

On the geometric-arithmetic mean inequality for matrices

M. ALIĆ*, P. S. BULLEN†, J. E. PEČARIĆ‡ AND V. VOLENEC§

Abstract. *In this paper refinements and converses of matrix forms of the geometric-arithmetic mean inequality are given.*

Key words: *arithmetic, geometric means, Hermitian matrices*

Sažetak. *U radu se daje poboljšanje i konverzija nejednakosti između geometrijske i aritmetičke sredine za matrice.*

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1. Introduction

Let $\alpha \in]0, 1[$ and let A, B be two $n \times n$ positive definite Hermitian matrices. The weighted arithmetic, geometric and harmonic means of A and B are respectively:

$$\begin{aligned} A \nabla B &= \alpha A + (1 - \alpha)B; & A \# B &= B^{1/2} (B^{-1/2} A B^{-1/2})^\alpha B^{1/2}; \\ A!B &= (\alpha A^{-1} + (1 - \alpha)B^{-1})^{-1}. \end{aligned} \quad (1)$$

The matrix form of the geometric-arithmetic mean inequality is well known:

$$A!B \leq A \# B \leq A \nabla B; \quad (2)$$

where the matrix inequality $A \geq B$ means that $A - B$ is positive semi-definite.

A converse inequality to a part of (2),

$$A!B \leq A \nabla B,$$

was obtained recently in [1].

In this paper we obtain further converses and refinements of (2).

*Department of Mathematics, University of Zagreb, Bijenička c. 30, HR-10 000 Zagreb, Croatia, e-mail: mladen.alic@cromath.math.hr

†Department of Mathematics, University of British Columbia, Vancouver BC, Canada V6T1Z2, e-mail: bullen@math.ubc.ca

‡Faculty of Textile Technology, University of Zagreb, Pierottijeva 6, HR-10 000 Zagreb, Croatia, e-mail: pecaric@mahazu.hazu.hr

§Department of Mathematics University of Zagreb, Bijenička c. 30, HR-10 000 Zagreb, Croatia, e-mail: volenec@cromath.math.hr

2. Preliminary lemmas

An essential tool in the proof of our main converse result is the following converse of the geometric- arithmetic mean inequality due to Dočev.

Lemma 1. *If $0 < m \leq a_i \leq M$, $1 \leq i \leq n$, $w_i > 0$, $1 \leq i \leq n$ and $W_n = \sum_{i=1}^n w_i$ then*

$$\frac{1}{W_n} \sum_{i=1}^n w_i a_i \leq K \left(\prod_{i=1}^n a_i^{w_i} \right)^{1/W_n}; \quad (3)$$

where if $\beta = M/m$ then

$$K = \frac{(\beta - 1)\beta^{1/(\beta-1)}}{e \log \beta}. \quad (4)$$

Furthermore, K regarded as a function of M is increasing, and regarded as a function of m is decreasing.

Proof. The inequality (3) is stated in [2, p.124]. The method of proof is that of [2, p.28, Remark(14)]; not as stated in the first reference. The last remark is an easy consequence of the proof and the concavity of the logarithmic function. \square

The following generalization of Bernoulli's inequality is due to Gerber [3].

Lemma 2. *If $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$, and $x > -1$ then*

$$\operatorname{sgn} \left((1+x)^\alpha - \sum_{i=0}^n \binom{\alpha}{i} x^i \right) = \operatorname{sgn} \left(\binom{\alpha}{n+1} x^{n+1} \right). \quad (5)$$

Substituting $t-1$ for x in (5) leads to

$$\binom{\alpha}{n+1} \left(t^\alpha - \sum_{i=0}^n \binom{\alpha}{i} (t-1)^i \right) \geq 0 \quad \text{if } t \geq 1 \quad (6)$$

$$(-1)^{n+1} \binom{\alpha}{n+1} \left(t^\alpha - \sum_{i=0}^n \binom{\alpha}{i} (t-1)^i \right) \geq 0 \quad \text{if } 1 \geq t > 0. \quad (7)$$

3. Converse results

Theorem 1. *Let A and B be two positive definite Hermitian matrices and let $0 < \alpha < 1$, then*

$$A \nabla B \leq K(A\#B),$$

where K is given by (4) with $\beta = \max\{\lambda_1, \lambda_n^{-1}\}$, where $\lambda_1 \geq \dots \geq \lambda_n$ are the eigenvalues of $B^{-1/2}AB^{-1/2}$.

Proof. Apply Lemma 1 with $n = 2$, $w_1 = \alpha$, $w_2 = 1 - \alpha$, $a_1 = \lambda_i$, $a_2 = 1$ to get

$$\alpha \lambda_i + 1 - \alpha \leq K_i \lambda_i^\alpha,$$

where K_i is given by (4) with

$$\beta_i = \frac{\max\{\lambda_i, 1\}}{\min\{\lambda_i, 1\}} = \max\{\lambda_i, \lambda_i^{-1}\}.$$

Since $\lambda_1 \geq \dots \geq \lambda_n$ we have, from *Lemma 1* that $K_i \leq K$, $1 \leq i \leq n$ where K is given by (4) with β as in the statement of the theorem. Hence

$$\alpha\lambda_i + 1 - \alpha \leq K\lambda_i^\alpha, \quad 1 \leq i \leq n. \quad (8)$$

If a positive definite matrix C has the representation $C = \Gamma D_\lambda \Gamma^*$, where Γ is unitary and D_λ is the diagonal matrix $\text{diag}\{\lambda_1, \dots, \lambda_n\}$, then (8) implies that

$$\alpha D_\lambda + (1 - \alpha)I \leq K D_\lambda^\alpha. \quad (9)$$

Pre- and post-multiplication of (9) by Γ and Γ^* respectively gives

$$\alpha C + (1 - \alpha)I \leq K C^\alpha. \quad (10)$$

Then setting $C = B^{-1/2} A B^{-1/2}$ in (10) we get

$$\alpha B^{-1/2} A B^{-1/2} + (1 - \alpha)I \leq \left(B^{-1/2} A B^{-1/2} \right)^\alpha. \quad (11)$$

Pre- and post-multiplication of (11) by $B^{1/2}$ completes the proof of the theorem. \square

Theorem 2. *With the hypotheses and notations of Theorem 1,*

$$A \# B \leq K(A!B).$$

Proof. Apply *Theorem 1* with A, B replaced by A^{-1}, B^{-1} , respectively. \square

4. Refinements

In this section we use the notations in section 1 but allow α to be an arbitrary real number.

Theorem 3. *Let A and B be two positive definite Hermitian matrices and let $\alpha \in \mathbb{R}$ then*

$$\binom{\alpha}{n+1} (A \# B - A \nabla B) \leq \binom{\alpha}{n+1} \sum_{i=2}^n \binom{\alpha}{i} B^{1/2} \left(B^{-1/2} (A - B) B^{-1/2} \right)^i B^{1/2} \quad \text{if } A > B; \quad (12)$$

$$(-1)^{n+1} \binom{\alpha}{n+1} (A \# B - A \nabla B) \geq (-1)^{n+1} \binom{\alpha}{n+1} \sum_{i=2}^n \binom{\alpha}{i} B^{1/2} \left(B^{-1/2} (A - B) B^{-1/2} \right)^i B^{1/2} \quad \text{if } A < B. \quad (13)$$

Proof. Let C be a positive definite Hermitian matrix then as in the proof of *Theorem 1*, but using (6) and (7), we get

$$\binom{\alpha}{n+1} \left(C^\alpha - \sum_{i=0}^n \binom{\alpha}{i} (C - I)^i \right) \geq 0 \quad \text{if } C > I \quad (14)$$

$$(-1)^{n+1} \binom{\alpha}{n+1} \left(C^\alpha - \sum_{i=0}^n \binom{\alpha}{i} (C-I)^i \right) \geq 0 \quad \text{if } I > C > 0. \quad (15)$$

Simple rewriting of (14) and (15) give:

$$\binom{\alpha}{n+1} \left(C^\alpha - \alpha C - (1-\alpha)I \right) \geq \binom{\alpha}{n+1} \sum_{i=2}^n \binom{\alpha}{i} (C-I)^i \quad \text{if } C > I \quad (16)$$

$$(-1)^{n+1} \binom{\alpha}{n+1} \left(C^\alpha - \alpha s C - (1-\alpha)I \right) \geq (-1)^{n+1} \binom{\alpha}{n+1} \sum_{i=2}^n \binom{\alpha}{i} (C-I)^i \quad (17)$$

if $I > C > 0$.

Substitution in (16) and 17) of $C = B^{-1/2}AB^{-1/2}$ and then pre- and post-multiplication by $B^{1/2}$ leads to (12) and (13). \square

The special case $n = 1$ of *Theorem 3* was proved in [4].

The case $n = 2$ of *Theorem 3* gives

Corollary 1. *Let A and B be two positive definite Hermitian matrices and let $\alpha \in \mathbb{R}$. If $A \geq B$ and if either $0 < \alpha < 1$ or $\alpha > 2$, then*

$$A \nabla B - A \# B \leq \frac{\alpha(1-\alpha)}{2} B^{1/2} (B^{-1/2}(A-B)B^{-1/2})^2 B^{1/2}.$$

If $\alpha < 0$ or $1 < \alpha < 2$, then the reverse inequality holds.

Using a generalization of Bernoulli's inequality proved in [5] we can generalize the corollary as follows.

Theorem 4. *Let A and B be two positive definite Hermitian matrices and let $\alpha, K \in \mathbb{R}$. If $A \geq KB$, with $0 < K \leq 1$ and if either $0 < \alpha < 1$ or $\alpha > 2$, then*

$$A \nabla B - A \# B \leq \frac{\alpha(1-\alpha)}{2} K^{\alpha-2} B^{1/2} (B^{-1/2}(A-B)B^{-1/2})^2 B^{1/2}.$$

If $\alpha < 0$ or $1 < \alpha < 2$, then the reverse inequality holds. The inequality also holds if $A \leq KB$, with $K \geq 1$ and if either $\alpha < 0$ or $1 < \alpha < 2$; the reverse inequality then holds if $0 < \alpha < 1$ or $\alpha > 2$.

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