P_n, P_{n-1}, P_{n-2})^\top$ to the vector $(P_{n+1}, P_n, P_{n-1})^\top$:*

$$\begin{pmatrix} P_{n+1} \\ P_n \\ P_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_n \\ P_{n-1} \\ P_{n-2} \end{pmatrix} \quad \Longrightarrow$$

$$char.Pol. : \begin{vmatrix} -x & 1 & 1 \\ 1 & -x & 0 \\ 0 & 1 & -x \end{vmatrix} = 0 = -x^3 + x + 1. \tag{9}$$

b) Generalisations of the equation defining the Golden Mean

V. de Spinadel's Metallic Means

A simple way to define generalisations of the Golden Mean deals with (positive) solutions of quadratic equations by allowing arbitrary (positive or negative) integer coefficients for the characteristic equation

$$x^2 - px - 1 = 0, \quad \text{or} \quad x^2 - x - q = 0, \quad (p, q \in \mathbb{Z}), \tag{10}$$

c.f. [8] and [9]. For the solutions of (10) with $p = 2, 3, 4, \ldots$ V. Spinadel coined the concepts "Silver Mean" σ_2, "Bronze Mean" σ_3, "Copper Mean" σ_4 resp. "Metallic Mean" σ_p for general $p \in \mathbb{N}$. The Silver Mean $\sigma_2 := 1 + \sqrt{2}$ occurs as the affine ratio of the 2nd diagonal of a regular octagon to its side and is, therefore, omnipresent in medieval architecture.

The Metallic Means take the values of periodic continued fractions of type

$$\sigma_p = p + \cfrac{1}{p + \cfrac{1}{p + \cfrac{1}{p + \ddots}}}, \quad p \in \mathbb{N}. \tag{11}$$

Obviously, problems in physics, mathematics, and geometry, which result in a quadratic equation with rational or especially integer coefficients can be interpreted as applications of what can be called the "Generalised Metallic Means Family", c.f. [14], where the authors generalise a result of G. Odom [3] connecting the Golden Mean to an equilateral triangle and its circumcircle, see Fig. 2, to regular polyhedra and their circum-sphere.

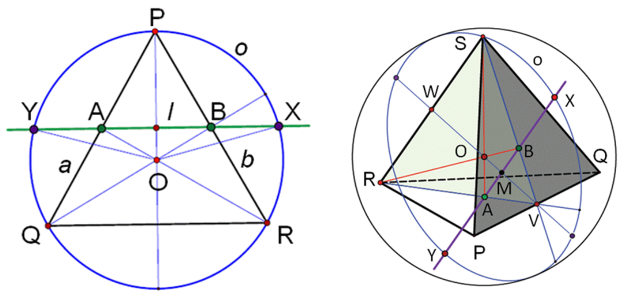

Figure 2: *Odom's discovery of the Golden Mean (left) and one of many possible generalisations (right).*

Generalising the equation (3) to irreducible equations of higher degree

H. van der Laan seems to be the first following this idea with using the cubic equation (7) for defining what he called the "plastic number" $\psi = 1,324717958\ldots$. But independent of him L. Rosenbusch [7] (see also [10]), too, looked for a 3D-analogue of the Golden Mean which he called "cubi ratio" $\rho := 0,6823278040\ldots$, and which is the (positive) solution of the cubic equation

$$x^3 + x - 1 = 0. \tag{7'}$$

Putting $y := \frac{1}{x}$ in (7) and (7') we get the equations $y^3 \pm y^2 - 1 = 0$, and a somehow natural generalisation of the equations (7) and (7') would be

$$x^d \pm x - 1 = 0 \quad \text{or} \quad x^d \pm x^{d-1} \pm \cdots \pm x - 1 = 0.$$

As many of the classical geometric problems from the ancient Greeks lead to cubic equations, one could consider general cubic equations, too, and construct the Fibonacci type sequence to it. This means to reconstruct the linear mapping (9) to a given characteristic polynomial

$$x^3 - px^2 - qx - r = 0, \quad (p, q, r \in \mathbb{R}), \tag{12}$$

with real solution x_0. As (12) is of degree $d = 3$, we put $S_{n+1} = aS_{n-2} + bS_{n-1} + cS_n$. Division by S_{n-2} and calculating the limits for $n \to \infty$ gives $x_0^3 = a + bx_0 + cx_0^2$, and we find $a := r$, $b := q$, $c := p$. Note that a Fibonacci type sequence should have at least two non-zero values a, b, c. These considerations can be formulated as

Result 1: Given an (irreducible) polynomial $P[x]$ of degree d, then it is possible to construct a Fibonnacci type sequence $\{\ldots, P_{n-d}, P_{n-d+1}, \ldots, P_{n-1}, P_n, \ldots\}$ to it, such that

$P[x]=0$ is the characteristic polynomial of the linear mapping

$$\{P_{n-d},P_{n-d+1},\dots,P_{n-1},P_n\}\mapsto\{P_{n-d+1},\dots,P_{n-1},P_n,P_{n+1}\}.$$

If the sequence P_{n+1}/P_n is convergent, then the limit $\lim_{n\to\infty}P_{n+1}/P_n$ is a (real) solution of $P[x]=0$. Therewith, we get an additional method to find Zeros of a polynomial, besides the well-known and more general methods "regula falsi" and "Newtons method".

Remark 9 *By a Tschirnhaus-Bring-Jerrard transform [1] it is always possible to transform an equation $x^d+p_1x^{d-1}+p_2x^{d-2}+\dots+p_{d-1}x+p_d=0$ into one with $p_1=p_2=0$. For a cubic equation this would result in $x^3=q$, which is not a proper form for constructing a Fibonacci type sequence.*
As an example, the equation $x^3=2$, describing the classical problem of doubling the cube, can be solved via the cubic equation $y^3-y^2+y-\frac{1}{3}=0$ with $y=\frac{1-x}{x}$.
Another example is the trisection of an angle α: Putting $x=\tan\alpha$, $a=\tan 3\alpha$ the corresponding cubic equation becomes $x^3-3ax^2-3x+a=0$.
The Cubus Simus, the "snub cube", resp. the regular heptagon are connected to the cubic equations $x^3+x^2+3x-1=0$ resp. $x^3+x^2-2x-1=0$, while $x^3-x^2+2x-1=0$ describes the axis-ratio of quarter-ellipses of a C^2-continuous bi-arc spiral consisting of such quarter ellipses. The latter equation can be transformed into (7) by the Tschirnhaus-Bring-Jerrard transformation and thus relates to van der Laan's Plastic Number ψ (c.f. [14]).

c) Continued fractions and their iterations

A theorem by Euler-Lagrange states that periodic continued fractions only lead to quadratic equations. This means that such fractions only can take values of the Generalized Metallic Means Family. From the decimal representation of, e.g., van der Laan's Plastic Number ψ, one could calculate finite approximations of the surely non periodic continued fraction of ψ. But one could generalise the concept "periodic continuous fraction" using reals instead of natural numbers as coefficients and iterate such a fraction as e.g.

$$a_{i+1}=a_i+\cfrac{1}{a_i+\cfrac{1}{a_i+\cfrac{1}{a_i+\ddots}}},\quad a_i\in\mathbb{R},\quad a_0=1. \tag{13}$$

In [12] the limit of this special sequence of periodic continued fractions is considered and turns out to be $\sqrt{2}/2$. This iteration process can also be seen as a sequence of quadratic equations (10) where the coefficient p is consecutively replaced by the (positive) solution of the former equation.

4 Fibonacci sequences and finite structures

a) Elliptic and spherical Geometry

In [13] visualisations of the Fibonacci sequence and the Golden Mean in non-Euclidean geometries were shown with one exception: the elliptic geometry. The elliptic lenght L of a line is π, and thus finite!
There seem to be two "natural" possibilities to handle Fibonacci numbers F_n^{el} in elliptic resp. spherical geometry:

(i) $F_n^{el}:=F_n mod L$.

(ii) $F_n^{el}:=F_n\cdot\frac{L}{F_{n+1}}$.

The way (i) seems not to be considered yet, while (ii) can be used to define the well-known "Golden Angle" as

$$\alpha:=2\pi-\lim_{n\to\infty}2F_n^{el}=137,520^\circ. \tag{14}$$

The usual definition of the Golden Angle devides the circumference of the unit circle into two angle-segments $2\pi-\alpha$ and α such that $(2\pi-\alpha):\alpha=\phi$. Obviously (ii) connects this definition to the circular model of the elliptic line. For the first stages $n=1,2,3$, of (ii) one may observe that the ratio of the regular $(n+1)$-gon's side s_{n+1} to its n-diagonal d_n takes remarkable values $r_n=d_n:s_{n+1}$, see Fig. 3: For the regular pentagon this ratio is the well-known Golden Mean, for the regular octagon it is the Silver Mean! As for $n\to\infty$ this ratio $r_n\to\infty$ it seems more reasonable to calculate the ratios $R_n=d_n:((n-1)\cdot s_{n+1})$ and their limit θ

$$\theta:=\lim_{n\to\infty}R_n=\lim_{n\to\infty}\frac{F_{n-1}\cdot\sin\left(\frac{\pi}{F_{n+1}}\right)}{\pi\cdot\frac{F_{n-1}}{F_{n+1}}}=\frac{\sin(\pi/\phi^2)}{\pi/\phi^2}$$
$$=0,78185913047\dots \tag{15}$$

which is somehow a "golden" number, too.

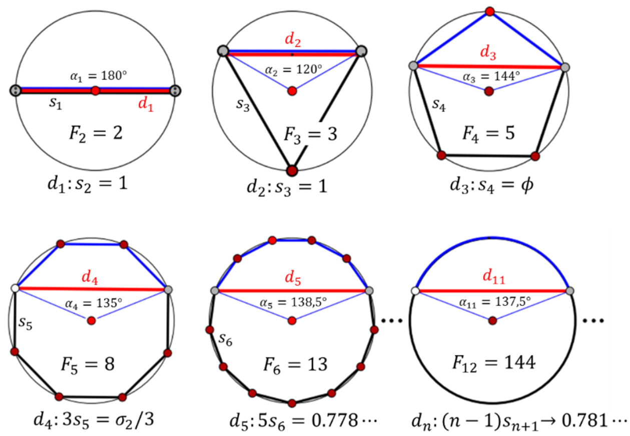

Figure 3: *F_{n+1}-gons and the ratios of diagonals d_n (red) and sides s_{n+1} (blue).*

b) Finite fields and Fibonacci numbers

We consider the Galois fields $GF(p)$, p prime, $p \in \mathbb{N}$. Here it seems to be natural to use the way (i) of the former subchapter (a) to define Fibonacci numbers F_n^p:

$$F_n^p := F_n \; mod p = F_{n-1}^p + F_{n-2}^p. \qquad (16)$$

Obviously the sequence of Fibonacci numbers F_n^p is periodic. The behaviour shall be visualised by the following examples:

$p=5$: **0**, **1**, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, **0**, **1**, 1, ...
Lenght of period is 20, all numbers of $GF(5)$occur.

$p=7$: **0**, **1**, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1, **0**, **1**, 1, ...
Lenght of period is 16, all numbers of $GF(7)$ occur.

$p=11$: **0**, **1**, 1, 2, 3, 5, 8, 2, 10, 1, **0**, **1**, 1, ...
Lenght of period is 10, not all numbers of $GF(11)$ occur! (4, 6, 7, 9 do not occur.)
Other start elements, e.g. 4, 6: **4**, **6**, 10, 5, 4, 9, 2, 0, 2, 2, **4**, **6**, ...
Again the length of period is 10. There are four numbers of $GF(11)$ that do not occur: 1, 3, 7, 8.

$p=13$: **0**, **1**, 1, 2, 3, 5, 8, 0, 8, 8, 3, 11, 1, 12, 0, 12, 12, 11, 10, 8, 5, 0, 5, 5, 10, **2**, **1**, 3, 4, 7, 11, 5, 3, 8, 11, 6, 4, 10, 1, 11, 5, 3, 8, 11, 6, 4, 10, 1, 11, 12, 10, 9, 6, 2, 8, 10, 5, 2, 7, 9, 3, 12, **2**, **1**, 3, 4, 7, ...
Lenght of period is 28 with 25 elements before the period, all numbers of $GF(13)$ occur!

$p=17$: **0**, **1**, 1, 2, 3, 5, 8, 13, 4, 0, 4, 4, 8, 12, 3, 15, 1, 16, 0, 16, 16, 15, 14, 12, 9, 4, 13, 0, 13, 13, 9, 5, 14, 2, 16, 1, **0**, **1**, 1, ...
Lenght of period is 36. There are three numbers of $GF(17)$ that do not occur: 7, 10, 11.

Trying to apply this method (16) also to a circle by taking $L = 360°$ and e.g. $10°$ as unit, one could consider Fibonacci numbers modulo 36, too:

*mod*36: **0**, **1**, 1, 2, 3, 5, 8, 13, 21, 34, 19, 17, 0, 17, 17, 34, 15, 13, 28, 5, 33, 2, 35, 1, **0**, **1**, 1, ...
Lenght of period is 23, only 15 numbers occur, 21 numbers are "gaps".

Remark 10 *As the quotient $F_{n+1}^p / F_n^p \in \{0, 1, \ldots, p-1\}$ is periodical, the limit does not make sense.*

One could also consider the way (i) of subchapter (a) and calculate Fibonacci numbers modulo π:

mod π: **0**, **1**, 1, 2, 3, 1.8584..., 1.7168..., 0.4336..., 2.1502..., 2.5840..., 1.5929..., ...

As π is transcendent and irrational, this sequence is not periodic.

Conclusion: The presented examples of Galois fields $GF(p)$ show that Fibonacci sequences are periodic independent of the pair of start values. As the set of Fibonacci numbers is countable, Fibonacci numbers modulo π must generate "gaps" in the interval $[0, \pi) \subset \mathbb{R}$. Here the questions arise, whether there occurs an attractor or not, and, whether there occur finite intervals as gaps or not.

5 Fibonacci sequences consisting of complex numbers

In Sec. 2. we mentioned that one can generalize the concept Fibonacci sequences by $a_n = F_n a_1 + F_{n-1} a_0$ with a_0, a_1 as general start values. Let us now take $a_0 := z_0$, $a_1 := z_1 \in \mathbb{C}$ and consider e.g. $z_0 := 1$, $z_1 := i$ as the pair of start values. We visualize the sequence $\{z_n\}$ in the Gauss plane (which is endowed with a Cartesian frame), see Fig. 4: The Fibonacci numbers $z_n = F_n + iF_{n+1}$ with odd index n are points of a branch of an equilateral hyperbola c, while the numbers with even index n belong to a branch of the conjugate hyperbola c'. Those hyperbolas have equations

$$y^2 - xy - x^2 = \mp 1. \qquad (17)$$

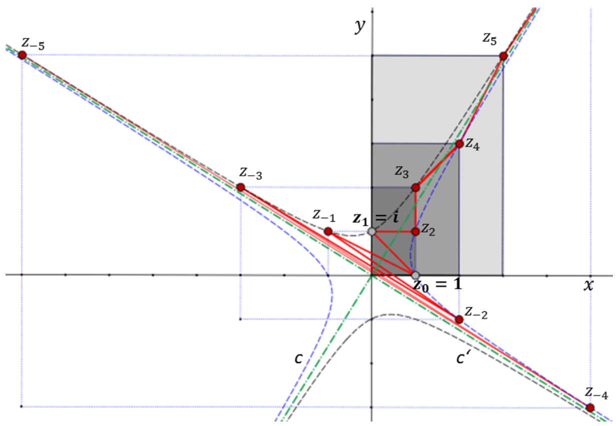

Figure 4: *The complex Fibonacci sequence to start values* $z_0 = (1,0)$, $z_1 = (0,1)$.

The stereographic projection $\cdot^\sigma$ of these hyperbolas (17) onto the Riemann sphere Σ obviously gives a pair of Viviani curves $c^\sigma, c'^\sigma \subset \Sigma$ with common tangents in their common double point U, which also acts as the centre of the stereographic projection. Thereby, the stereographic images of the "Fibonacci points" z_n of the Gauss plane belong to one loop of each of the two Viviani curves.

Calculating $z = \lim z_{n+1}/z_n$ leads to the complex quadratic equation $z^2 - z - 1 = 0$ with real coefficients. Its solutions

$s_1 = a_1 + ib_1$, $s_2 = a_2 + ib_2$ fulfil

$$s_1 s_2 = 1 + 0.i, \quad s_1 + s_2 = 1 + 0.i$$
$$\implies \quad s_2 = 1/s_1 = \frac{a_1 - ib_1}{a_1^2 + b_1^2}$$
$$\implies \quad a_1 + ib_1 + \frac{a_1 - ib_1}{a_1^2 + b_1^2} = 1 + 0.i$$
$$\implies \quad b_{1,2} = 0, \quad a_{1,2} = \frac{1}{2}(1 \pm \sqrt{5})$$
$$\implies \quad s_1 = \phi + 0.i, \quad s_2 = -\phi^{-1} + 0.i.$$

Result 2: Independent of the complex start elements, the limit $z = \lim z_{n+1}/z_n$ of the quotient returns at the (real) Golden Mean ϕ.

Remark 11 *From arbitrary (different) start values z_0, z_1 we get $z_n = F_n z_1 + F_{n-1} z_0$, $(n \in \mathbb{Z})$. We may speak of z_n being a "Fibonacci combination" of z_0, z_1. Consequently, it follows that the Fibonacci points z_n belong to a straight line, if $z_1 = z_0 \cdot r_1$, $r_1 \in \mathbb{R}$. In this case the stereographic image on the Riemann sphere Σ is circular.*

Remark 12 *We might relate $\mathbb{C}$ to the two-dimensional vector space $\mathbb{R}^2$, as indicated by its visualisation in the Gauss plane. Then the expression $z = \lim z_{n+1}/z_n$ must be replaced by $\|\vec{z}\| = \lim \|\vec{z_{n+1}}\|/\|\vec{z_n}\|$. A short calculation shows that this limit again takes the value $\phi \in \mathbb{R}$.*

Remark 13 *Replacing $\mathbb{C}$ by the ring D of dual numbers $d := a + \varepsilon b$, $(\varepsilon^2 = 0,\ a, b \in \mathbb{R})$ or the ring A of double numbers $h := a + jb$, $(j^2 = 1,\ a, b \in \mathbb{R})$ does not change the visualisation of Fibonacci numbers in the respective Gauss plane. Figure 4 can be considered as a visualisation of dual or double Fibonacci numbers as well. As far as a Fibonacci number is not a zero divisor in D resp. A, the quotients d_{n+1}/d_n resp. h_{n+1}/h_n make sense and their limits for $n \to \infty$ are again $\phi \in \mathbb{R}$. We show this for the case of Fibonacci dual numbers $d_{n+1} = F_n d_1 + F_{n-1} d_0$:*

$$d = \lim_{n\to\infty} \frac{d_{n+1}}{d_n} = \lim_{n\to\infty} \frac{F_n d_1 + F_{n-1} d_0}{F_{n-1} d_1 + F_{n-2} d_0} = \phi \cdot \frac{\phi d_1 + d_0}{\phi d_1 + d_0}$$
$$= \begin{cases} \phi & \text{for} \quad \phi d_1 + d_0 \notin N = \{0 + \varepsilon b, b \in \mathbb{R}\} \\ - & \text{for} \quad \phi(a_1 + \varepsilon b_1) + a_0 + \varepsilon b_0 \in N \end{cases}$$

From the last equation follows that the condition for a not declared limit reads as

$$\phi a_1 + a_0 \in N, \quad b_1, b_0 \in \mathbb{R} \quad \text{arbitrary.} \tag{18}$$

The stereographic images of the pair of conjugate equilateral hyperbolas (17) passing through the Fibonacci dual numbers are points of curves of degree 4 on a quadratic cylinder, while Fibonacci double numbers belong to curves on a one-sheeted hyperboloid, c.f. [2].

6 Fibonacci type sequences defined as vector combinations

Remark 12 in gives a hint how to define Fibonacci type sequences of vectors:

Recursive definition: Given a set of initial vectors $\{v_0, v_1, \ldots, v_p\} \subset \mathbb{R}^k$, $p, k \in \mathbb{N}$ and a set of scalars $\{r_0, \ldots, r_p\}$, then

$$v_{p+r+1} := r_0 v_{0+q} + r_1 v_{1+q} + \cdots + r_p v_{p+q}, \quad q \in \mathbb{N}_0. \tag{19}$$

The sequence obviously can be extended to negative values of $q \in \mathbb{Z}$.
A Fibonnacci type combination of two independent vectors gives "Fibonacci vectors" of a two-dimensional vector space. The Padovan combinations (6) and (8) (i.e., in (19) we put $r_0 = 1, r_1 = 1, r_2 = 0$) applied to three independent start vectors delivers "Padovan vectors" of a 3-space, which are recursively defined by

$$v_n = v_{n-1} + v_{n-2} = P_{n-2} v_0 + P_n v_1 + P_{n-1} v_2. \tag{20}$$

We present an example using the basis vectors of $\mathbb{R}^3$ as the start vector triplet, i.e

$$v_0 = (1,0,0)^\top, \quad v_1 = (0,1,0)^\top, \quad v_2 = (0,0,1)^\top,$$

such that (20) becomes

$$v_{n+1} = (P_{n-3}, P_{n-1}, P_{n-2})^\top. \tag{21}$$

For the following visualisations in Figure 5, it seems to be necessary to calculate the Padovan numbers at least for $-40 \le n \le 19$, see Table 1 :

Table 1: *List of Padovan numbers for $-40 \le n \le 19$*

-40 ... -31	145	-89	56	-7	-33	49	-40	16	9
-30 ... -21	-24	25	-15	1	10	-14	11	-4	-3
-20 ... -11	7	-7	4	0	4	-3	1	1	-2
-10 ... -1	2	-1	0	1	-1	1	0	0	1
0 ... 9	**0**	**1**	**1**	1	2	2	3	4	5
10 ... 19	7	9	12	16	21	28	37	49	65

Remark 14 *The Padovan numbers resp. vectors with negative index n are recursively defined by*

$$Q_{m+1} = Q_{m-2} - Q_m, \quad (m = -n) \quad \text{resp.} \quad v_{m+1} = v_{m-2} - v_m. \tag{22}$$

Thereby, it follows that $\lim_{m\to\infty} \frac{Q_{m-1}}{Q_m} = \frac{1}{\psi}$ is a solution of $y^3 + y^2 - 1 = 0$.

Remark 15 *The (real) vector* $(1,\psi^2,\psi)^\top$, $\psi = 1,32\ldots$ *the van der Laan number, points to* v_∞. *Equation (7) for van der Laan's Plastic Number* $x_1 = \psi$ *has two complex conjugate solutions of the quadratic equation*

$$x^2+\psi x+(\psi^2-1)=0, \tag{23}$$

$$x_{2,3}=\frac{1}{2}\cdot(-\psi\pm\sqrt{4-3\psi^2})\approx -0,662359\pm i\cdot 1,12456. \tag{24}$$

The correspoding complex vectors $(1,x_{2,3}^2,x_{2,3})^\top$ *represent complex ideal points of the spiral set of points described by (21) for* $m\to\infty$, *see Fig. 5 visualising the top view of that point set. The star-shaped blue polygon tends to a pentagram for* $m\to\infty$. *(The black, almost straight lined polygon describes the top view of the first n points,* $n>2$.*) One might calculate the angle* $\alpha := \angle(v_{n-1}-v_n)(v_{n+1}-v_n)$ *of the limit 3D-pentagram's spikes. It turns out that one gets*

$$\cos\alpha=\frac{\psi(2\psi-1)}{\psi^2+\psi+1}\quad\Longrightarrow\quad\alpha\approx 68,0\ldots^\circ.$$

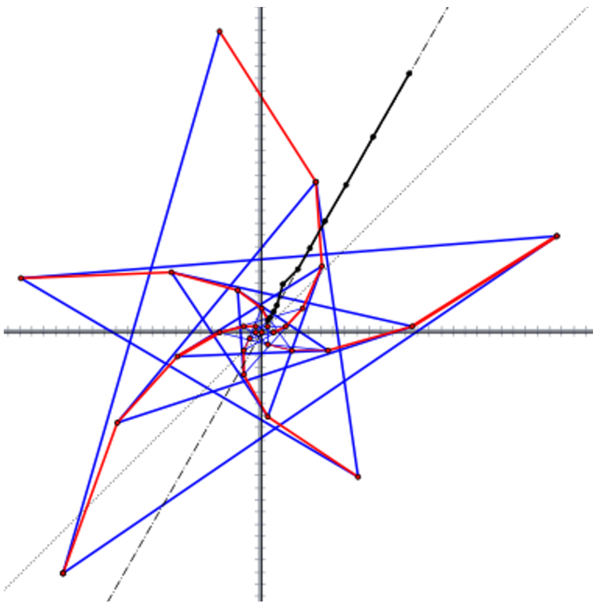

Figure 5: *Top view of the Padovan type point set resulting from (20).*

Remark 16 *Obviously one can interpret the set of vectors* v_n *(19) as homogeneous coordinates of points of a projective space* $\pi = \mathrm{P}(\mathbb{R}^k)$. *Again we focus on the example of Padovan vectors as treated above. We replace the vector symbol* v_n *by the symbol* V_n *for projective points and continue the "Padovan example" above rewriting (21) as*

$$V_{n+1}=(P_{n-3},P_{n-1},P_{n-2})\mathbb{R}. \tag{25}$$

7 Normed Fibonacci combinations and Conclusion

The interpretation in Remark 16 stimulates to consider Fibonacci or Padovan type combinations of *Matrices* or even (algebraic) *equations* too. The Fibonacci combination of e.g. two such geometric objects Obj_0, Obj_1 of the same type results in a set of objects belonging to a pencil

$$Obj_n = F_n Obj_1 + F_{n-1} Obj_0. \tag{26}$$

Similarly, the Padovan-combination of three such objects results in a set of these objects belonging to a two-parameter manifold. As one can interpret the coefficients F_n, F_{n-1} in (26) as *projective coordinates* within the pencil of objects, the limit object Obj_∞ for $n\to\infty$ also makes sence. As an example we show this situation for the Fibonacci combination of two circles c_0, c_1:

$$c_0\ldots(x-m_0)^2+y^2=r_0^2,\qquad c_1\ldots(x-m_1)^2+y^2=r_1^2,$$

$$\begin{pmatrix} m_0^2-r_0^2 & -m_0 & 0\\ -m_0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\qquad \begin{pmatrix} m_1^2-r_1^2 & -m_1 & 0\\ -m_1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

To receive the normed equation for the circle c_n we have to "norm" the Fibonacci combination as in (26):

$$c_n=\frac{F_{n-1}}{F_{n-1}+F_n}c_0+\frac{F_n}{F_{n-1}+F_n}c_1, \tag{27}$$

and we get

$$c_\infty=\frac{1}{\phi^2}c_0+\frac{1}{\phi}c_1. \tag{28}$$

Using power rules for ϕ, we receive the normed equation for that limit circle c_∞

$$c_\infty\ldots\left(x-\left(\frac{m_0}{\phi^2}+\frac{m_1}{\phi}\right)\right)^2+y^2=\frac{r_0^2}{\phi^2}+\frac{r_1^2}{\phi}-\frac{1}{\phi^3}(m_0-m_1)^2. \tag{29}$$

From (29), we read off that the center M_∞ of c_∞ divides the segment between the centres M_0, M_1 of c_0, c_1 in the Golden Ratio. Note that the radius r_∞ and thus c_∞ can be complex, if c_0, c_1 span a hyperbolic pencil of circles! In case c_∞ is real, it must intersect the circles of the elliptic pencil of circles conjugate to that spanned by c_0, c_1 orthogonally. In case c_0, c_1 span an elliptic or parabolic pencil of circles, c_∞ has to pass throught the basis point(s) of that pencil. So it is easy to construct c_∞ to a given pair c_0, c_1, see Fig. 6.

In the following, we focus on the norming process used in (27) and we will speak of "normed Fibonacci combinations" of two (geometric) objects.

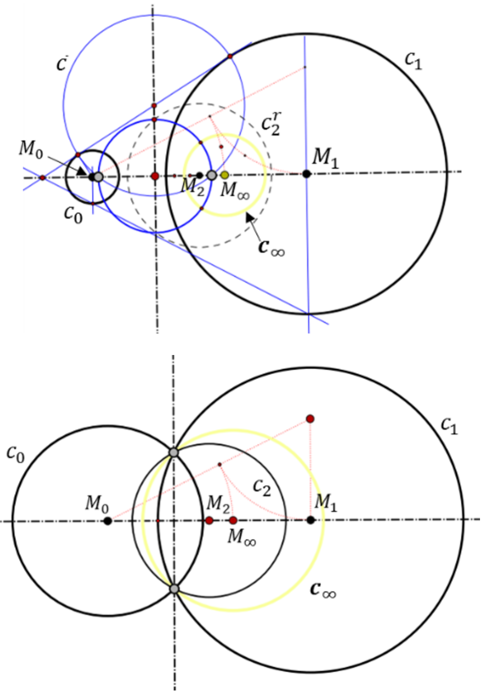

Figure 6: *Limit circle c_∞ to The Fibonacci combination of two circles c_0, c_1.*
top: c_0, c_1 span a hyperbolic pencil of circles, $\overline{c}$ is a circle of the (elliptic) conjugate pencil, c_∞ is real, but c_2 is imaginary with the "real representative" c_2^r.
bottom: c_0, c_1 span an elliptic pencil of circles. All circles c_n are real.

a) Let the given objects Obj_0, Obj_1 be $(n \times m)$-matrices. For $n = m$, they can be interpreted as the coordinate representation of collineations, while for $n > m$ they represent linear mappings. A Fibonacci combination (26) of such matrices represents a collineation resp. a linear mapping which depends on the ratio $F_n : F_{n-1}$ and the two initial matrices Obj_i. Thus, the resulting mapping also has the "normed Fibonacci representation"

$$Obj_n = \frac{F_{n-1}}{F_{n+1}} Obj_0 + \frac{F_n}{F_{n+1}} Obj_1 \tag{30}$$

and we call Obj_n the "n^{th} Fibonacci mean" of Obj_0 and Obj_1. For the object Obj_∞ the name "Golden Mean of Obj_0 and Obj_1" could be coined.

b) Let the given objects Obj_0, Obj_1 be vectors, the coordinates of which fulfil a (quadratic) condition, then the coordinates of the Fibonacci or Padovan type combination Obj_n will not fulfil this condition anymore! Examples of such vectors are, e.g., the Plücker coordinates of lines of a projective 3-space, the spear- and cycle- resp. circle-coordinates in Laguerre, Möbius, and Liegeometry (see [2]). Here a "norming" of the resulting vector Obj_n analogue to that used in (25) would allow to extend the concept "pencil" or "two-parameter manifold" as well as "Fibonacci mean" and "Padovan mean" also to these cases.

   For example, in the case of Line Geometry the Fibonacci combination of two skew lines g_0, g_1 results in a set of linear complexes within the pencil of linear complexes spanned by g_0, g_1. Here the norming process could consider the axes a_n of these complexes Obj_n, if the space of action is the projective closure of the Euclidean 3-space. As the set of axes of a pencil of linear complexes comprises the rulings of a Plücker conoid, the axis a_∞ of the "limit complex" Obj_∞ is a line "in Golden Ratio" between g_0 and g_1. We might call a_∞ the Golden Mean line of the lines g_0 and g_1.

c) Coming back to numbers we put $Obj_i := \log q_i$. Now (30) becomes

$$\log q_n = \frac{F_{n-1}}{F_{n+1}} \log q_0 + \frac{F_n}{F_{n+1}} \log q_1, \qquad n \in \mathbb{N}, \quad q_0, q_1 \in \mathbb{R}^+, \tag{31}$$

   what means that

$$q_n = \sqrt[F_{n+1}]{q_0^{F_{n-1}} \cdot q_1^{F_n}} \tag{32}$$

   and finally

$$q_\infty = q_0^{1/\phi^2} \cdot q_1^{1/\phi} \tag{33}$$

   Hereby, we receive as a final

Result 3: The *geometric mean* of two (ordered) numbers q_0, q_1 can be interpreted as the *third normed Fibonacci mean* q_3 of these numbers, while q_∞ is their *Geometric Golden Mean.*

Concluding remark. The presented material does not at all cover all possibilities to generalise the Fibonacci sequence. By focussing on both, the Fibonacci sequence and the Padovan sequence the material connects to known facts which are seen from a perhaps new point of view. Obviously the presented methods can be applied to any Fibonacci (resp. Padovan) type sequence. It might be worth to treat the sequence leading to V. de Spinadel's Silver Mean more explicite and in the above presented way.

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