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ON DERIVATING OF AN ELASTIC STABILITY MATRIX FOR A TRANSVERSELY – CRACKED BEAM – COLUMN BASED ON TAYLOR EXPANSION

IZVOD MATRICE ELASTIČNE STABILNOSTI STUPA SA POPREČNOM PUKOTINOM NA OSNOVI TAYLOROVOG RAZVOJA

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Abstract

This paper progresses the implementation of a simplified computational model for predicting buckling load P_{cr} for slender beam – type structures with transverse cracks. The work presents an upgrade of the finite element model that has already successfully proved itself as being capable of producing applicable results by exhibiting considerably good agreement with those values from more precise and complex computational models. The major handicap of the previously applied finite element was the absence of a clear analytical form as it was presented (due to its complex form) only by the main steps of the numerical procedure. For that reason, the main advance of the presented finite element lies in its clear analytical form. In order to achieve this improvement the location of the crack was limited to the mid – span position.
Although both (old and new) cracked beam – column finite elements are derived at on the basis of a fairly accurate approximation of the governing differential equation's solution (GDE), the novel finite element even produces slightly better results. It further allows for flexible utilization and also yields a small compact computational model, thus exhibiting itself as very suitable for inverse identification problems. Numerical examples covering several structures with different boundary conditions are briefly presented in order to confirm the competence of the newly derived finite element. The results obtained using the presented finite element are further compared with corresponding values from references thus clearly proving the quality of the newly derived finite element.

Keywords: *Transversely cracked beam – column, Stability problems, Buckling load, Finite element method*

Sažetak

Ovaj rad poboljšava uporabu pojednostavljenog komputacijskog modela u predviđanju sile izvijanja P_{cr}-za vitke konstrukcije grednog tipa s poprečnom pukotinom.
Rad predstavlja nadogradnju modela konačnog elementa za koji je već uspješno dokazano da daje primjenjive rezultate pokazujući dobro slaganje s vrijednostima dobivenih preciznijim i složenijim komputacijskim modelima.
Glavni nedostatak prethodno primijenjenog konačnog elementa je nepostojanje jasnog analitičkog oblika, jer je (zbog složenosti) predstavljen samo po glavnim koracima numeričkog postupka. Iz tog razloga, glavna prednost predstavljenog konačnog elementa leži u jasnom analitičkom obliku. Da bi se postiglo to poboljšanje mjesto pukotine je ograničeno na sredinu elementa.
Iako su oba konačna elementa (stari i novi) za napuknuti stup izvedena na osnovi prilično točne aproksimacije rješenja diferencijalne jednadžbe za poprečne pomake, novi daje i nešto bolje rezultate. To dodatno omogućuje fleksibilnu uporabu i daje mali kompaktni komputacijski model, zbog čega je vrlo pogodan za probleme inverzne identifikacije.
Numerički primjeri koji obuhvaćaju nekoliko konstrukcija s različitim rubnim uvjetima ukratko su prikazani kako bi potvrdili učinkovitost novo izvedenog konačnog elementa.

Dobiveni rezultati dalje se uspoređuju s odgovarajućim vrijednostima iz literature čime se jasno dokazuje postignuta kvaliteta.

Ključne riječi: *Poprečno puknuti stupovi, problemi stabilnosti, sila izvijanja, metoda konačnih elemenata*

1. Introduction

1. *Uvod*

Initial theoretical research into elastic flexural buckling was preceded by Euler's 1759 treatise [1] on column flexural buckling.
That work introduced the first analytical method for predicting the reduced strengths of slender columns. Based on his studies, solutions for single non – cracked columns under various boundary conditions have been given in analytical forms. For structures consisting of more than one element the search for an analytical solution is complex and therefore numerical solutions, especially the finite elements method, are usually applied to these kinds of problems. The beam finite elements can be easily implemented for beam structural elements without cracks, since the stiffness and generalized geometrical stiffness matrixes are already known generally. However, the situation changes significantly if the elements are transversely cracked. 2D or 3D finite element analysis must be implemented in order to achieve an accurate model of the structure. This allows a detailed discrete description of the crack but it is also very time consuming and requires considerable design and computational effort. However, in some engineering analyses, simplified computational models have to be implemented due to the fact that a substantial amount of the data is either unknown or irrelevant for the considered analysis. A typical representative example of such problems is the inverse identification of cracks, where the presence, the location and the depth of the crack should be detected from the limited measured response on an actual structure.
Okamura [2] presented an appropriate simplified model for these kinds of problems that is widely used for the computation of transverse displacements regarding transversely cracked slender beams. In order to expand utilization of Okamura's model from simple single structural element beam structures (where the corresponding differential equation for transverse displacements can also be solved analytically) to more complex frame – type structures, several finite elements were further developed by Biondi [3], Palmeri [4] and Skrinar et al. [5], even for multi – cracked and stepped elements [6].
Unfortunately, the main orientation of preceding works have been primarily devoted towards implementation of the computational model in problems where transverse displacements resulted, either from transverse load or (in the case of a single side crack) from centric tensile load, Skrinar [7].
Some studies also examined the simplified model's behavior regarding buckling problems of a slender beam with a transverse crack.
The inclusion of the compressive forces' effect into analysis namely completes the computational model and thus increases its applicability. The finite element method, a purely numerical approach, has advanced towards being the appropriate approach for these problems. Within the finite element method approach two strategies are evoluted.
Firstly, a geometrical stiffness matrix for a beam finite element with a transverse crack and a constant axial force was derived at [8].
This matrix was later improved by introducing a linear distribution of axial compressive force [9]. As both geometrical stiffness matrixes provide a complement to the stiffness matrix of a cracked beam element's finite element, subjected to pure bending, all three matrixes were therefore derived at by the same static interpolation functions.
Consequently, to achieve the convergence of the results several finite elements were required when modeling a structural member which was not appropriate from the inverse identification point of view. In order to overcome this problem the beam – column finite element model was studied where the axial force directly impacted the stiffness matrix [10]. However, this approach faced a drawback of interpolation function not being simple polynomials any longer. In order to eliminate this the trigonometric terms in the interpolation functions were replaced by

Taylor's expansions. The implementation of these expanded interpolation functions resulted in a non – exact solution – based cracked beam column finite element (WSCBCFE), [10]. Its form was enormous, due to the complex expressions in the interpolation functions, and never presented or published. Despite this deficiency of clear form, this element presented practical progress. Namely, when comparing the beam – column finite element model with the standard cracked beam element, accompanied by the geometrical stiffness matrix, it becomes evident that not only the computational model for buckling problems becomes smaller but also the accuracy has simultaneously increased. Namely, the discrepancies of the results obtained with WSCBCFE compared with the results obtained with much more complex and time consuming 2D finite element models have evidently decreased despite the evident differences in the computational effort.

This paper now eliminates the main obstacle of practical implementation of the WSCBCFE – the absence of clear analytical form.

2. Simplified model for cracked beams

2. *Pojednostavljeni model grede s pukotinom*

One of the simplest plain models for mathematically describing a cracked structure's response behavior is a model where the crack is introduced as a rotational linear spring connecting the non – cracked parts of the structure modeled as elastic elements, as presented by Okamura et al. [2].

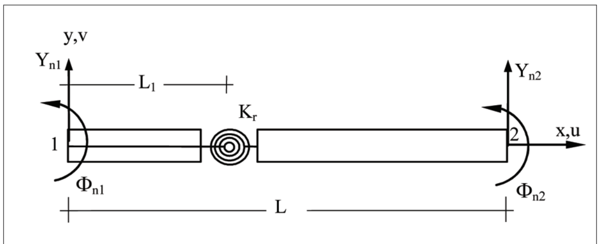

Figure 1 *The beam finite element with a crack represented by a linear rotational spring*

Slika 1 *Konačni element nosača sa pukotinom predstavljenom kao linearna rotacijska oprugra*

Transverse displacements are thus additionally influenced by the position L_1 of the crack as well as its depth *d*. The stiffness K_r of the rotational spring for rectangular cross – sections depends on the height of the non – cracked cross – section *h*, the relative depth of the crack *d/h*, and the product of Young's modulus *E* with the moment of inertia of the non – cracked cross – section *I* i.e. flexural rigidity *EI*. Okamura et al. introduced the earliest definition for rotational stiffness and is the only definition taking Poisson's ratio n into account.

3. Derivation of the cracked beam – column finite element

3. *Derivacija konačnog elementa stupa s pukotinom*

When studying the elastic buckling load of a column, it is necessary to determine the ultimate load at which the structure remains in equilibrium at the deformed position. When considering a differential element with an infinitesimal length of beam – column, it is possible to derive at a beam – column equation for the coordinate *x*, i.e. a governing differential equation that relates to transverse displacement v, axial compressive force P, the geometrical and mechanical properties of the cross section (united in flexural rigidity EI), and the applied transverse load q. The general solution of the beam – column equation, which is a fourth – order ordinary differential equation with constant coefficients, is given by:

$$v(x) = A_1 \cdot \cos\left(\sqrt{\frac{P}{EI}} \cdot x\right) + A_2 \cdot \sin\left(\sqrt{\frac{P}{EI}} \cdot x\right) +$$

$$A_3 \cdot \sqrt{\frac{P}{EI}} \cdot x + A_4 + \text{particular integral} \qquad (1)$$

where A_1, A_2, A_3 and A_4 are integration constants obtained from boundary conditions (which are commonly assumed to be ideal), while the particular integral depends on the value of transverse load q. The elastic buckling load is afterwards obtained by calculating the ultimate axial load $P=P_{cr}$ at which the column remains in equilibrium at a deformed position even when q=0. For the simplest situation, i.e. for non – cracked single slender beams and q=0, analytical expressions for P_{cr} are given for various frequent boundary conditions (pin – ended column, fixed – ended or clamped column,

propped cantilever and cantilever), in many textbooks. The interpolation function in the form of Eq.(1) is also implemented in the derivation of a non – cracked beam – column finite element's stiffness matrix.
However, since the crack separates the beam into two elastic parts, the transverse displacements cannot be described by a single function anymore and therefore two displacement functions are required. Consequently, dual coupled differential equations for the parts on the left (v_1) and right (v_2) sides of the crack have to be solved simultaneously. Their general solutions preserve the form of Eq.(1) by the consideration of zero transverse load q (and with the abbreviation $\alpha = \sqrt{\frac{P}{EI}}$):

$$v_1(x) = A_1 \cdot \cos(\alpha \cdot x) + A_2 \cdot \sin(\alpha \cdot x) + A_3 \cdot \alpha \cdot x + A_4 \qquad 0 \le x \le L_1 \tag{2}$$

$$v_2(x) = B_1 \cdot \cos(\alpha \cdot x) + B_2 \cdot \sin(\alpha \cdot x) + B_3 \cdot \alpha \cdot x + B_4 \qquad L_1 \le x \le L \tag{3}$$

The rotations are consequently given as (i=1,2):

$$\varphi_i(x) = \frac{dv_i(x)}{dx} \tag{4}$$

Bending moments along the element are further (i=1,2):

$$M_i(x) = EI \cdot \frac{d^2 v_i(x)}{dx^2} \tag{5}$$

Finally, the shear forces along the element are given as (i=1,2):

$$V_i(x) = -\left(EI \cdot \frac{d^3 v_i(x)}{dx^3} + P \cdot \frac{dv_i(x)}{dx} \right) \tag{6}$$

These solutions that serve as interpolation functions for the cracked beam – column finite element implement eight unknown constants altogether. Four of them are derived at from conditions already known from the non – cracked beam – column finite element as they compress the actual boundary kinematic conditions. Since the finite element has standard four degrees of freedom for transverse displacements, Fig. 1, these conditions are:

$$v_1(0) = Y_{n_1} \tag{7}$$

$$v_2(L) = Y_{n_2} \tag{8}$$

$$\varphi_1(0) = \Phi_{n_1} \tag{9}$$

$$\varphi_2(L) = \Phi_{n_2} \tag{10}$$

where upward nodal translations (Y_{n1} and Y_{n2}) and counterclockwise nodal rotations (Φ_{n1} and Φ_{n2}) are taken as positive.
The remaining four unknown constants are obtained from the continuity conditions at the crack location ($x=L_1$). These conditions are the equality of displacement:

$$v_1(L_1) = v_2(L_1), \tag{11}$$

the condition for discrete increase of rotations:

$$\varphi_1(L_1) + \frac{M_1(L_1)}{K_r} = \varphi_2(L_1), \tag{12}$$

the equality of bending moments:

$$M_1(L_1) = M_2(L_1), \tag{13}$$

and the equality of shear forces:

$$V_1(L_1) = V_2(L_1). \tag{14}$$

Each boundary and continuity condition leads to a linear equation and therefore the missing constants are obtained from a set of eight linear equations. Afterwards, the transverse displacement functions (i=1,2) are rewritten as:

$$v_i(x) = \{N_i\} \cdot \begin{Bmatrix} Y_{n_1} \\ \Phi_{n_1} \\ Y_{n_2} \\ \Phi_{n_2} \end{Bmatrix} = \{N_i\} \cdot \{q\} \tag{15}$$

where vector represents the vector of discrete nodal displacements and rotations.

Once the interpolation functions $v_i(x)$ for the transverse displacements are known, the complete deformation energy can be expressed in terms of unknown nodal displacements as:

$$U = \frac{1}{2} \cdot \int_{x=0}^{L_1} EI \cdot \left(\frac{\partial^2 v_1}{\partial x^2} \right)^2 \cdot dx + \frac{1}{2} \cdot \int_{x=L_1}^{L} EI \cdot \left(\frac{\partial^2 v_2}{\partial x^2} \right)^2 \cdot dx + \frac{1}{2} \cdot K_r \cdot \left(\left. \frac{\partial v_1}{\partial x} \right|_{L_1} - \left. \frac{\partial v_2}{\partial x} \right|_{L_1} \right)^2 \quad (16)$$

where both integrals and the last term represent the strain energy in both the elastic parts and the crack, respectively. By introducing Eq. (15) into Eq. (16) the beam – column stiffness matrix is thus:

$$[K] = EI \cdot \int_{x=0}^{L_1} \{N_1''\}^T \cdot \{N_1''\} \cdot dx + EI \cdot \int_{x=L_1}^{L} \{N_2''\}^T \cdot \{N_2''\} \cdot dx + K_r \cdot \left(\{N_1'\} - \{N_2'\} \right)^T \cdot \left(\{N_1'\} - \{N_2'\} \right) \quad (17)$$

In the derivation of the WSCBCFE finite element the trigonometrical terms sin(α×x) and cos(α×x) were replaced by the adequate polynomial expansion. Although this brought a huge generalised form of matrix, the matrix is actually reduced into a very small and compact computational model when numerical values were introduced.
Since the local elements stiffness' matrixes can be transformed into a global coordinate system and further assembled into a global stiffness matrix of the complete structure $[K_s]$, the critical buckling load P_{cr} is then finally obtained from the non – trivial solution of a system of linear homogeneous equations:

$$[K_s] \cdot \{Y\} = \{0\} \quad (18)$$

where {Y} represents the structure's vector of displacements and rotations within the considered degrees of freedom.
Although the WSCBCFE finite element already yielded excellent solutions from the engineering point of view its main drawback was the absence of a closed form solution. Therefore, to obtain such a form a compromise had to be reached to allow for the analytical solution to be obtained.
In order to gain the form of the stiffness matrix where the coefficients can be written analytically, the location of the crack was exclusively limited to the element's mid – span i.e. L_1=L/2. Consequently, this location's fixation somehow represents a handicap for situations where the crack is located elsewhere as it requires the implementation of an additional non – cracked beam – column finite element to adequately model the cracked beam. However, on the other hand, an analytically written form of the stiffness matrix clearly presents the overweighting advantage.
The decision made not only allowed for the stiffness matrix to be written in pure analytical form but also essentially reduced the number of different coefficients appearing within the stiffness matrix. The whole matrix namely consists of just four diverse terms and can be written in general form as:

$$[K] = \begin{bmatrix} k_{11} & k_{12} & -k_{11} & k_{12} \\ k_{12} & k_{22} & -k_{12} & k_{24} \\ -k_{11} & -k_{12} & k_{11} & -k_{12} \\ k_{12} & k_{24} & -k_{12} & k_{22} \end{bmatrix} \quad (19)$$

In order to optimize the ratio between the stiffness matrix's accuracy on the one hand and the lengths as well as complexities of the derived at expressions on the other, initially some numerical preliminary studies were executed. These studies focused on the minimum number of required terms in the Taylor's series and their impact on the accuracy obtained. Since the simplified model's GDEs' analytical solutions' values of critical buckling force P_{cr} for the crack located at the mid – span were already given [10] for three different boundary conditions (cantilever, pin – ended column and propped cantilever) those results served as Benchmark values. Special attention was devoted to the propped cantilever structure which had the more complex deformation line with two radii of curvature.
The studies initiated by implementing 8 terms within each of Taylor's expansion of trigonometric terms. Then the number of terms was sequentially increased by 1 and the studies were terminated with 20 terms where no

significant improvement of the results was evident for all three considered structures. The results are given in Table 1.

Table 1 *The convergence study of the buckling force's results according to the number of terms in Taylor expansion*

Tablica 1 *Studija konvergencija rezultata sile izvijanja u zavisnosti od broja članova u Taylorovi ekspanziji*

	cantilever [N]	simply supported beam [N]	propped cantilever [N]
GDEs' solutions	46352.9	174534.5	372817.2
8 terms	46329.4	167099.5	306228.5
10 terms	46353.4	175225.4	466392.1
11 terms	46353.0	174662.4	387125.0
12 terms	46352.9	174497.3	369930.2
13 terms	46352.9	174528. 1	371626.9
14 terms	46352.9	174536.0	373078.2
15 terms	46352.9	174534.7	372914.9
16 terms	46352.9	174534.4	372800.0
17 terms	46352.9	174534.5	372811.1
18 terms	46352.9	174534.5	372818.1
19 terms	46352.9	174534.5	372817.5
20 terms	46352.9	174534.5	372817.1

It is clearly evident from this table that among all three structures the cantilever structure monotonically converged as first. The limit value was already achieved by implementing solely 12 terms, while the corresponding values for the simply supported beam and the propped cantilever exhibited errors of 0.02% and 0.77%, respectively. These values proved (as expected) that the propped cantilever structure required more terms in the expansions. It is further apparent from the table that the results for the propped cantilever exhibited non – monotonic convergence.
While implementing 15 terms in polynomial expansions the error for the propped cantilever already dropped to under 0.03%. The result for the simply supported beam was almost perfect while the increase in the number of terms had no influence on the cantilever's result.
Finally, since it is further obvious from the table that the implementation of more than 15 terms had just a slight impact on the results, it was decided to implement 15 terms' expansions in the derivation of the final expression.

The corresponding derived at expressions of the stiffness' matrix coefficients are thus given as:

$$k_{11} = \frac{1}{\beta} \cdot \sum_{n=0}^{7} \frac{3 \cdot P \cdot (-1)^n}{(2 \cdot n)!} \alpha^{2 \cdot n} \cdot L_1^{2 \cdot n - 1} \tag{20}$$

$$k_{12} = \frac{1}{\beta} \cdot \sum_{n=0}^{8} \frac{-3 \cdot P \cdot (-1)^n}{(2 \cdot n - 1)!} \alpha^{2 \cdot n - 2} \cdot L_1^{2 \cdot n - 2} \tag{21}$$

$$k_{22} = \frac{1}{\beta \cdot \gamma} \cdot (k_{221} + k_{222} - k_{223}) \tag{22}$$

$$k_{24} = \frac{1}{\beta \cdot \gamma} \cdot (k_{223} + k_{242}) \tag{23}$$

with the following abbreviations:

$$\beta = \sum_{N=0}^{6} \frac{2 \cdot (-1)^N}{\left(\prod_{n=1}^{N} 2 \cdot n\right) \cdot \prod_{n=2}^{N+1} (2 \cdot n + 1)} \alpha^{2 \cdot N + 2} \cdot L_1^{2 \cdot N + 2}$$

$$k_{222} = \sum_{N=0}^{6} \frac{8 \cdot P \cdot K_r \cdot (-1)^N \cdot \alpha^{2 \cdot N + 2}}{\left(\prod_{n=2}^{N+1} (2 \cdot n + 1)\right) \cdot \left(\frac{7 \cdot N^6}{2880} - \frac{11 \cdot N^5}{320} + \frac{119 \cdot N^4}{576} - \right.}$$

$$\frac{\cdot L_1^{2 \cdot N + 1}}{\left. \frac{41 \cdot N^3}{64} + \frac{1679 \cdot N^2}{1440} - \frac{6 \cdot N}{5} + 1 \right)}$$

4. Numerical examples

4. *Numerički primjeri*

The efficiencies of the presented expressions were confirmed by completely re – analyzing the cracked structures from the reference [10] where comprehensive compilations were given of the results from various approaches. These cracked structures were: cantilever, pin – ended column and propped cantilever. For all three structures, differing only in boundary conditions, the length L was 10 m, the cross – section was a rectangle with dimensions 0.10/0.20 m. The Young modulus was 30 GPa and the Poisson's ratio was 0.3. Single 0.1 m deep cracks were separately introduced on all structures at 9 locations within the structure over 1 m distances. From among all existing definitions for rotational spring, the definition given by Okamura was selected due to the fact that it is the only one that takes the

Poisson's ration into account.
Each structure was analyzed by two noticeably different models. The first model was a 2D FEM computational model established within the commercial finite element program COSMOS/m. The model consisted of 20,000 8 noded quadrilateral finite elements with approximately 122,000 degrees of freedom. In this model accurate descriptions of the cracks were realized.
The second computational model was the simplified model with the crack modeled by rotational spring, and various analyzing approaches were implemented. Firstly, GDEs of all three structures were analyzed and the corresponding characteristic equation was solved for each structure. This provided the first set of the results for the simplified computational model. Afterwards, finite element analyses were performed using WSCBCFE.
These results from previous analyses are repeated in the second, third and fourth columns of Tables 2, 3 and 4 that belong to the cantilever, simply supported beam and propped cantilever, respectively. They offer a reliable platform for autonomous testing of the new finite element's results.
The last column in these tables presents the results obtained by the newly – presented finite element.
The new cracked beam – column finite element computational models for each structure and location of the crack generally consisted of two 2 noded beam finite elements. The only exception was the situation where the crack was located at the mid – span and only one newly – presented FE was sufficient.
For the cantilever and simply supported beam 4 unconstrained degrees of freedom were required (and just 2 in the case where the crack was located at the mid – span). Alternatively, 3 unconstrained degrees of freedom were required for the propped cantilever (and just 1 in the case where the crack was located at the mid – span).
It is evident that the results in the last three columns belonging to the same computational model and analyzed by different approaches really exhibit very good matching.
As the last three columns of Tables 2, 3 and 4 belong to the simplified computational model they are directly comparable. Among these three columns the first column recapitulates the results from models' governing differential equations providing the model's most accurate results. The last two columns belong to finite element solutions of the governing differential equations: the WSCBCFE and the newly derived at finite element, respectively.
It is obvious from the results for the cantilever structure (Table 2) that for a given crack's location all the results that belong to the simplified model were completely identical. Therefore, for this structure the only advantage of the new finite element over the WSCBCFE is the analytical form.

Table 2 *Critical buckling force P_{cr} [N] for a cracked cantilever obtained from different approaches for various positions of the crack (L_1 is the distance from the clamped end)*

Tablica 2 *Kritična uklonska sile P_{cr} [N] za napuknutu konzolu dobivena iz različitih pristupa za različite pozicije pukotine (L_1 je udaljenost od pričvršćenog kraja)*

L_1	2D FEM	GDES	WSCBCFE	new FE
1 m	43495.9	43815.6	43815.6	43815.6
2 m	43907.9	44206.5	44206.5	44206.5
3 m	44515.1	44784.2	44784.2	44784.2
4 m	45284.2	45515.6	45515.6	45515.6
5 m	46166.5	46352.9	46352.9	46352.9
6 m	47092.5	47229.3	47229.3	47229.3
7 m	47971.3	48058.5	48058.5	48058.5
8 m	48698.7	48743.1	48743.1	48743.1
9 m	49177.2	49192.7	49192.7	49192.7

However, it is apparent from the results for the cracked pin – ended beam (Table 3) that just the results from the new element were identical (with one rather minuscule exception) to the results from the GDE's. Therefore, the results that belong to the new finite element exhibited slightly better agreement to the results from GDE's as the WSCBCFE's results.
Finally, for the propped cantilever (Table 4) the results from WSCBCFE as well as the new finite element did not exhibit such an ideal agreement with the results from GDEs. However, although the results were still very good from the engineering point of view, the results from the new finite element clearly exhibited noticeably better accuracy than WSCBCFE's values. Therefore, for this kind of structure the advantages of the new finite element over the WSCBCFE are of the analytical form as well as better results.
When comparing both finite element approaches regarding all three structures: 2D FEM

Table 3 *Critical buckling force P_{cr} [N] for a cracked pin – ended column obtained from different approaches for various positions of the crack (L_1 is the distance from the left end)*

Tablica 3 *Kritična sila P_{cr} [N] za puknuti stup sa zglobnim spojevima na kraju dobivena iz različitih pristupa za različite pozicije u pukotinu (L_1 je udaljenost od lijevog kraja)*

L_1	2D FEM	GDES	WSCBCFE	new FE
1 m	194468.5	194899.8	194899.7	194899.8
2 m	187702.1	188529.4	188529.3	188529.4
3 m	180210.2	181426.6	181426.6	181426.6
4 m	174880.0	176336.3	176336.2	176336.3
5 m	173001.1	174534.5	174534.4	174534.7
6 m	174880.0	176336.3	176336.2	176336.3
7 m	180210.2	181426.6	181426.5	181426.6
8 m	187702.1	188529.4	188529.3	188529.4
9 m	194468.4	194899.8	194899.7	194899.8

Table 4 *Critical buckling force P_{cr}[N] for a cracked propped cantilever obtained from different approaches for various positions of the crack (L_1 is the distance from the clamped end)*

Tablica 4 *Kritična sila P_{cr} [N] za puknutu naslonjenenu konzola dobivena iz različitih pristupa za različite pozicije u pukotinu (L_1 je udaljenost od pričvršćenog kraja)*

L_1	2D FEM	GDES	WSCBCFE	new FE
1 m	373382.6	375839.1	375816.1	375843.6
2 m	392774.6	394383.1	394351.6	394383.2
3 m	402708.6	403813.8	403778.5	403813.8
4 m	392016.1	393719.8	393691.0	393722.1
5 m	370003.7	372817.2	372797.4	372914.9
6 m	354094.7	357623.0	357607.4	357625.4
7 m	352639.4	356308.3	356291.4	356308.4
8 m	367366.6	370460.8	370436.6	370460.8
9 m	391092.2	392904.8	392872.2	392907.0

computational model (consisting of 20,000 8 noded quadrilateral finite elements and approximately 122,000 degrees of freedom/equations) versus the new cracked beam – column finite element model (up to 2 beam finite elements and up to 4 unconstrained degrees of freedom), it is evident from the results that the enormous difference in computational effort was barely reflected in the results. The maximum difference regarding the structures appeared for the cropped cantilever (1.04%), while for the cantilever and pin – ended column the maximum discrepancies were 0.74% and 0.89%, respectively.

5. Conclusions

5. *Zaključak*

Okamura's simplified computational model of a cracked beam was once more implemented for the prediction of buckling load P_{cr} for slender beam – type structures with a transverse crack. Previously presented solutions have outlined two potential approaches. The first approach – the search for an analytical expression for critical buckling load by solving governing equations had shown itself to be applicable to a rather small collection of simple and moderate structures. In order to expand the applicability of the model a special cracked beam – column finite element, derived at on the basis of an adequate approximation of the governing differential equation's solution for displacements, has been derived at. Although this approach was applicable to more generalized structures it suffered from serious practical limitation: namely the absence of a clear analytical form. This handicap has been overcome in the presented paper where the progress was achieved by presenting a clear analytical and compact form of the stiffness matrix for the situation where the crack is located at the mid – span of the finite element. In addition, not only that the stiffness matrix of the new finite element is given in full analytical form, it also produces somewhat more accurate results.

By deriving this cracked beam – finite element's closed – analytical form the simplified model's versatility was essentially improved. Namely, this is the only approach that is flexible enough for the analyses of frame type structures yet, on the other hand, it is also small and compact enough to be potentially applicable in inverse problems.

As the derived at beam – column finite element has the standard four degrees of freedom for transverse displacements it is directly compatible with the standard finite element for a non – cracked case and therefore can be easily

included within any existing software.
Despite clear advancement the orientation for future research is obvious as potential further research interest should examine the expansion of an already existing stiffness matrix to an arbitrary location of the crack's location simultaneously preserving, if possible, the compact analytical form.

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