# Least orthogonal absolute deviations problem for generalized logistic function *

Tomislav Marošević†

**Abstract**. *We consider the existence of optimal parameters for generalized logistic model by least orthogonal absolute deviations, and prove the existence of such optimal solution, under the monotonicity condition on the data.*

**Key words:** *generalized logistic function, least orthogonal absolute deviations, total least squares*

**Sažetak**. **Najmanja ortogonalna apsolutna odstupanja za generaliziranu logističku funkciju.** *Razmatra se problem egzistencije optimalnih parametara za generaliziranu logističku funkciju u smislu najmanjih ortogonalnih apsolutnih odstupanja, te dokazuje egzistenciju takvog optimalnog rješenja, uz uvjet monotonosti podataka.*

**Ključne riječi:** *generalizirana logistička funkcija, najmanja ortogonalna apsolutna odstupanja, najmanji potpuni kvadrati*

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## 1. Introduction

The generalized logistic function

$$f(t;b,c) = \frac{A}{(1+be^{-c\gamma t})^{\frac{1}{\gamma}}}\,, \qquad b>0\,, \tag{1}$$

is often used in applied research (see [5], [13]). It is the solution of the so-called Nelder's model

$$y' = cy\Big(1-\big(\frac{y}{A}\big)^{\gamma}\Big)\,, \qquad A,\gamma>0\,,$$

(see [8]). The constant $A>0$ denotes the saturation level, and the constant $\gamma>0$ is the so-called asymmetry coefficient (see [10]).

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†Faculty of Electrical Engineering, University of Osijek, Kneza Trpimira 2b, HR-31 000 Osijek, Croatia, e-mail: Tomislav.Marosevic@etfos.drava.hr

The unknown parameters $b$, $c$ have to be determined on the basis of the given data $(w_i, t_i, f_i)$, $i = 1, \dots, m$ $(m > 2)$, where $t_i$ are the measured values of the independent variable, $f_i$ are the measured values of the dependent variable, and $w_i > 0$ are the data weights.
In the case when only the measured values $f_1, \dots, f_m$ of the dependent variable contain unknown additive errors $\varepsilon_i$, the estimation of the parameters $b$, $c$ is usually done in the sense of ordinary least squares, i.e. the $L_2$-norm (see [3], [6]), by minimizing the functional

$$G(b,c) = \sum_{i=1}^{m} w_i \Big( f_i - \frac{A}{(1+be^{-c\gamma t_i})^{1/\gamma}} \Big)^2 .$$

Some results on the existence problem for generalized logistic function in the sense of ordinary least squares can be found in [2], [5], [10].

The least squares criterion is most suitable in cases where errors are normally distributed with mean zero and variance $\sigma^2 I$. When errors do not follow the normal (Gaussian) distribution, the use of other $L_p$-norms $(p \neq 2)$ is recommended (see [3]).

In general case, significant errors exist both in measurements $t_i$ of the independent variable and in measurements $f_i$ of the dependent variable ([1]). If one assumes that $t_i$ has the unknown additive error $\delta_i$ and $f_i$ has the unknown additive error $\varepsilon_i$, then we have the model

$$f_i = \frac{A}{(1+be^{-c\gamma t_i})^{1/\gamma}} + \varepsilon_i \,, \qquad i = 1, \dots, m,$$

and the parameters $b$, $c$ can be estimated by the total least squares approach (see [1], [4]), which leads to minimization of the functional

$$T(b,c,\delta_1,\dots,\delta_m) = \sum_{i=1}^{m} w_i \Big( \big( f_i - \frac{A}{(1+be^{-c\gamma t_i})^{1/\gamma}} \big)^2 + \delta_i^2 \Big) . \tag{2}$$

Note that $T(b,c,\delta_1,\dots,\delta_m)$ is the weighted sum of squares of the distances from the data points to the curve $\Gamma_f$, where $t \mapsto f(t;b,c) = \frac{A}{(1+be^{-c\gamma t})^{1/\gamma}}$.

In this paper we consider the existence problem of optimal parameters for the generalized logistic function in the sense of the least orthogonal absolute deviations (LOAD, see [9], [12]). The LOAD approach leads to minimization of the weighted sum of orthogonal distances from measurement points to the model function (see [7]). In this case, the problem of estimating the parameters $b$ and $c$ is reduced to the problem of minimizing the functional

$$F(b,c,\delta) = \sum_{i=1}^{m} w_i \sqrt{ \Big( f_i - \frac{A}{(1+be^{-c\gamma t_i})^{1/\gamma}} \Big)^2 + \delta_i^2 } \,, \tag{3}$$

where $\delta = [\delta_1, \dots, \delta_m]^T \in \mathbb{R}^m$. We are going to prove, for monotonic data, the existence of a point $(b^*, c^*, \delta^*)$ which minimizes the functional $F$. The idea of the proof is based on [7] and [11].

## 2. The existence problem

Assume we are given the data $(w_i, t_i, f_i)$, $i = 1, \ldots, m$, $m > 2$, $t_1 < t_2 < \ldots < t_m$. Further, suppose that $f_i > 0$, $i = 1, \ldots, m$, and that we apply the generalized logistic model function. Let us observe that if $f_1 \le f_2 \le \ldots \le f_m$, then it is appropriate to describe such data by an increasing function. Also, if $f_1 \ge f_2 \ge \ldots \ge f_m$, then it is suitable to use a decreasing function.

With regard to the existence problem of optimal parameters $b^*$, $c^*$ of the generalized logistic function in the sense of least orthogonal absolute deviations, we have the following theorem.

**Theorem 1.** *Let the given data* $(w_i, t_i, f_i)$, $i = 1, \ldots, m$, $m > 2$, $t_1 < \ldots < t_m$, *be such that* $0 < f_i < A$, $\forall i$.
*(i) If* $f_1 \le f_2 \le \ldots \le f_m$ *, then there exists a point* $(b^*, c^*, \delta^*) \in S \times \mathbb{R}^m$,

$$S = \{(b, c) \in \mathbb{R}^2 : b > 0\,,\ c \ge 0\},$$

*at which the functional* $F$ *defined by (3) attains its infimum on the set* $S \times \mathbb{R}^m$.
*(ii) If* $f_1 \ge f_2 \ge \ldots \ge f_m$ *, then there exists a point* $(b^*, c^*, \delta^*) \in D \times \mathbb{R}^m$,

$$D = \{(b, c) \in \mathbb{R}^2 : b > 0\,,\ c \le 0\},$$

*at which the functional* $F$ *defined by (3) attains its infimum on the set* $D \times \mathbb{R}^m$.

**Proof. Case (i).** Since $F \ge 0$, there exists $F^* := \inf\limits_{(b,c,\delta) \in S \times \mathbb{R}^m} F(b, c, \delta)$.

(If $f_1 = f_2 = \ldots = f_m$, then $F\Big(\big(\frac{A}{f_1}\big)^\gamma - 1, 0, \mathbf{0}\Big) = 0$, and the functional $F$ has its global minimum for $b = \big(\frac{A}{f_1}\big)^\gamma - 1$, $c = 0$, $\delta = \mathbf{0}$; hence, the assertion of theorem was proved. So, let us suppose further that $f_1 < f_m$.)

Let $(b_n, c_n, \delta^n)$ be a sequence in $S \times \mathbb{R}^m$ such that

$$F^* = \lim_{n\to\infty} F(b_n, c_n, \delta^n) = \lim_{n\to\infty} \sum_{i=1}^m w_i \sqrt{\left(f_i - \frac{A}{(1 + b_n e^{-c_n \gamma (t_i + \delta_i^n)})^{1/\gamma}}\right)^2 + (\delta_i^n)^2}\,. \tag{4}$$

Note that the sequences $(\delta_i^n)$, $i = 1, \ldots, m$, are bounded. If it were not, we would have $\limsup\limits_n F(b_n, c_n, \delta^n) = \infty$, which contradicts the assumption (4).

We are going to show, by contradiction, that the sequence $(c_n)$ is also bounded. Assume that the sequence $(c_n)$ is unbounded. By the Bolzano-Weierstrass theorem, we may assume (without loss of generality, by taking appropriate subsequence if necessary) that $c_n \to \infty$ and

$$\lim_{n\to\infty} \delta_i^n = \delta_i^*\,, \quad i = 1, \ldots, m.$$

But, let us show that the infimum of the functional F cannot be attained in such way. We are going to find a point in $S \times \mathbb{R}^m$ at which the functional $F$ defined by (3) attains a value which is smaller than $\lim\limits_{n\to\infty} F(b_n, c_n, \delta^n)$.

Denote $M := \{1, \ldots, m\}$. Note that when $c_n \to \infty$, sequences $(b_n e^{-c_n \gamma (t_i + \delta_i^n)})$, $i \in M$, (or respectively, without loss of generality, their corresponding subsequences) have the following property:

If $\lim_{n\to\infty} b_n e^{-c_n\gamma(t_{i_0}+\delta_{i_0}^n)} = r < \infty$ for some $i_0 \in M$, then $\lim_{n\to\infty} b_n e^{-c_n\gamma(t_i+\delta_i^n)} = 0$ for every $i \in M$ such that $t_i + \delta_i^* > t_{i_0} + \delta_{i_0}^*$.

Namely, if $t_i + \delta_i^* > t_{i_0} + \delta_{i_0}^*$, then

$$\lim_{n\to\infty} b_n e^{-c_n\gamma(t_i+\delta_i^n)} = \lim_{n\to\infty} b_n e^{-c_n\gamma(t_{i_0}+\delta_{i_0}^n)} \cdot \lim_{n\to\infty} e^{-c_n\gamma(t_i+\delta_i^*-t_{i_0}-\delta_{i_0}^n)} = 0\,.$$

Similarly, if $\lim_{n\to\infty} b_n e^{-c_n\gamma(t_{l_0}+\delta_{l_0}^n)} = \infty$ for some $l_0 \in M$, then $\lim_{n\to\infty} b_n e^{-c_n\gamma(t_i+\delta_i^n)} = \infty$ for $i \in M$ such that $t_i + \delta_i^* < t_{l_0} + \delta_{l_0}^*$.

Because of that, for the sequences $(b_n e^{-c_n\gamma(t_i+\delta_i^n)})$, $i \in M$, (or respectively, by the Bolzano-Weierstrass theorem, for their corresponding subsequences), we may assume that one of the two following cases appears when $c_n \to \infty$.

**Subcase (i1)**. All the sequences $(b_n e^{-c_n\gamma(t_i+\delta_i^n)})$, $i \in M$, are unbounded and $\lim_{n\to\infty} b_n e^{-c_n\gamma(t_i+\delta_i^n)} = \infty$, $\forall i \in M$. In this case we have

$$F^* = \lim_{n\to\infty} F(b_n, c_n, \delta^n) = \sum_{i\in M} w_i \sqrt{f_i^2 + (\delta_i^*)^2} > \sum_{i\in M} w_i |f_i - f_1| = F\Big( (\frac{A}{f_1})^{1/\gamma} - 1, 0, \mathbf{0} \Big),$$

which contradicts the assumption (4). Therefore, in this subcase the infimum of the functional $F$ cannot be attained.

**Subcase (i2)**. There exists at least one $i_0 \in M$ such that:
the sequence $(b_n e^{-c_n\gamma(t_{i_0}+\delta_{i_0}^n)})$ is bounded, $\lim_{n\to\infty} b_n e^{-c_n\gamma(t_{i_0}+\delta_{i_0}^n)} = r_{i_0} < \infty$ and

$$\lim_{n\to\infty} b_n e^{-c_n\gamma(t_i+\delta_i^n)} = \begin{cases} \infty, & \text{for } i \in K_-, \\ 0, & \text{for } i \in K_+, \end{cases}$$

where

$$K_- := \{ i \in M : t_i + \delta_i^* < t_{i_0} + \delta_{i_0}^* \}, \qquad K_+ := \{ i \in M : t_i + \delta_i^* > t_{i_0} + \delta_{i_0}^* \}\,.$$

In order to show that the infimum of the functional F cannot be attained in such way, we are going to find a point in $S \times \mathbb{R}^m$ at which the functional $F$ defined by (3) attains a value which is smaller than $F^* = \lim_{n\to\infty} F(b_n, c_n, \delta^n)$.

By using the notations $\tau^* := t_{i_0} + \delta_{i_0}^*$ and $K := \{ i \in M : t_i + \delta_i^* = \tau^* \}$, we obtain

$$\lim_{n\to\infty} F(b_n, c_n, \delta^n) \geq \sum_{i\in K_-} w_i \sqrt{f_i^2 + (\delta_i^*)^2} + \sum_{i\in K} w_i |\delta_i^*| + \sum_{i\in K_+} w_i \sqrt{(f_i - A)^2 + (\delta_i^*)^2} =: F_0\,. \tag{5}$$

Let us show that there exists a point in the set $S \times \mathbb{R}^m$ at which the functional $F$ attains value which is smaller than $F_0$. For that purpose, let us define functions $\hat{b}(c)$, $\hat{\delta}_i(c)$, $i = 1, \ldots, m$, from $\mathbb{R}^+$ to $\mathbb{R}$:

$$\hat{b}(c) = b_0 e^{c\gamma\tau^*},$$

$$\hat{\delta}_i(c) = \begin{cases} \delta_i^*, & i \in K_- \cup K_+ \\ \tau^* - t_i - \frac{1}{c\gamma} \ln \frac{(\frac{A}{f_i})^\gamma - 1}{b_0}, & i \in K\,, \end{cases}$$

where

$$b_0 = \begin{cases} \max\{(\frac{A}{f_i})^\gamma - 1 \,:\, i \in K \,\&\, t_i \geq \tau^*\}, & \text{if } \{i \in K : t_i \geq \tau^*\} \neq \emptyset, \\ (\frac{A}{f_m})^\gamma - 1, & \text{if } \{i \in K : t_i \geq \tau^*\} = \emptyset. \end{cases}$$

From there it follows

$$\begin{aligned} F(\hat{b}(c), c, \hat{\delta}(c)) &= \sum_{i \in K_- \cup K_+} w_i \sqrt{\left(f_i - \frac{A}{(1 + b_0 e^{-c\gamma(t_i + \delta_i^* - \tau^*)})^{1/\gamma}}\right)^2 + (\delta_i^*)^2} \\ &\quad + \sum_{i \in K} w_i \left| \delta_i^* - \frac{1}{c\gamma} \ln \frac{(\frac{A}{f_i})^\gamma - 1}{b_0} \right| . \end{aligned} \tag{6}$$

For a sufficiently large positive $c$ we have

$$\sqrt{(f_i)^2 + (\delta_i^*)^2} > \sqrt{\left(f_i - \frac{A}{(1 + b_0 e^{-c\gamma(t_i + \delta_i^* - \tau^*)})^{1/\gamma}}\right)^2 + (\delta_i^*)^2}, \qquad \text{for } i \in K_- ,$$

$$\sqrt{(f_i - A)^2 + (\delta_i^*)^2} > \sqrt{\left(f_i - \frac{A}{(1 + b_0 e^{-c\gamma(t_i + \delta_i^* - \tau^*)})^{1/\gamma}}\right)^2 + (\delta_i^*)^2}, \qquad \text{for } i \in K_+ .$$

Further, by the property of monotonic data and by the definition of $b_0$, it is not difficult to see that for a sufficiently large $c$

$$|\delta_i^*| \geq \left| \delta_i^* - \frac{1}{c\gamma} \ln \frac{(\frac{A}{f_i})^\gamma - 1}{b_0} \right| , \qquad \text{for } i \in K .$$

Therefore, by comparing (5) and (6), from above inequalities it follows that for a sufficiently large positive $c$

$$F(\hat{b}(c), c, \hat{\delta}(c)) < F_0 \leq F^* ,$$

which contradicts the assumption (4). This means that, when $c_n \to \infty$, the infimum of the functional $F$ cannot be obtained.

Hence, the sequence $(c_n)$ is bounded. By the Bolzano-Weierstrass theorem we may assume that the sequence $(c_n)$ is convergent. Let $c_n \to c^*$.

Then, it follows from (3) that the sequence $(b_n)$ is bounded, too. If not, i.e. if $b_n \to \infty$, it would be

$$\lim_{n \to \infty} F(b_n, c_n, \delta^n) \geq \sum_{i \in M} w_i |f_i| > \sum_{i=1}^{m} w_i |f_i - f_1| = F\big((\frac{A}{f_1})^\gamma - 1, 0, \mathbf{0}\big),$$

which contradicts the assumption (4). Analogously, we can assume that $b_n \to b^* \geq 0$.

By continuity of the functional $F$, we get

$$\inf_{(b,c,\delta) \in S \times \mathbb{R}^m} F(b, c, \delta) = \lim_{n \to \infty} F(b_n, c_n, \delta^n) = F(b^*, c^*, \delta^*) .$$

Furthermore, for $b=0$, $c\in[0,\infty\rangle$ and $\delta\in\mathbb{R}^m$ we have

$$F(0,c,\delta)=\sum_{i=1}^{m} w_i\sqrt{(f_i-A)^2+(\delta_i)^2}>\sum_{i=1}^{m}|f_i-f_m|=F\Big((\frac{A}{f_m})^\gamma-1,0,\mathbf{0}\Big)\,.$$

Therefore, we conclude that $b^*>0$, i.e. $(b^*,c^*)\in S$.

**Case (ii).** Let $c<0$. Since

$$\sqrt{\left(f_i-\frac{A}{(1+be^{-c\gamma(t_i+\delta_i)})^{\frac{1}{\gamma}}}\right)^2+\delta_i^2}=\sqrt{\left(f_i-\frac{A}{(1+be^{-(-c)\gamma(-t_i-\delta_i)})^{\frac{1}{\gamma}}}\right)^2+(-\delta_i)^2}\,,$$

and the data $(-t_i,f_i)$ $i=1,\dots,m$ have the increasing property, this case reduces to *Case (i)*. $\square$

Let $A<0$. Since the symmetrical expressions

$$\Big(f_i-\frac{A}{(1+be^{-c\gamma(t_i+\delta_i)})^{1/\gamma}}\Big)^2=\Big((-f_i)-\frac{(-A)}{(1+be^{-c\gamma(t_i+\delta_i)})^{1/\gamma}}\Big)^2,\qquad i=1,\dots,m\,,$$

hold, from *Theorem 1.* we get the following corollary.

**Corollary 1.** *Let the given data $(w_i,t_i,f_i)$, $i=1,\dots,m$, $m>2$, $t_1<\ldots<t_m$, such that $A<f_i<0$, $\forall i$.*
*(i) If $f_1\ge f_2\ge\ldots\ge f_m$, then there exists a point $(b^*,c^*,\delta^*)\in S\times\mathbb{R}^m$, $S=\{(b,c)\in\mathbb{R}^2 : b>0\,,\,c\ge 0\}$, at which the functional $F$ defined by (3) attains its infimum on the set $S\times\mathbb{R}^m$.*
*(ii) If $f_1\le f_2\le\ldots\le f_m$, then there exists a point $(b^*,c^*,\delta^*)\in D\times\mathbb{R}^m$, $D=\{(b,c)\in\mathbb{R}^2 : b>0\,,\,c\le 0\}$, at which the functional $F$ defined by (3) attains its infimum on the set $D\times\mathbb{R}^m$.* $\square$

**Remark 1.** *In the approximation problem for the generalized logistic function one can also show analogously that* Theorem 1. *is applicable in the case of the approximation by the total least squares criterion, i.e.* Theorem 1. *is applicable for the functional $T$ defined by (2).*

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