A multidimensional generalization of the Steinhaus theorem*

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Abstract. *Steinhaus has shown that the subset of* $\mathbb{R}$ *of the form* $A + B = \{a + b : a \in A, b \in B\}$ *contains an interval provided both sets* $A \in \mathbb{R}$ *and* $B \in \mathbb{R}$ *have positive Lebesgue measure.*

In the present paper the author generalizes this fact to subsets of $\mathbb{R}^n$*,* $n \in N$*, satisfying the above conditions. The statement is proved in a new way using a very interesting lemma.*

Key words: *measurable set, Lebesgue measure, outer Lebesgue measure, compact set, closed set*

Sažetak.**Višedimenzionalna generalizacija Steinhausova teorema**. *Steinhaus je dokazao da podskup od* $\mathbb{R}$ *oblika* $A+B = \{a+b : a \in A, b \in B\}$ *sadrži interval, ako skupovi* $A \in \mathbb{R}$ *i* $B \in \mathbb{R}$ *imaju pozitivnu Lebesgueovu mjeru.*

U ovom radu autor tu tvrdnju poopćava na podskupove od $\mathbb{R}^n$*,* $n \in \mathbb{N}$*. Tvrdnja je dokazana na novi način, pomoću zanimljivih lema.*

Ključne riječi: *izmjeriv skup, Lebesgueova mjera, vanjska Lebesgueova mjera, kompaktan skup, zatvoren skup*

AMS subject classifications: 28A05

Received June 26, 1997 Accepted November 25, 1997

Properties of measurable and Baire sets have been studied in the last 70 years. Some of the basic notions in both theories are "small", "thin" or "negligible" set in comparison to a "big" or "thick" set. Let us list some of the basic terms connected with measure and category.

The set $A \subset \mathbb{R}$ is said to be measurable in the Lebesgue sense if and only if $m_e(S) = m_e(S \cap A) + m_e(S \cap A^C)$ for every $S \subset \mathbb{R}$, $A^C = \mathbb{R}\backslash A$. Here m_e denotes outer Lebesgue measure defined in the following way:

$$m_e(A) = \inf_{\substack{A \subset G \\ G \text{ open}}} \{m(G)\}.$$

*The lecture presented at the MATHEMATICAL COLLOQUIUM in Osijek organized by Croatian Mathematical Society – Division Osijek, June 6, 1997.

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The set A is said to be thick on the interval I if it has non-empty intersection with every subinterval of the interval I. A set which is not thick in any interval is said to be nowhere-thick. If a set can be represented as a finite or countable union of nowhere-thick sets we say that the set is of the first category, otherwise, it is of the second category. If the set $A \subset \mathbb{R}$ can be written in the form $A = (G \backslash P) \cup Q$, where G is an open set, and P and Q are sets of the first category, we say that A is a Baire set or that it has the Baire property.

We shall use the following notations:

τ - the family of open sets

$\mathcal{B}$ - the family of Borel sets, $\mathcal{B} =< \tau >$

$\mathcal{M}$ - the family of measurable sets in the sense of Lebesgue

$\mathcal{M}^+ = \{A \in \mathcal{M} : m(A) > 0\}$

$\mathcal{B}_a$ - the family of Baire sets (sets possessing the Baire property)

$\mathcal{B}_a^+ = \{A \in \mathcal{B}_a : A \text{ is of the second category}\}$

n - the family of sets of measure zero (a set $A \subset \mathbb{R}$ is of measure zero, if for every $\epsilon > 0$ there exists a sequence of intervals $(I_n, n \in \mathbb{N})$ such that $A \subset \bigcup\limits_n I_n$ and $\sum m(I) < \epsilon$)

p -the family of sets of the first category.

For the purpose of comparing families of measurable and Baire sets we give a review of some known facts in the following table:

$\mathcal{M}$= measurable sets	$\mathcal{B}_a$ = Baire sets
$\mathcal{M} =< \mathcal{B}, n >$, the smallest σ-algebra containing $\mathcal{B}$ and n	$\mathcal{B}_a =< \mathcal{B}, p >$, the smallest σ-algebra containing $\mathcal{B}$ and p
$A \in \mathcal{M} \Rightarrow A = S \cup N$ where $S \in F_\sigma$, $N \in n$ (F_σ)= countable union of closed sets	$A \in \mathcal{B}_a \Rightarrow A = T \cup P$ where $T \in G_\delta$, $P \in p$ (G_δ)= countable intersection of open sets
$A \in \mathcal{M} \Rightarrow A = T \backslash N$, where $T \in G_\delta$, $N \in n$	$A \in \mathcal{B}_a \Rightarrow A = S \backslash P$, where $T \in F_\sigma$, $P \in p$

It is easy to see that: $\mathcal{M} \not\subset \mathcal{B}_a$ and $\mathcal{B}_a \not\subset \mathcal{M}$, $\mathcal{M}^+ \not\subset \mathcal{B}_a^+$ and $\mathcal{B}_a^+ \not\subset \mathcal{M}^+$, therefore, it is clear that some results for measurable sets are not necessarily true for Baire sets. Moreover, there exists a set $B \subseteq \mathbb{R}$ (Bernstein) such that both sets B and B^C intersect every uncountable closed subset of $\mathbb{R}$ and $B \notin \mathcal{M}$, $B \notin \mathcal{B}_a$.

In 1920 Steinhaus proved the following theorem:

Theorem 1. *If $A, B \subset \mathbb{R}$ and $A, B \in \mathcal{M}^+$, then the set $A+B = \{a+b : a \in A, b \in B\}$ contains an interval.*

On the other hand, Piccard gave in 1939 the following statement:

Theorem 2. *If $A, B \subset \mathbb{R}$ and $A, B \in \mathcal{B}_a^+$, then the set $A+B$ contains an interval.*

These results were generalized in 1976 by Sander:

Theorem 3. *If $A, B \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $A \in \mathcal{M}^+$ and $m_e(B) > 0$, then $A+B$ contains an n-dimensional interval (an open sphere).*

Theorem 4. *If $A, B \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $A \in \mathcal{B}_a^+$ and B is of the second category, then $A+B$ contains an n-ball.*

In the same year J. Smital proved that if $A, B \subset \mathbb{R}$, $m_e(A) > 0$ and B is thick in $\mathbb{R}$, then $m_e((A+B)\cap I) = m(I)$ for every interval I.

It is proved in the author's Ph.D. thesis that:

Theorem 5.

1. $\begin{matrix} m_e(A) > 0 \\ m_e(B) > 0 \end{matrix} \not\Rightarrow A+B$ *contains an interval*

2. $m_e(B) > 0 \not\Rightarrow B$ *is thick on some interval I.*

In 1986 H. Miller proved the following statement:

Theorem 6. *If A and B are subsets of the set $\mathbb{R}$, $m_e(A) > 0$ and $m_e(B) > 0$, then there exists an interval I such that $m_e((A+B)\cap I) = m(I)$.*

In a joint paper, H. Miller and the author generalized this result in the following two directions:

1. they considered the problem in $\mathbb{R}^n$, $n \in \mathbb{N}$, $n > 1$.

2. instead of "+", they considered a more general function f satisfying some conditions.

Theorem 7. *If $A, B \subset \mathbb{R}^n$, $m_e(A) > 0$ and $m_e(B) > 0$, then there exists a cube such that $m_e((A+B)\cap K) = m(K)$.*

Proof. According to *Lemma 1.3* from [1], Y_N covers a cube $K = T(a+b, q(h))$ in the sense of Vitali, where $q(h)$ is a positive number for which $(2h+q(h))^n < \frac{3}{2}(2h)^n$.

Therefore, for the given N, $N \geq 2$, according to the Vitali theorem on the cover (see *Theorem 1.1* in [1]), there exist mutually disjoint cubes H_1, H_2, ..., H_i from Y_N such that $m(K \setminus \bigcup\limits_{j=1}^{i} H_j) < \frac{1}{N}$ and $m(K) = (q(h))^n = t$.

For a sufficiently great N let K_N denote the cube $T\left((a+b, q(h) - \frac{2}{N^{\frac{1}{n}}}\right)$. Then $m(K_N) = \left(q(h) - \frac{2}{N^{\frac{1}{n}}}\right)^n = t_N$, so that $m(K \setminus \cup (H_j : H_j \subset K)) < \frac{1}{N} + (t - t_N)$. Therefore,

$$m(\cup(H_j : H_j \subset K)) > t - (\frac{1}{N} + (t - t_N)) = \left(q(h) - \frac{2}{N^{\frac{1}{n}}}\right)^n - \frac{1}{N}.$$

From the above and by using *Lemma 1.2* in [1] we have:

$$\begin{aligned} m_e((A+B)\cap K) \geq \ & m_e(\cup((A+B)\cap H_j : H_j \subset K)) \\ > \ & \left(\left(q(h) - \frac{2}{N^{\frac{1}{n}}}\right)^n - \frac{1}{N}\right) \cdot \left(1 - \frac{n2^n}{N^{\frac{1}{n}}} + \frac{2^n}{N}\right) \end{aligned}$$

for a sufficiently great N. The converse is obvious. $\square$

Theorem 8. (A generalized Steinhaus theorem) *If A and B are measurable subsets of $\mathbb{R}^n$ of positive Lebesgue measure, then the set $A+B=\{a+b : a\in A, b\in B\}$ contains a cube.*

Proof. Without loss of generality, we can assume that A and B are compact sets of $\mathbb{R}^n$ of positive Lebesgue measure. Then $A+B$ is a compact subset of $\mathbb{R}^n$. Notice that any closed subset of $\mathbb{R}^n$ is either nowhere thick or it contains a cube. If $A+B$ is nowhere thick, then $m_e((A+B)\cap K) < m(K)$ for every cube K, which opposes the previous theorem, so $A+B$ must contain a cube. $\square$

Remark 1. *It was known that there exist measurable sets $A, B\subset\mathbb{R}^n$, such that $A+B$ is not measurable. H. Miller generalized this result so that instead of "+" he considered a function $f:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$, which satisfies some conditions. It would be interesting to mention that the analogue of Kurepa's result holds, that is, there exist subsets $A, B\subset\mathbb{R}^n$ with the Baire property, so that $A+B$ does not have the Baire property.*
As already seen, classical results of Steinhaus and Piccard show that $A+B=\{a+b : a\in A, b\in B\}$ contains an interval provided:

(a) *A and B are measurable sets of real numbers each having positive Lebesgue measure, or*

(b) *A and B are sets of real numbers of the second category with the Baire property.*

The conditions (a) and (b) are sufficient for the set $A+B$ to contain an interval, but they are not necessary. For example, $C+C$ contains an interval, where C is the Cantor set.

In [4] the author, B. Guljaš and H. Miller gave an answer to the question raised by H. Miller several years ago: "Should $g(C,C)$ contain an interval if the function $g : R\times R\to R$ satisfies appropriate conditions?" All proofs for a positive answer were given in two ways: analytically and geometrically. Along with several previously proved lemmas, the following was proved:

Theorem 9. *Suppose $g\in C^1([0,1]^2)$. If there exist $x, y\in C$ such that $D_1g(x_0,y_0)\neq 0$, and $D_2g(x_0,y_0)\neq 0$, and if one of the following conditions holds:*

(i) $|D_1g(x_0,y_0)/D_2g(x_0,y_0)|\neq 3^n$ *for each* $n\in\mathbb{Z}$

(ii) (x_0,y_0) *is a local extremum point of the function D_1g/D_2g, then $g(C,C)$ contains an interval.*

It would be interesting to mention that there exist sets $A, B\subseteq\mathbb{R}$ such that $A-B$ contains an interval, and $A+B$ does not contain an interval (see [5]).

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